CHAPTER ONE:

TRIGONOMETRY

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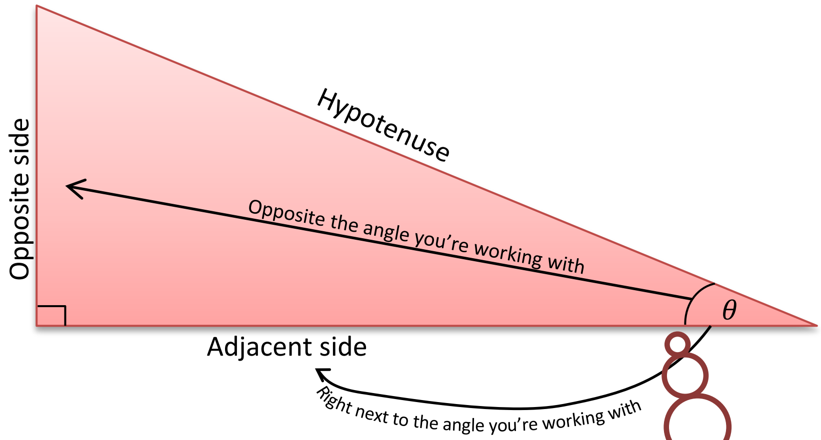
1.1 Quick REVISION OF GRADE 10 work:

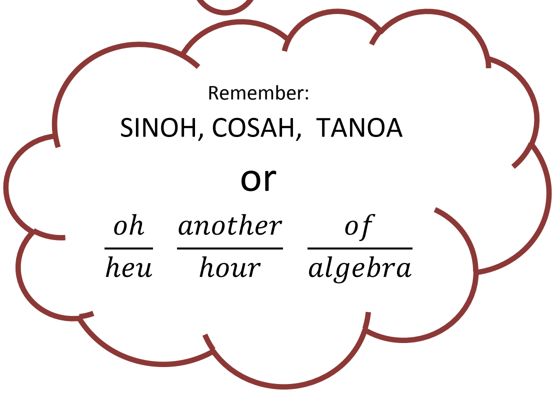
1.1.1 Solving TRIANGLES using SIN, COS AND TAN



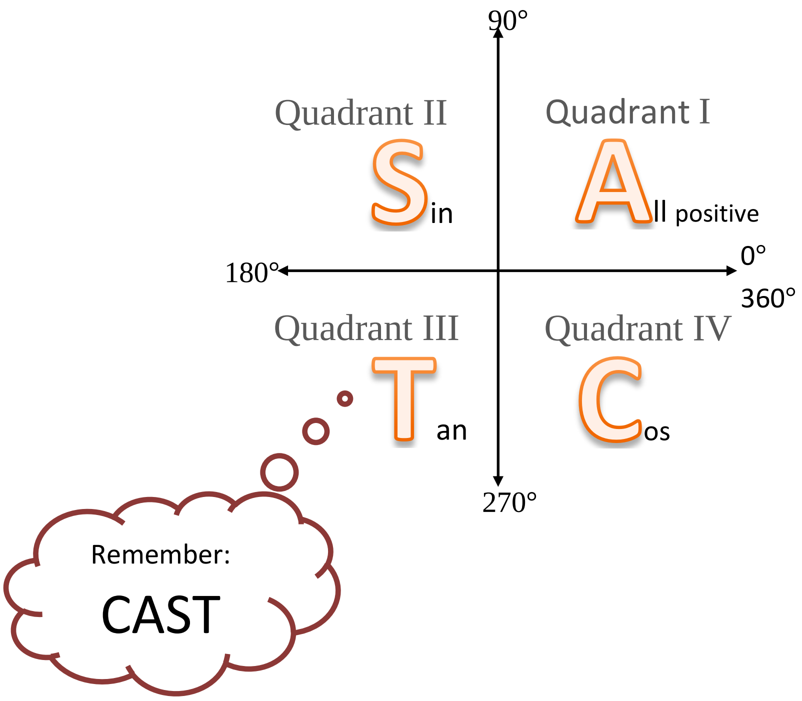
$$\sin\theta = \frac{opposite\ side}{hypotenuse}$$

$$\cos\theta = \frac{adjacent\ side}{hypotenuse}$$

$$\tan\theta = \frac{opposite\ side}{adjacent\ side}$$



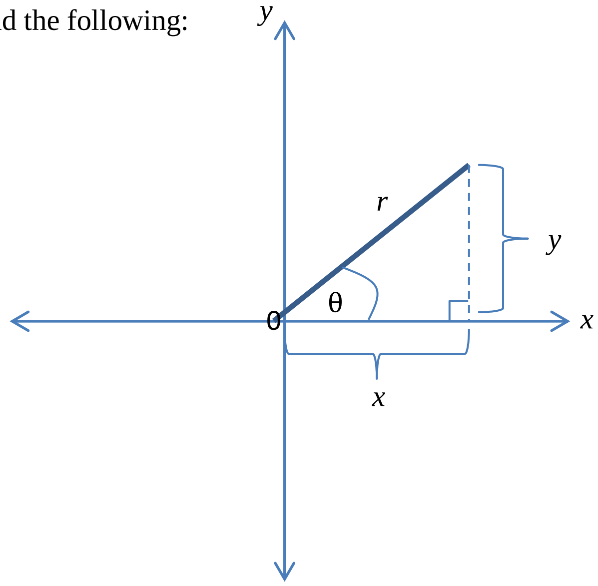
1.1.2 The CARTESIAN PLANE



1.1.3 Using PYTHAGORAS

Pythagoras is quite useful when you want to calculate a value of a function with NO angle given and a value when certain specifications are given – example 1. It is also used to give values in terms of variables given (in some cases a certain angle would be involved) – example 2.

Firstly understand the following:



therefore…

$\sin\theta = \frac{y}{r}$ $\qquad$ $\cos\theta = \frac{x}{r}$ $\qquad$ $\tan\theta = \frac{y}{x}$

remember: $\qquad x^2 + y^2 = r^2$

x and y can be positive or negative as they are coordinates

r can never be negative as it is a length (hypotenuse)

Specifications

Example

Given that $\boldsymbol{\sin\theta = \frac{3}{5}}$, and $\boldsymbol{90° < \theta < 360°}$, find the value of $\cos\theta + \tan\theta$ **without the use of a calculator**.

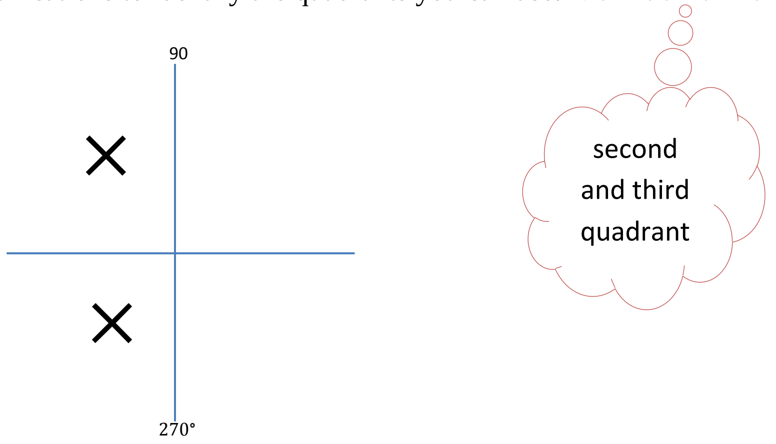
Unknown angle, known value.

the words: **without a calculator** – this means you cannot use the calculator to calculate the angle and then use the angle to calculate the values.

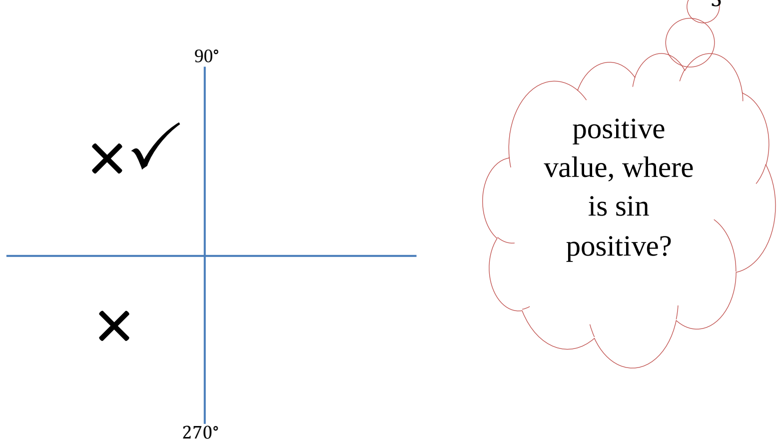
Answer:

Step 1: Sketch the situation on a Cartesian plane (this would help in determining the signs of the function values).

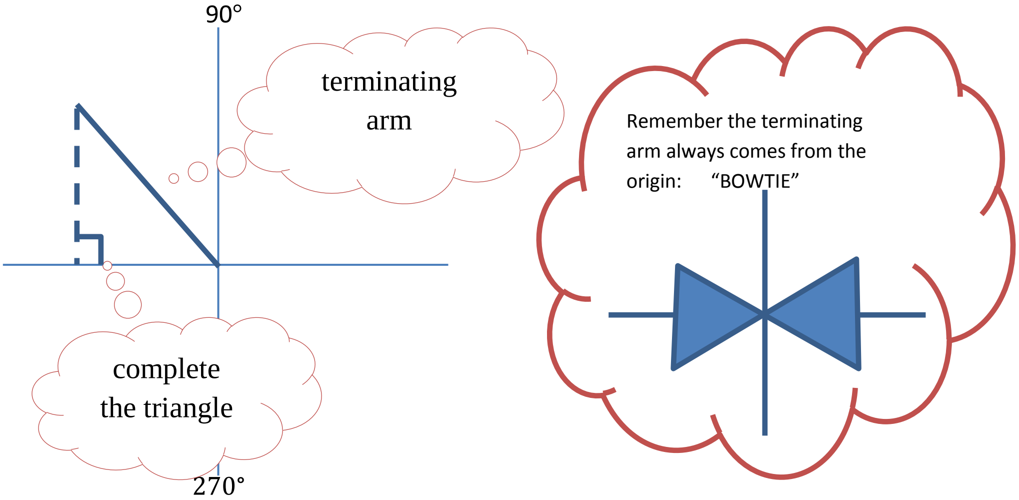
→ use specifications to identify the quadrants you can use: $90° < \theta < 270°$:



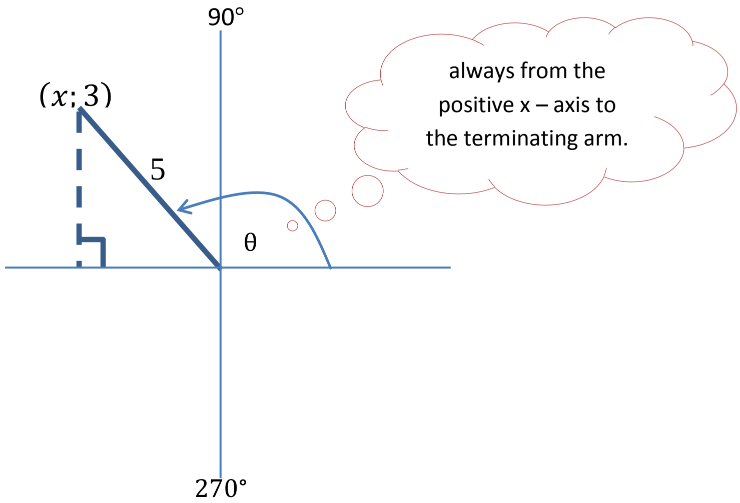
→ now use the info given to determine the exact quadrant: $\sin\theta = \frac{3}{5}$:



→ now draw your terminating arm and complete a right angled triangle towards the x – axis:



→ now fill in $(x; y)$ and r and put the angle in place: $\sin\theta = \frac{3}{5} = \frac{y}{r}$



Step 2: Use PYTHAGORAS to calculate the missing value, taking note of the quadrant in which you are working. (in this case you are working in the second quadrant where x is negative).

$$x^2 + y^2 = r^2$$

$$\therefore x^2 = r^2 - y^2$$

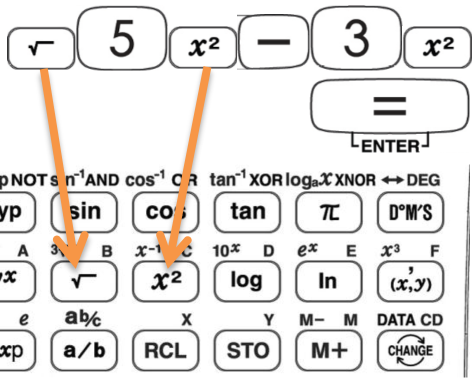
$$\therefore x^2 = (5)^2 - (3)^2$$

$$\therefore x = \pm\sqrt{(5)^2 - (3)^2}$$

$$\therefore x = \pm 4$$

$$\therefore x = -4 \text{ only}$$

Press



Step 3: Summarize the values of x, y and r. Use these values to calculate the values asked.

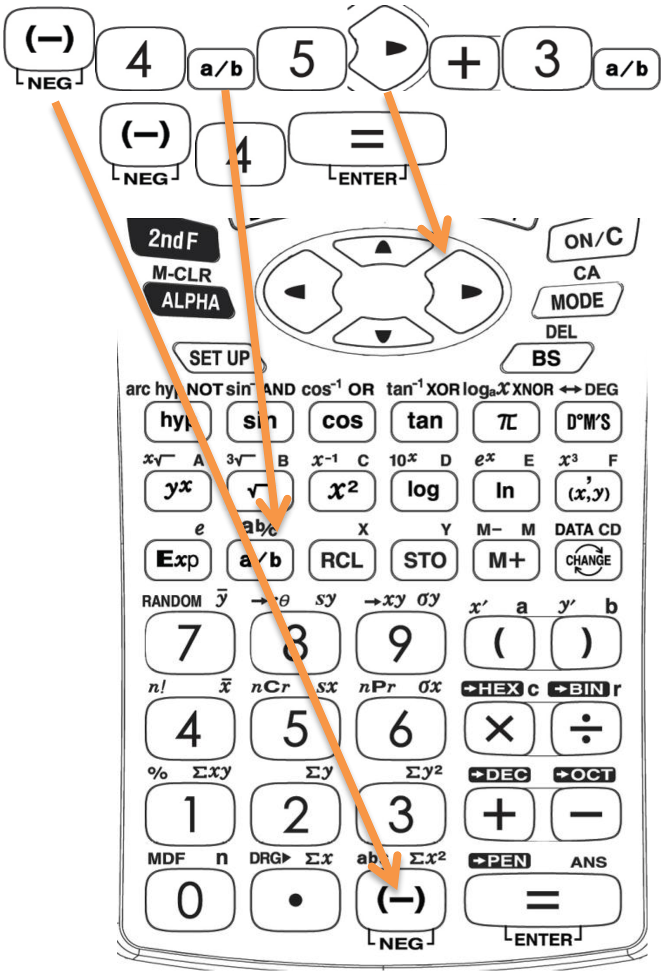
$x = -4$
$y = 3$
$r = 5$

$$\cos\theta + \tan\theta \qquad \rightarrow \left(\frac{x}{y} + \frac{y}{x}\right)$$

$$= -\frac{4}{5} + \frac{3}{-4}$$

$$= -1\frac{11}{20}$$

Press the following:



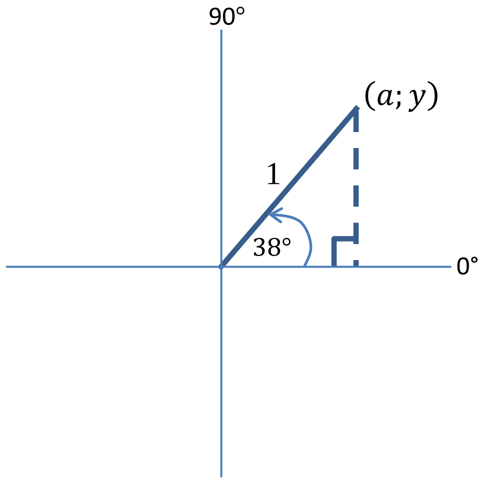
Example

If $\cos 38° = a$, find the value of the following in terms of a:

a. $\tan 38°$ b. $\sin 38°$

Answer:

$$\cos 38° = a = \frac{a}{1} = \frac{x}{r}$$



$x^2 + y^2 = r^2$

$\therefore y^2 = r^2 - x^2$

$\therefore y = \pm\sqrt{(1)^2 - (a)^2}$

$\therefore y = \sqrt{1 - a^2}$ only

$$\boxed{\begin{array}{c} x = a \\ y = \sqrt{1-a^2} \\ r = 1 \end{array}}$$

a. $\tan 38° = \frac{y}{x} = \frac{\sqrt{1-a^2}}{a}$

b. $\sin 38° = \frac{y}{r} = \frac{\sqrt{1-a^2}}{1} = \sqrt{1-a^2}$

Activity 1:

a. If $\sin\beta = -\frac{5}{13}$ and $0° < \beta < 270°$, determine the value of $\frac{1}{\tan\beta}$.

b. If $\cos\theta > 0$ and $\tan\theta = 1$, determine the value of $\frac{\sin\theta + \tan\theta}{\cos\theta}$.

c. Determine the value of $\sin^2\alpha$ if $\cos\alpha = \frac{2}{3}$.

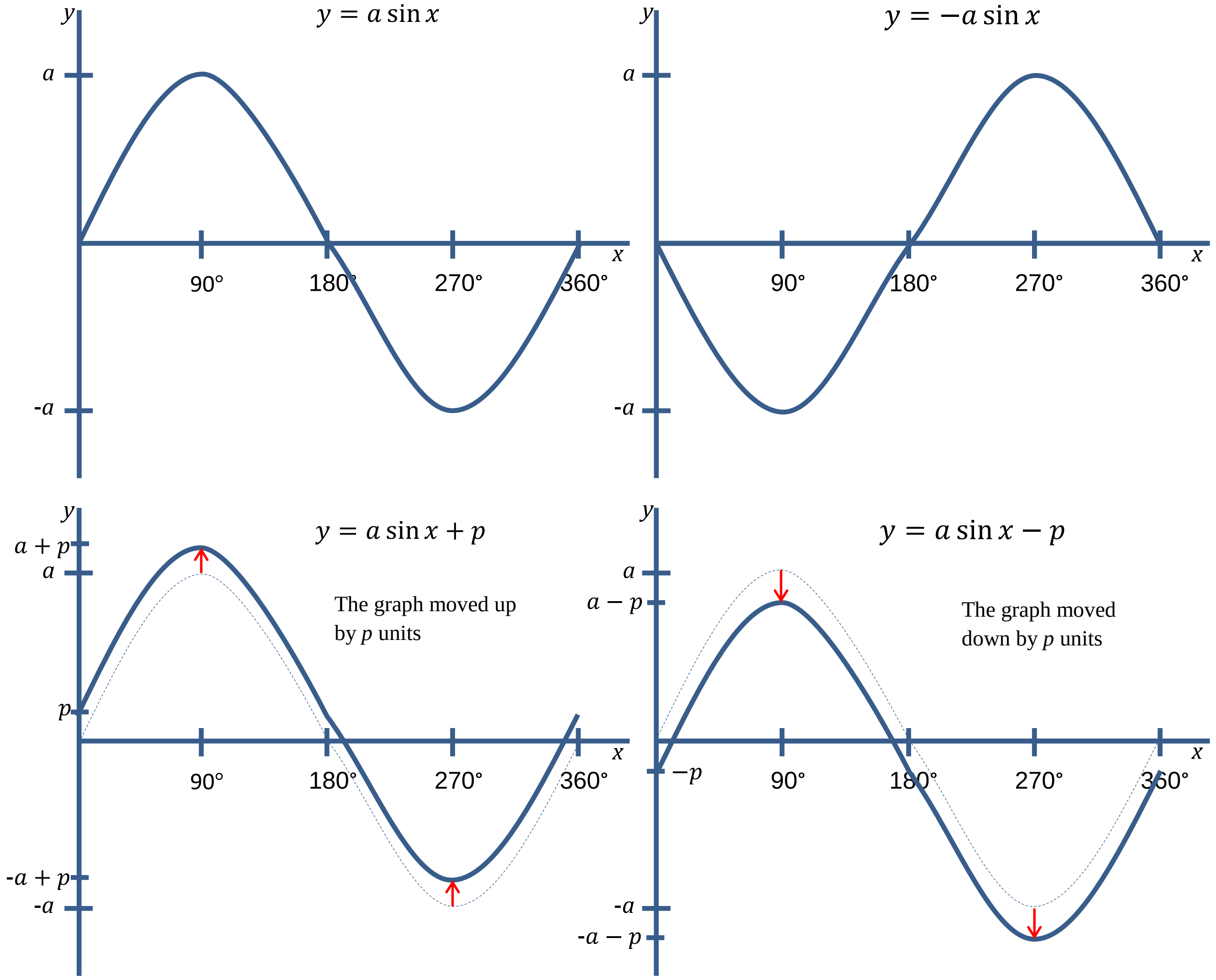
$\sin^2\alpha = (\sin\alpha)^2$

d. If $2\cos\gamma - 1 = \frac{1}{2}$ and $\tan\gamma < 0$, evaluate $\sin\gamma \, . \cos\gamma - \tan\gamma$.

e. If $\sin A = \frac{5}{7}$ and $\cos B = \frac{3}{4}$ and $\cos A < 0$; $\sin B > 0$, determine the value of $\frac{\tan A - \tan B}{\tan A \tan B + 1}$.

1.1.4 Basic Trig FUNCTIONS

THE SIN GRAPH:



Notes on the Sin graph:

- Shape: Funny S on it's side
- The 'a' is called the AMPLITUDE – The distance between the middle of the graph and the Maximum or Minimum of the graph.
- The 'p' is the amount of units that the graph is shifting vertically (up or down).
- In general, you get one Sin graph every 360°, this occurrence is called, the PERIOD of the graph.
- Four significant points: 0°⟶Start at origin;

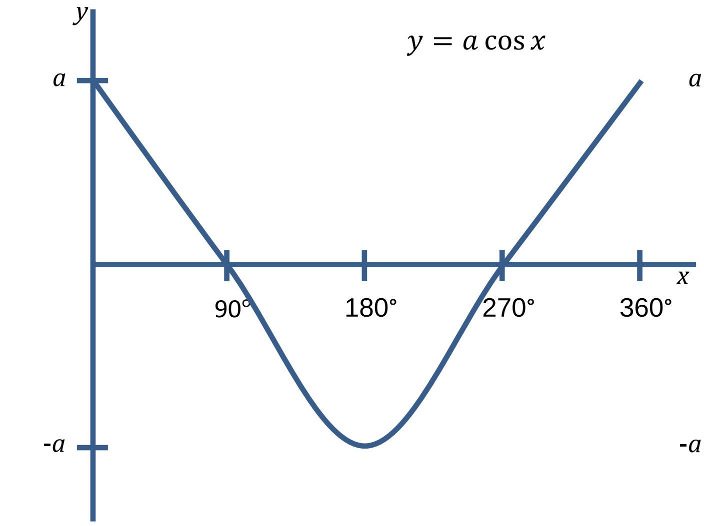
  90°⟶Maximum/Minimum at a/-a;

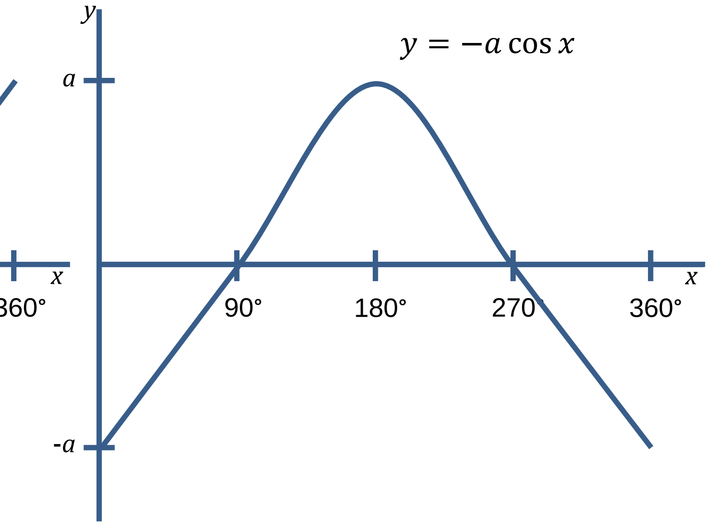
  180°⟶Cut through the x axis;

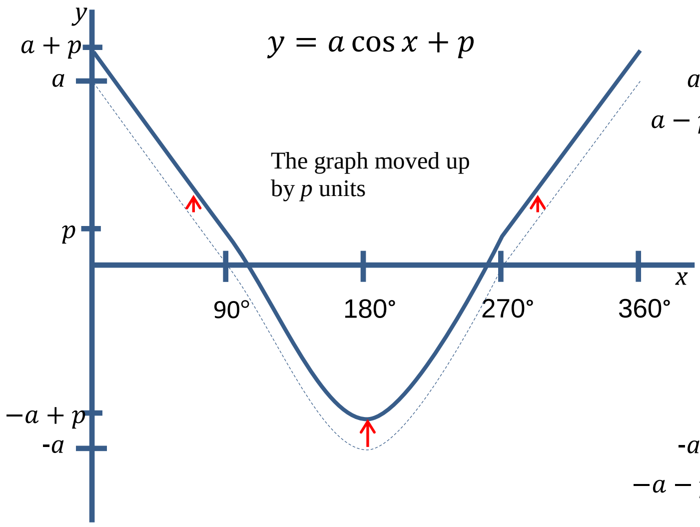
  270°⟶Maximum/Minimum at a/-a;

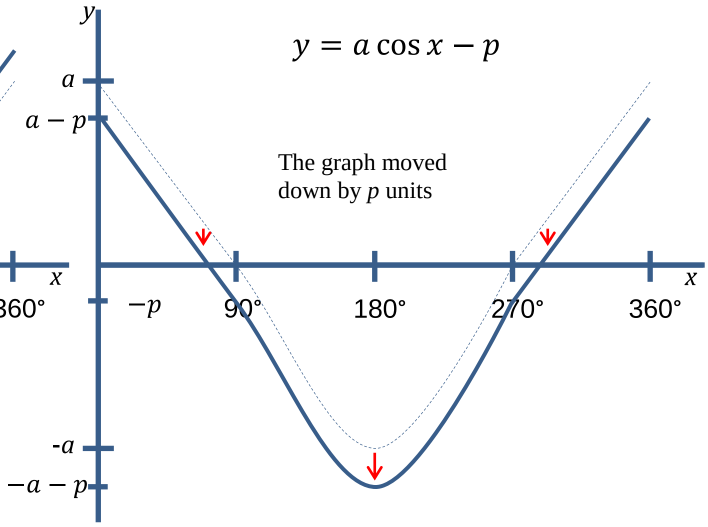
  360°⟶Stop at x axis.

THE COS GRAPH:









Notes on the Cos graph:

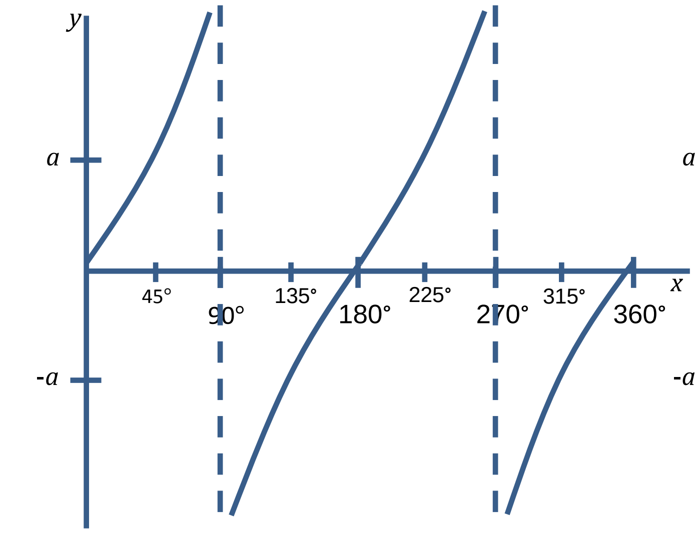
- Shape: V when positive, A when negative
- 'a' is the AMPLITUDE – Previously explained.
- 'p' is the vertical shift – Previously explained.

Refer to sin graph

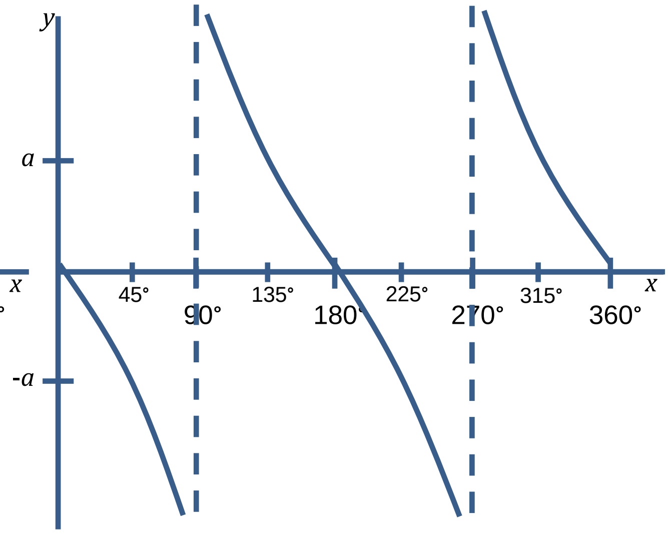
- In general, the Cos graph completes within 360°, therefore, the PERIOD of the Cos graph is also 360°.
- Four significant points: 0°⟶Start at Maximum/Minimum , a/-a;
  90°⟶Cut through the x axis;
  180°⟶Maximum/Minimum at a/-a;
  270°⟶Cut through the x axis;
  360°⟶Stop at Maximum/Minimum, a/-a.

THE TAN GRAPH:

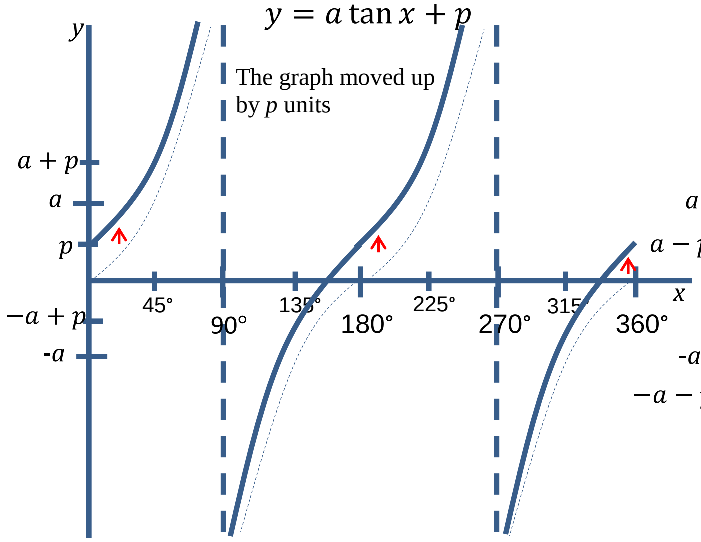
$y = a \tan x$



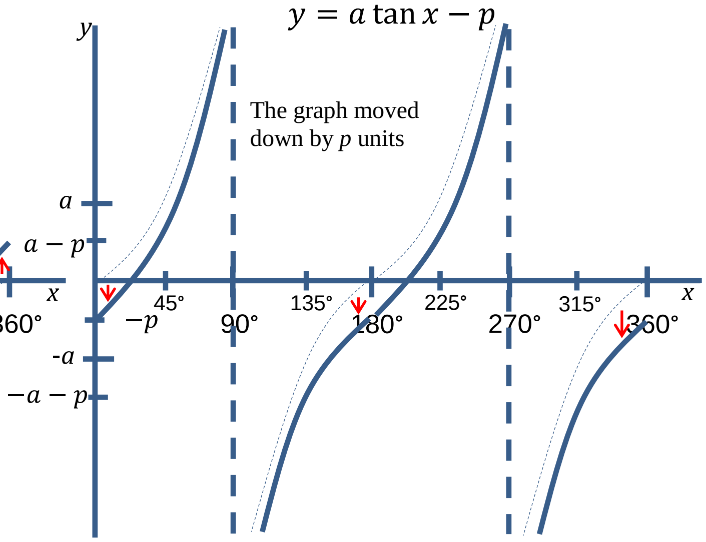
$y = -a \tan x$



$y = a \tan x + p$



$y = a \tan x - p$



Notes on the Tan graph:

- Shape: Completely different to the Sin and Cos graph, it has no maximum or minimum with a slight bend in the middle.
- The *'a'* is where 45°; 135°; 225° and 315° would work together, i.e. the graph would go through the coordinates $(45°; a)$; $(135°; -a)$; etc.
- The *'p'* is the amount of units that the graph is shifting vertically (up or down).
- The Tan graph has ASYMPTOTES which are the *x* values where the graph does not exist, think of it as an electric fence that your graph cannot touch. These ASYMPTOTES are at $90°; 270°$; etc. They HAVE to be INDICATED on your graph at all times with a broken line.
- In general, the Tan graph completes a full cycle between asymptotes within 180°, i.e. the distance between the asymptotes is 180°. Thus the distance between the asymptotes is now called the PERIOD.
- A few significant points: Every 45° something happens:
  - $0° \longrightarrow$ Start at the origin
  - $45° \longrightarrow$ Goes through $(45°; a)$
  - $90° \rightarrow$ Asymptote
  - $135° \rightarrow$ Goes through $(135°; -a)$
  - $180° \rightarrow$ Cut though the *x* axis
  - $225° \rightarrow$ Goes through $(225°; a)$
  - $270° \rightarrow$ Asymptote
  - $315° \rightarrow$ Goes through $(315°; -a)$
  - $360° \rightarrow$ Stop on *x* axis

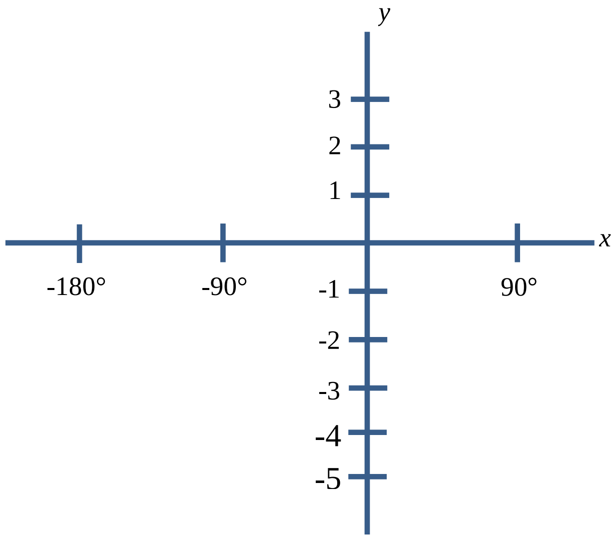
Example

Sketch the graph $y = 3\sin x - 2; x \in [-180°; 90°]$

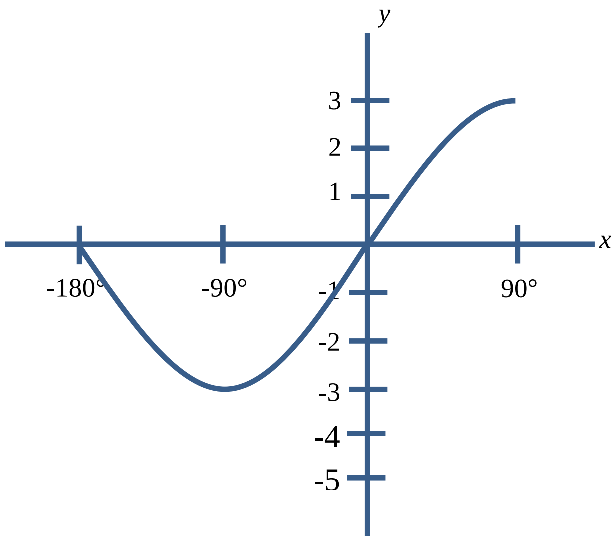
Answer:

If you were to do this manually, you would have to keep in mind, all the significant points and shift them accordingly, this is how:

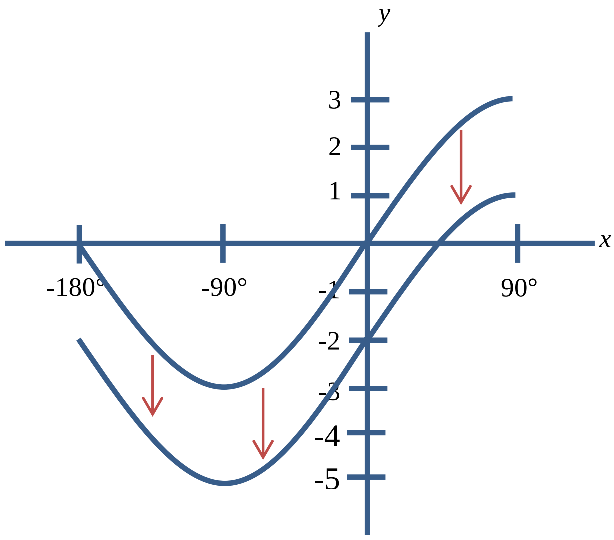
STEPS:



1. Prepare the Cartesian plane, remember to label you axes clearly! (Also remember that you are going to shift the graph 2 units down, therefore, provide enough space for shifting.)



2. Sketch $y = 3\sin x$ roughly on the prepared plane, keeping in mind all the significant points (see above).



3. Now shift this graph down by 2 units, thereby having finished sketching $y = 3\sin x - 2$.

You can also save some time and use a calculator to help you:

How to use the SHARP EL 535 WT CALCULATOR to make these steps in Trig graphs easy:

Buttons to press | On Screen

MODE

<MODE>

0:NORMAL 1:STAT

3

TABLE MODE

Function?

3 **sin** **RCL** **RCL** **−** **2**

3sinX-2

= (ENTER)

Wherever you want to start reading coordinates, you press this value at X_Start, in our case: -180

X_Step is used to press in the units you want the x – values to be apart, in the sin graph, something happens every 90°, however, to get a nice curve, we'll take 45°

X_Start:
0.
X_Step:
1.

− **1** **8** **0** **=** (ENTER)

4 **5**

X_Start:
-180.
X_Step:
45

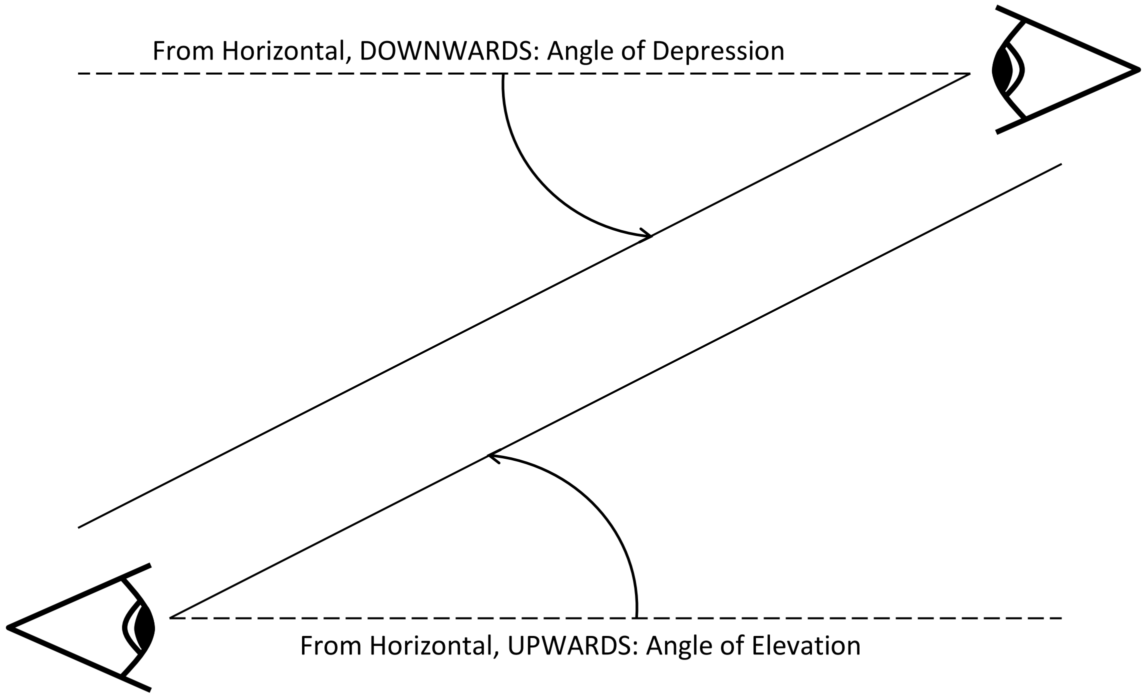
= (ENTER)

This screen now gives you coordinates, the *x* coordinate under X and the *y* coordinate under ANS, simply plot the points and connect the dots. You can scroll up or down to get the other coordinates.

X	ANS
-180	-2
-135	-4.1213
-90	-5

1.1.5 Angle of DEPRESSION AND ELEVATION

You always measure an angle of Depression or Elevation from a HORIZONTAL LINE downwards or upwards.

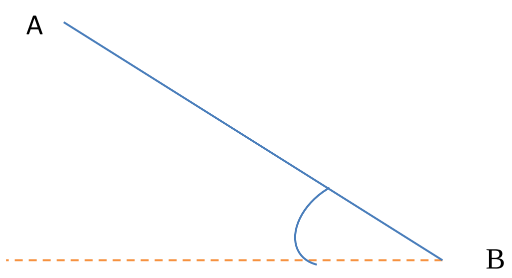


No matter what the question, always get two points to work from and check whether the story is mentioning an ANGLE OF DEPRESSION or an ANGLE OF ELEVATION.

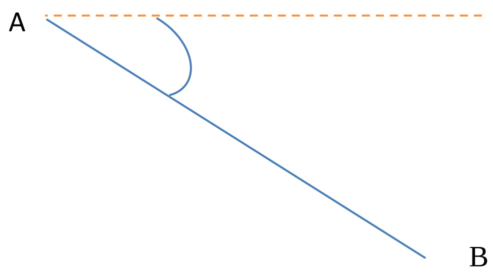
According to that you draw a straight horizontal line from the bottom point (in the case of an ANGLE OF ELEVATION), or from the top point (in the case of an ANGLE OF DEPRESSION), connect the two points and use the angle inbetween your horizontal line and connected line.

Wording of story sums can be confusing so here are a few examples:

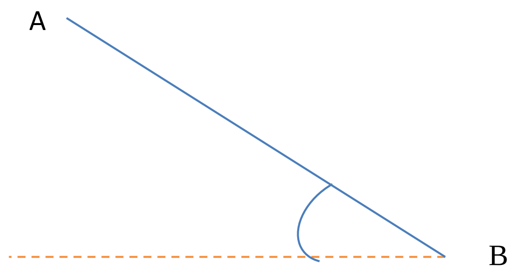
Angle of Elevation from A to B



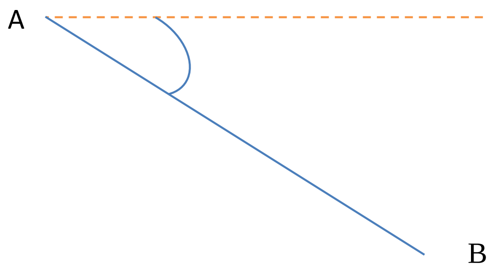
Angle of Depression from A to B



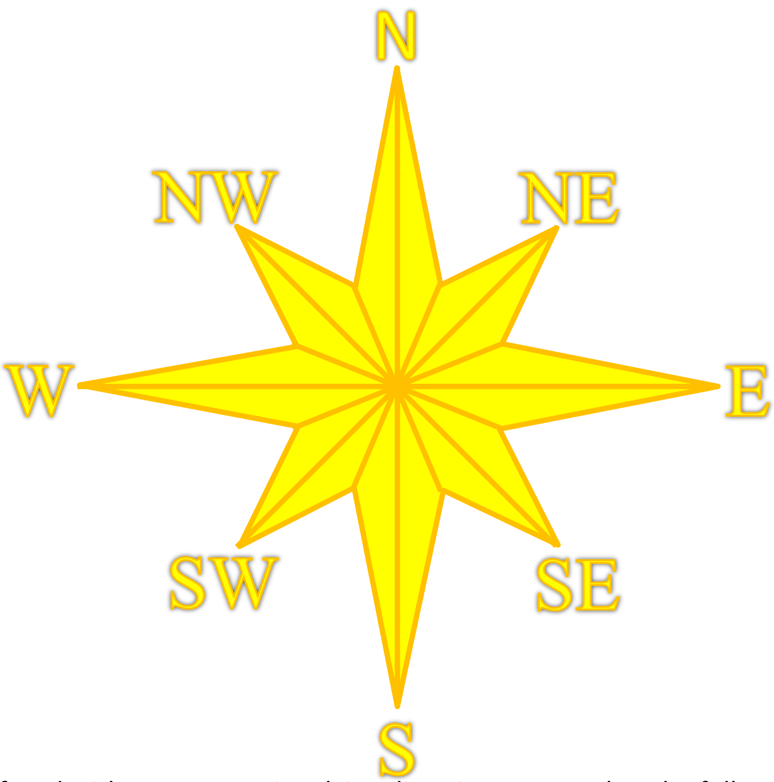
Angle of Elevation from B to A



Angle of Depression from B to A



1.1.6 NORTH, EAST, SOUTH, WEST



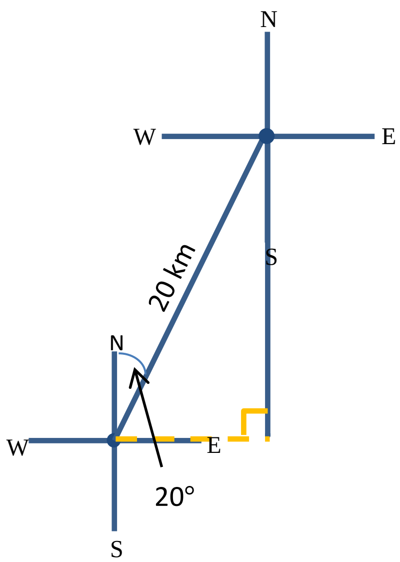
When faced with a story sum involving directions, remember the following:

- Always start in the middle of the compass points, wherever you end, draw another set of compass points.
- When using the compass, you measure degrees in an clockwise direction.

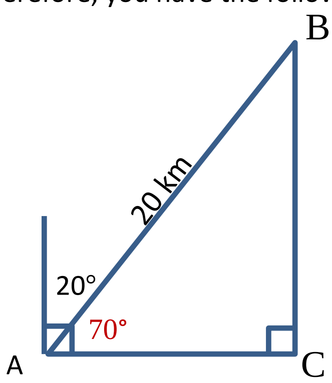
The following **example** will illustrate how easy this can be:

A ship travels 20° east of north for 20 km, it then turns and travels south until it's current position is in line with it's original position, how far away is the ship from it's original starting point, and how far has the ship travelled in total?

Sketch (This sketch is what YOU would have to draw)



Therefore, you have the following:



Answer:

The question is first asking the distance AC, then the distance A to B to C, i.e. you have to calculate AC and BC:

$\cos \mathrm{A} = \frac{\mathrm{AC}}{\mathrm{AB}}$
$\therefore \mathrm{AC} = \mathrm{AB} \cos \mathrm{A}$
$\therefore \mathrm{AC} = 20 \cos 70°$
$\therefore \mathrm{AC} = 6.8$ km

Therefore, the ship is 6.8 km from its original starting point.

$\sin \mathrm{A} = \frac{\mathrm{BC}}{\mathrm{AB}}$
$\therefore \mathrm{BC} = \mathrm{AB} \sin \mathrm{A}$
$\therefore \mathrm{BC} = 20 \sin 70°$
$\therefore \mathrm{BC} = 18.8$ km

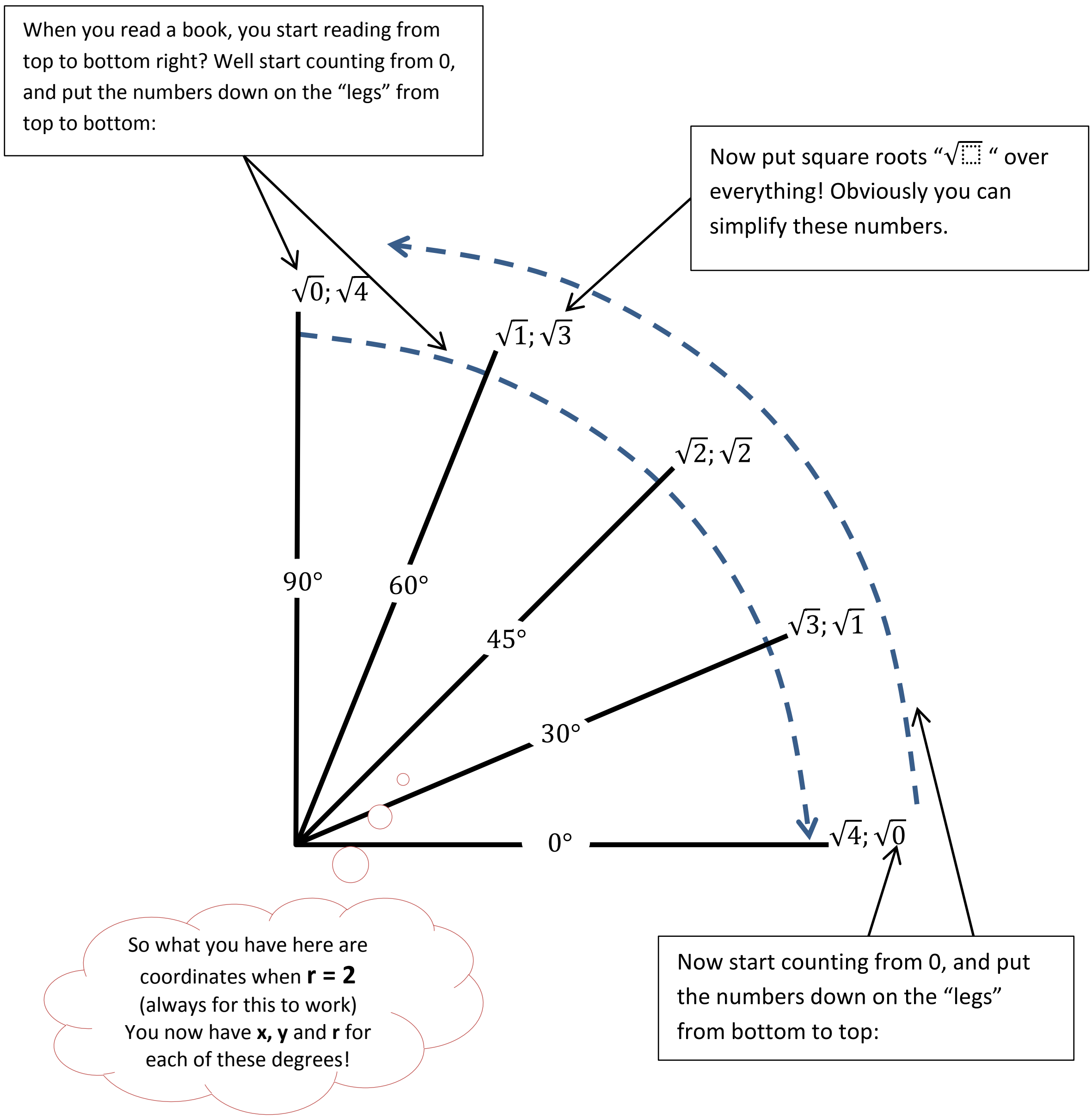
Therefore, the distance the ship travelled was:
$20 + 18.8$
$= 38.8$ km in total

1.2 Revision of GRADE 11 work:

1.2.1 SPECIAL ANGLES

Special angles are angles that you can find the value of without using a calculator, these are: 0°; 30°; 45°; 60° and 90°. You can also add to this list: 180°; 270° and 360°, as these are significant values on graphs.

Here's an easy way of remembering Special angles:



Example

Find the value of the following **without the use of a calculator**:

a. $\sin 0°$ b. $\sin 30° \times \cos 60°$ c. $\dfrac{\tan 30°}{\sin 45°}$

d. $\dfrac{\sin 60°}{\cos 60°}$ e. $\sin^2 45° + \cos^2 45\,°$

Answers:

a. $\sin 0° \; \left(\dfrac{y}{r}\right)$

$= \dfrac{0}{2}$

$= 0$

b. $\sin 30° \times \cos 60°$

$= \dfrac{1}{2} \times \dfrac{1}{2}$

$= \dfrac{1}{4}$

c. $\dfrac{\tan 30°}{\sin 45°}$

$= \dfrac{\frac{1}{\sqrt{3}}}{\frac{\sqrt{2}}{2}}$

$= \dfrac{1}{\sqrt{3}} \times \dfrac{2}{\sqrt{2}}$

$= \dfrac{1}{\sqrt{6}}$

d. $\dfrac{\sin 60°}{\cos 60°}$

$= \dfrac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}$

$= \dfrac{\sqrt{3}}{2} \times \dfrac{2}{1}$

$= \sqrt{3}$

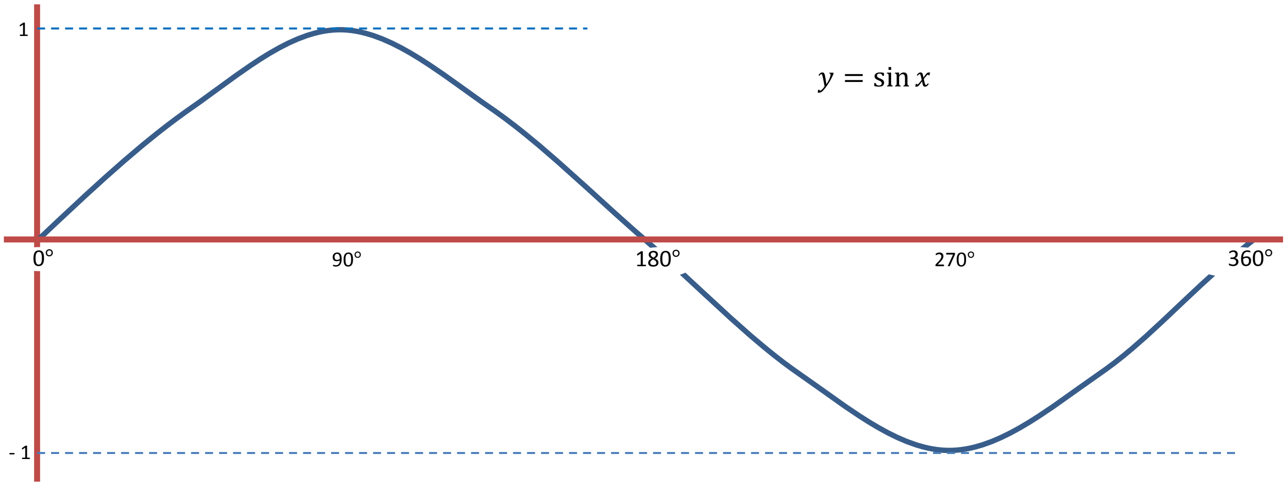
e. $\sin^2 45° + \cos^2 45$

$= \left(\dfrac{\sqrt{2}}{2}\right)^2 + \left(\dfrac{\sqrt{2}}{2}\right)^2$

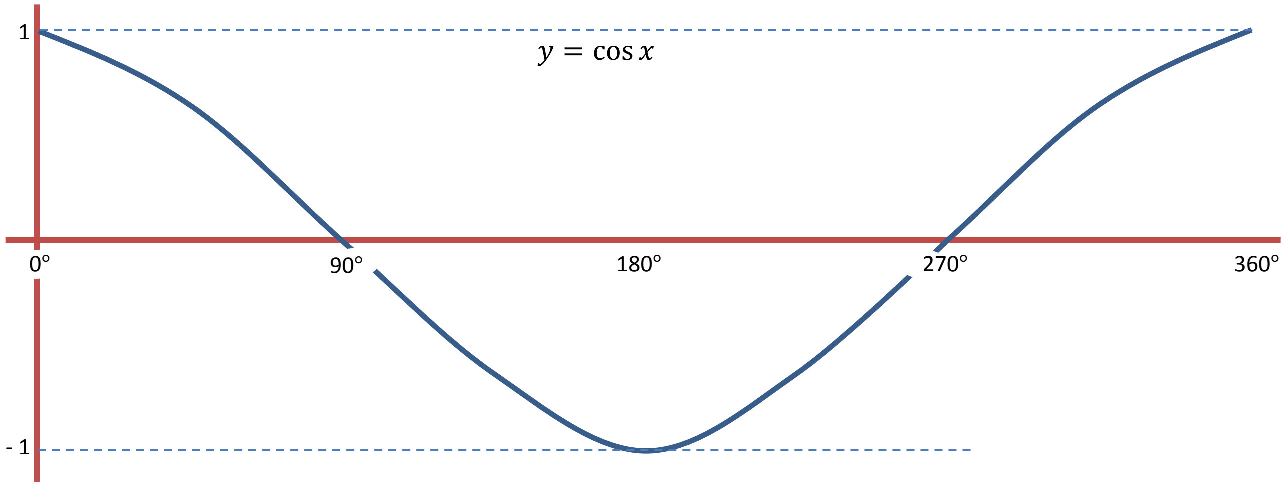
$= \dfrac{2}{4} + \dfrac{2}{4}$

$= 1$

Adding 180°, 270° and 360°



According to the sine graph: $\sin 90° = 1;\ \sin 180° = 0; \sin 270° = -1; \sin 360° = 0$



According to the cos graph: $\cos 90° = 0; \cos 180° = -1\,; \cos 270° = 0; \cos 360° = 1$

Activity 2

Calculate the value of the following without the use of a calculator:

a. $\frac{\sin 30°.\cos 45°}{\tan 60°}$

b. $\frac{\sin 30°}{\cos 30°} \times \frac{1}{\tan 30°}$

c. $\sin^2 60° + \cos 60°$

d. $\tan 45°$

e. $\tan 90°$

f. $\frac{\sin 90°}{\cos 90°}$

1.2.2 SIMPLIFYING AND PROVING IDENTITIES

Identities are used to prove one side equal to another. They are used in a number of different ways. You'll find that this is where the "ALGEBRA" part comes in. We're going to use identities, together with special angles, reduction formulae (still to come) and co-functions (still to come) to do some serious trig!

> Study work: $\frac{\sin\theta}{\cos\theta} = \tan\theta$ (Have a look at the previous exercise and find a proof of this…)
>
> $\sin^2\theta + \cos^2\theta = 1$ (Again, have a look at the exercise for a proof…)
>
> *You'll have to know these off by heart and be able to use them back and forth,*
>
> *also:* According to basic ALGEBRA:
>
> If $\sin^2\theta + \cos^2\theta = 1$ then
>
> $\sin^2\theta = 1 - \cos^2\theta$ and
>
> $\cos^2\theta = 1 - \sin^2\theta$

So what do we do with this knowledge?

We can SIMPLIFY or PROVE LHS = RHS!

Example:

Simplify:

1. $\tan\theta\,.\cos\theta$ 2. $\frac{1-\cos\theta}{\sin^2\theta}$

Answers:

	Steps:
1. $\tan\theta\,.\cos\theta$	
$= \frac{\sin\theta}{\cos\theta}.\cos\theta$	1. Turn everything into sin and cos
$= \sin\theta$	2. Use Algebraic knowledge to simplify as normal.

NOTE: If you get stuck, substitute sin with y and cos with x, simplify and substitute back. (Don't make a habit of this though, it's only for emergencies and you are NOT allowed to show any y's or x's in your steps when writing exams!)

2. $\frac{1-\cos\theta}{\sin^2\theta}$

$= \frac{1-\cos\theta}{1-\cos^2\theta}$

1. Seeing as though you cannot change anything or cancel anything, see if you can EXPAND something.

$= \frac{1-\cos\theta}{(1-\cos\theta)(1+\cos\theta)}$

2. Use Algebraic knowledge to simplify as normal.

$= \frac{1}{1+\cos\theta}$

Example

Prove the following:

1. $\frac{\sin^2 x.\cos x.\tan x}{1-\cos^2 x} = \sin x$

2. $\frac{1+2\sin x.\cos x}{\sin x+\cos x} = \cos x\,(\tan x + 1)$

Answers:

Steps

1. LHS $= \frac{\sin^2 x.\cos x.\tan x}{1-\cos^2 x}$

1. Choose a side to start with, preferably a side that you can actually manipulate with identities and algebraic knowledge.

$= \frac{\sin^2 x.\cos x.\frac{\sin x}{\cos x}}{1-\cos^2 x}$

2. Change everything to sin and cos.

$= \frac{\sin^2 x.\sin x}{\sin^2 x}$

3. Use algebraic knowledge and square identities to simplify.

$= \sin x$

= RHS

4. Always end with this statement. (Note: If you do not end up with the other side, you are welcome to take the other side and simplify it as well, after you've found that both sides eventually simplify to the same expression, you end your statement with LHS = RHS)

2. LHS = $\frac{1+2\sin x.\cos x}{\sin x+\cos x}$

$= \frac{\sin^2 x+\cos^2 x+2\sin x.\cos x}{\sin x+\cos x}$

1. Sometimes you need to expand "1", especially when you feel that you are "stuck" or cannot algebraically do anything at this stage.

$= \frac{\sin^2 x+2\sin x.\cos x+\cos^2 x}{\sin x+\cos x}$

2. Rearrange the top expression, it now looks like a quadratic equation (same as: $y^2 + 2yx + x^2$), now you can factorise as you would normally in a fraction.

$= \frac{(\sin x+\cos x)^2}{\sin x+\cos x}$

$= \sin x + \cos x$

Note: This does not look like the RHS, now, simplify the RHS.

RHS $= \cos x\,(\tan x + 1)$

$= \cos x \left(\frac{\sin x}{\cos x} + 1\right)$

$= \sin x + \cos x$

$\therefore$ LHS = RHS

Activity 3

3.1 Simplify the following:

a. $\frac{\sin x}{\tan x}$

b. $\cos^2 y\,(1 - \sin^2 y)$

c. $\frac{\tan^2\alpha}{\sin^2\alpha}(\cos^2\alpha - 1)$

d. $\sin x - \tan x\,.\cos x$

e. $\tan^2 y\,.\cos^2 y + \frac{\sin^2 y}{\tan^2 y}$

f. $\frac{(\tan^4\beta-1)(1-\sin^2\beta)^2}{\sin^2\beta-\cos^2\beta}$

3.2 Prove the following:

a. $\tan A\,.\sin^2 A\,.\cos A = \sin^3 A$

b. $\cos^2 B\,.\tan^2 B - 1 = -\cos^2 B$

c. $\frac{1}{1-\sin F} + \frac{1}{1+\sin F} = \frac{2}{\cos^2 F}$

d. $\frac{1-2\sin^2\gamma}{\sin\gamma.\cos\gamma} = \frac{1-\tan\gamma}{\tan y}$

1.2.3 REDUCTION FORMULAE

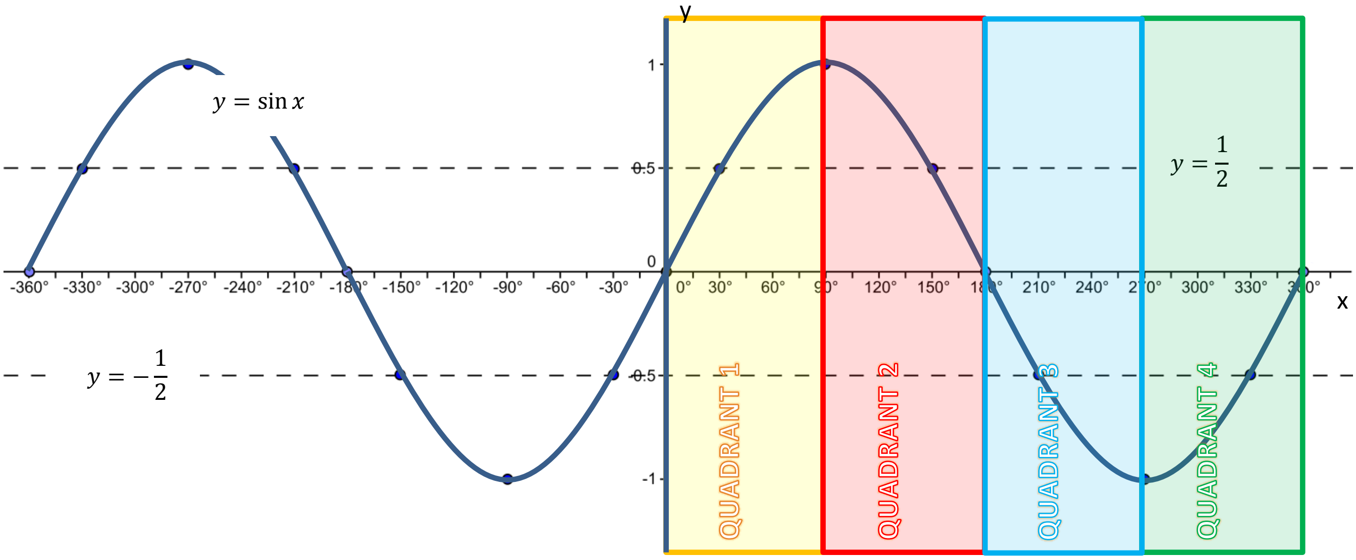
$(\mathbf{180° \pm \boldsymbol{\theta}}); (\mathbf{360° \pm \boldsymbol{\theta}})$

Reduction Formulae are generally used for one purpose … to REDUCE an angle to a more workable size.

What constitutes a workable size?

- You can use Reduction Formulae to reduce angles for e.g. 135° to 45° and then use special angles to find the value of the function without using a calculator.
- You can reduce an angle to its smallest size to cancel a certain function with another.
- To make equations easier, etc

How it works: Remember trig functions/graphs? Here's an example of how reduction formulae works, this example is also applicable to trig equations, which we will revise later in this chapter.



Look at the graph and determine where the graph $y = \sin x$ cuts the line $y = \frac{1}{2}$.
The two graphs cut at the following x – values: -330°;-210°; 30° and 150°.

In Quadrant 1, you have $\sin 30° = \frac{1}{2}$

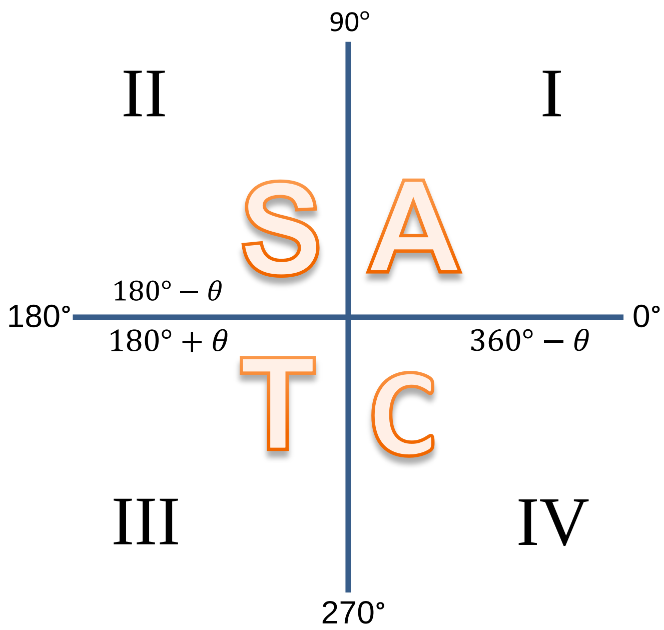
In Quadrant 2, you have $\sin 150° = \frac{1}{2}$, (note: $180° - 30° = 150°$)

In Quadrant 3, you have $\sin 210° = -\frac{1}{2}$, (note: $180° + 30° = 210°$)

In Quadrant 4, you have $\sin 330° = -\frac{1}{2}$, (note: $360° - 30° = 330°$)

So in actual fact, you'll get all the x – values that would produce $\frac{1}{2}$ or $-\frac{1}{2}$ by simply adding or subtracting 30° from different quadrant points!

Summary:



180°- θ
$\sin(180° - \theta) = \sin\theta$
$\cos(180° - \theta) = -\cos\theta$
$\tan(180° - \theta) = -\tan\theta$

180°+ θ
$\sin(180° + \theta) = -\sin\theta$
$\cos(180° + \theta) = -\cos\theta$
$\tan(180° + \theta) = \tan\theta$

360° - θ
$\sin(360° - \theta) = -\sin\theta$
$\cos(360° - \theta) = \cos\theta$
$\tan(360° - \theta) = -\cos\theta$

Example

Find the value of $\sin 135°$ without the use of a calculator.

Answer:

			Steps:
$\sin 135°$	→Quadrant II	1.	Find out which quadrant these degrees are in.
$= \sin(180° - 45°)$		2.	Find out what you have to add or subtract in that quadrant to get to the smallest degrees possible.
$= \sin 45°$		3.	Now, use the smaller angle.
$= \frac{\sqrt{2}}{2}$		4.	Simplify

Example

Simplify the following: $\frac{\sin(180°+\theta).\cos(360°-\theta)}{\tan(180°-\theta)} + \cos^2\theta$

Answer:

$\frac{\sin(180°+\theta).\cos(360°-\theta)}{\tan(180°-\theta)} + \cos^2\theta$	1.	Get rid of the brackets by using the
$= \frac{-\sin\theta.\cos\theta}{-\tan\theta} + \cos^2\theta$		rules stated above.
$= \frac{\sin\theta.\cos\theta}{\frac{\sin\theta}{\cos\theta}} + \cos^2\theta$	2.	Now use identities to simplify this
$= \frac{\sin\theta.\cos\theta}{1} \times \frac{\cos\theta}{\sin\theta} + \cos^2\theta$		expression.
$= \cos^2\theta + \cos^2\theta$		
$= 2\cos^2\theta$		

Co-Functions

Complementary – angles are angles that add up to 90°. 60° and 30° add up to 90°.
Now check this out:

$\sin 30° = \frac{1}{2}; \cos(90° - 30°) = \cos 60° = \frac{1}{2}$

$\sin 45° = \frac{\sqrt{2}}{2}; \cos(90° - 45°) = \cos 45° = \frac{\sqrt{2}}{2}$

$\sin 60° = \frac{\sqrt{3}}{2}; \cos(90° - 60°) = \cos 30° = \frac{\sqrt{3}}{2}$

$\therefore \boldsymbol{\sin\theta = \cos(90° - \theta)}$

and

$\sin 30° = \frac{1}{2}; \cos(90° + 30°) = \cos(120°) = -\frac{1}{2}$

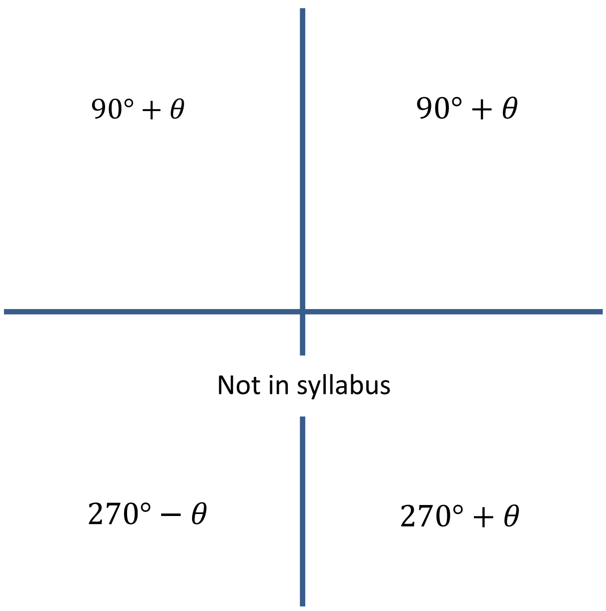
$\sin 45° = \frac{\sqrt{2}}{2}; \cos(90° + 45°) = \cos(135°) = -\frac{\sqrt{2}}{2}$

$\sin 60° = \frac{\sqrt{3}}{2}; \cos(90° + 60°) = \cos(150°) = -\frac{\sqrt{3}}{2}$

$\therefore \boldsymbol{\sin\theta = -\cos(90° + \theta)}$

Therefore sin and cos are co-functions of each other, meaning that when you add the angles with the functions, and get 90°, the one function and angle equals the other function and co-angle…

Let's summarize:



<u>$90° - \theta$</u>

$\sin(90° - \theta) = \cos\theta$
$\cos(90° - \theta) = \sin\theta$
$\tan(90° - \theta) = \cot\theta$

<u>$90° + \theta$</u>

$\sin(90° + \theta) = \cos\theta$
$\cos(90° + \theta) = -\sin\theta$
$\tan(90° + \theta) = -\cot\theta$

Remember: COT is not in your current syllabus

Example

Simplify the following:

1. $\frac{\sin(90°-\theta).\tan(90°+\theta)}{\cos(90°+\theta)}$ 2. $\frac{\cos 135°.\tan 45°.\sin 20°}{\tan 210°.\cos 70°}$

Answers:

First quadrant, sin is positive

Second quadrant, tan is negative but...not in syllabus!, Change to sin and cos

1. $\frac{\sin(90°-\theta).\tan(90°+\theta)}{\cos(90°+\theta)}$

Second quadrant, cos is negative

$$= \frac{\cos\theta.\frac{\sin(90°+\theta)}{\cos(90°+\theta)}}{-\sin\theta}$$

$$= \frac{\cos\theta.\frac{\cos\theta}{-\sin\theta}}{-\sin\theta}$$

$$= \frac{\cos^2\theta}{\sin^2\theta}$$

$$= \frac{1}{\tan^2\theta}$$

Steps:

1. Determine in which quadrant you are working first
2. Decide on the sign of the current function
3. Then change the function if you are dealing with a co-function
4. Now use identities to simplify

TADA!

2. $\frac{\cos 135°.\tan 45°.\sin 20°}{\tan 210°.\cos 70°}$

$$= \frac{\cos(180°-45°).\tan 45°.\sin 20°}{\tan(180°+30°).\cos(90°-20°)}$$

$$= \frac{-\cos 45°.\tan 45°.\sin 20°}{\tan 30°.\sin 20°}$$

$$= \frac{-\frac{\sqrt{2}}{2}.1}{\frac{1}{\sqrt{3}}}$$

$$= \frac{-\sqrt{2}}{2} \times \frac{\sqrt{3}}{1}$$

$$= -\frac{\sqrt{6}}{2}$$

1. Find the appropriate formulae (Reduction or co-functions) to reduce the angle to the smallest angle possible.
2. Now reduce the angles
3. Use special angles to find the value of each function and cancel all non-special angles.
4. Simplify as usual

Angles smaller than 0° or greater than 360°

- When you have a negative angle, you have two choices:

Method 1:

Always remember that negative angles are in the 4th quadrant:

e.g. 1. $\sin(-\theta)$ → sin is negative in the 4th quadrant
$= -\sin\theta$

2. $\cos(-\theta)$ → cos is positive in the 4th quadrant
$= \cos\theta$

3. $\tan(\theta - 180°)$ → the angle is in the wrong order, switch by taking out a negative.
$= \tan[-(180° - \theta)]$ → Now you have a negative angle, tan is negative in the 4th quadrant
$= -\tan(180° - \theta)$ → Carry on as usual…
$= -(-\tan\theta)$
$= \tan\theta$

Method 2:

When you add or subtract 360° to anything, the value would not change, think about turning 360°, you would still be facing the same side as you have before you started turning, so when you have a negative angle, add 360° until you have a positive angle.

e.g. 1. $\sin(-\theta)$
$= \sin(-\theta + 360°)$
$= \sin(360° - \theta)$
$= -\sin\theta$

2. $\cos(-\theta)$
$= \cos(-\theta + 360°)$
$= \cos(360° - \theta)$
$= \cos\theta$

3. $\tan(\theta - 180°)$
$= \tan(\theta - 180° + 360°)$
$= \tan(\theta + 180°)$
$= \tan(180° + \theta)$
$= \tan\theta$

As you can see, this method can be quite long…

When you have an angle larger than 360°, just subtract 360° until you reach an angle within the range of the Cast Diagram [0°;360°]

e.g. 1.

$$\begin{aligned}&\sin 495°\\&= \sin(495° - 360°)\\&= \sin 135°\\&= \sin(180° - 45°)\\&= \sin 45°\\&= \frac{\sqrt{2}}{2}\end{aligned}$$

2.

$$\begin{aligned}&\cos(-1080°)\\&= \cos(1080°)\\&= \cos(1080° - 3.360°)\\&= \cos 0°\\&= 1\end{aligned}$$

Activity 4

Simplify the following:

a. $\frac{\sin(-\theta).\cos(360°-\theta).\tan(180°-\theta)}{\cos(-\theta)\tan(180°+\theta).\cos(90°+\theta)}$

b. $\frac{\cos(90°-\theta).\tan\theta.\cos(360°-\theta)}{\tan(180°-\theta).\sin(90°-\theta).\sin(-\theta)}$

c. $\frac{\cos(90°+\alpha).\tan(-\alpha).\cos(\alpha-360°)}{\sin(180°-\alpha).\tan(180°+\alpha).\cos(-\alpha)}$

d. $\frac{\sin 130°.\tan 120°.\cos 330°}{\cos 320°}$

e. $\frac{\cos 240°+\tan 315°}{\cos 135°.\sin 225°}$

f. $\tan 210° + \frac{\sin 120°+\cos 330°}{\sin 210°.\cos 150°}$

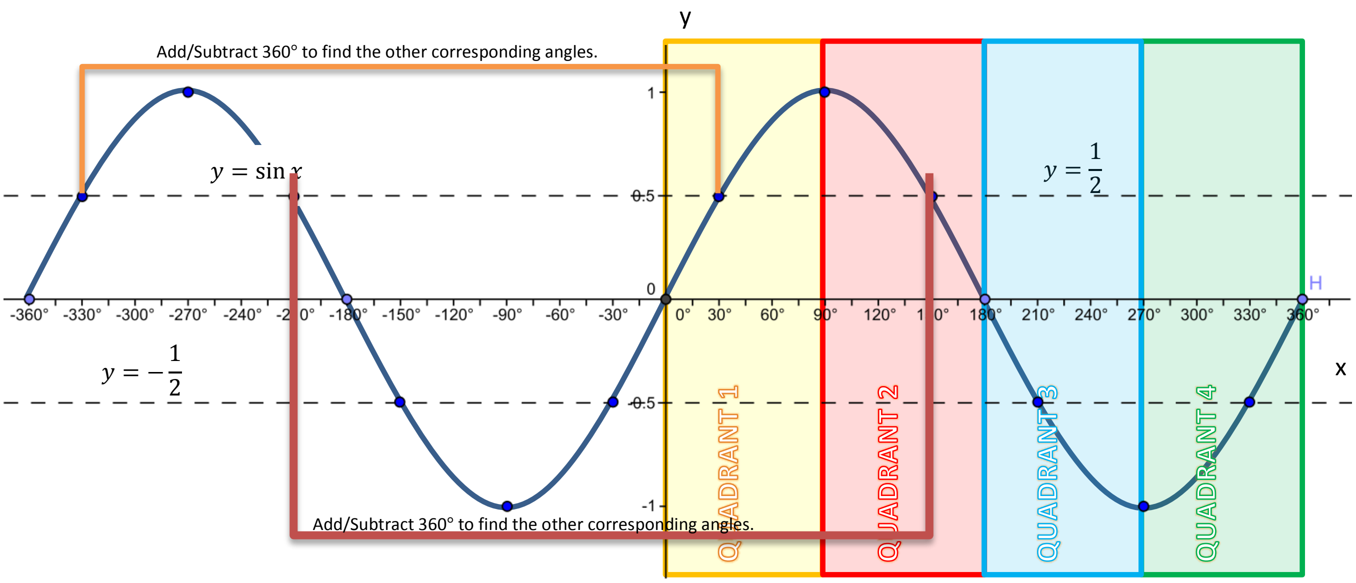
1.2.4 THE GENERAL SOLUTION

The General solution is about finding a formula that will help you find all angles that fit a specific criteria within a certain given domain. You'll have to be able to do two things using General Solution.

1. Find the General Solution only.

2. Find the General Solution and use it to solve an angle for all values within a specific given domain.

The GENERAL SOLUTION for SIN



Similar to the theory behind Reduction formulas:

If you have to solve the following: $\sin x = 0.5$ for $x \in [-360°; 360°]$

Press 2nd F sin 0 • 5 : answer: 30° → this is your Reference angle

Look at the sketch: to find another answer, simply add or subtract 360° (remember angles greater than 360° - Reduction formulae revision)

Finding the other two solutions: If you look at the sketch between 0° and 180°, you'll see that the reference angle is mirrored, you can find the mirror angle by subtracting your reference angle from 180°.
Therefore: $180° - 30° = 150°$
Now add or subtract 360° to find the other answers.

You don't need a hectic sketch to find all the answers!!! Look at this…

Example

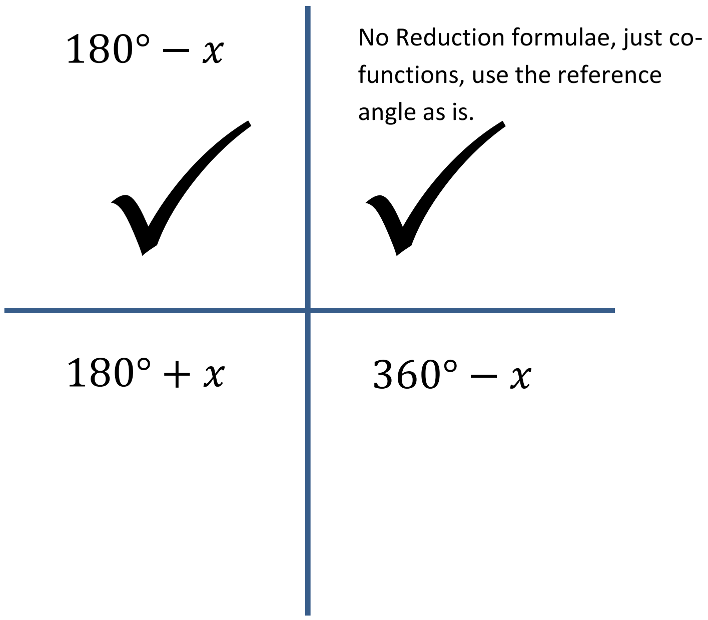
Solve for x if $\sin x = 0.5$ and $x \in [-360°; 360°]$

Answer:

STEPS

1. Identify the quadrant in which you can work.
2. Identify the Reduction formulae you can use in those quadrants.

$\sin x = 0.5$ → 0.5 is positive, therefore, you are working in the first and second quadrant



Reference angle: 30°

3. Use your calculator to find the reference angle (always use a POSITIVE VALUE)
4. Use your quadrants, reduction formulae and reference angle to find the GENERAL SOLUTION

Mirrored angle

Quadrant I

$x = 30° + k.360°$

Quadrant II

$x = 180° - 30° + k.360°$

Original Reference angle

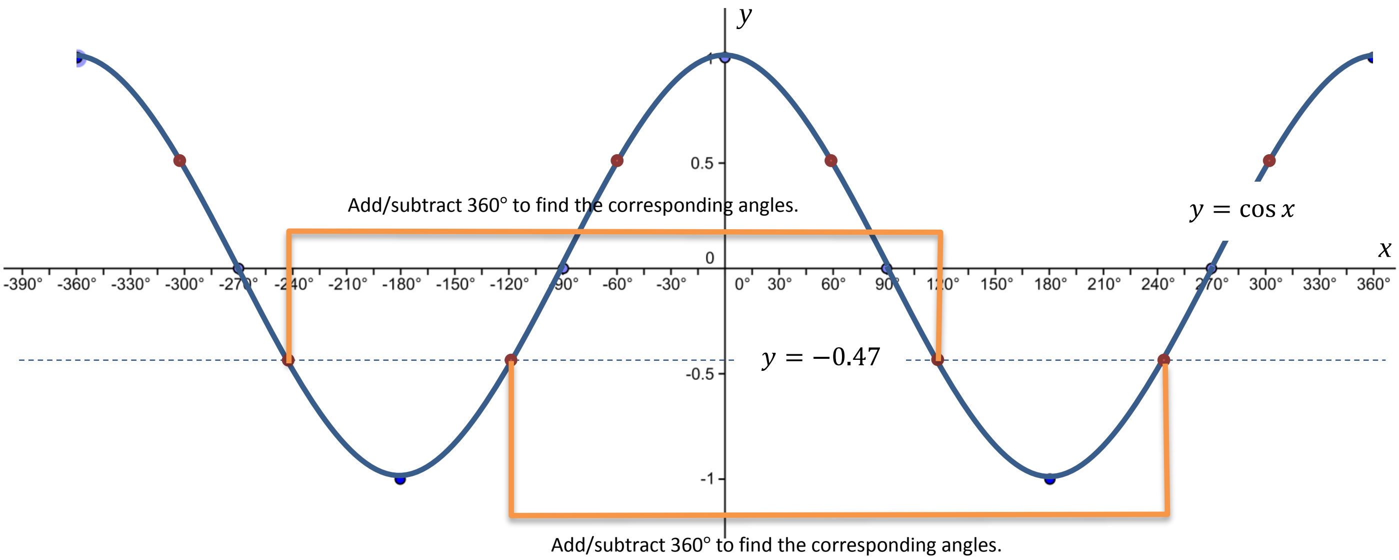
Adding or subtracting 360° a certain number of times until you have reached the limits "domain" given, hence "k", where "k" can be positive (adding) or negative (subtracting).

$\therefore x = 30° + k.360°$ $\quad x = 150° + k.360°$ → Simplified

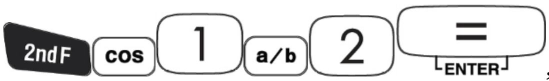
$\therefore x = 30°;\ -330°$ or $x = 150°;\ -210°$

5. Use the GENERAL SOLUTION to find all applicable angles.

GENERAL SOLUTION for COS



The GENERAL SOLUTION for cos is very similar to sin. Look at the sketch,

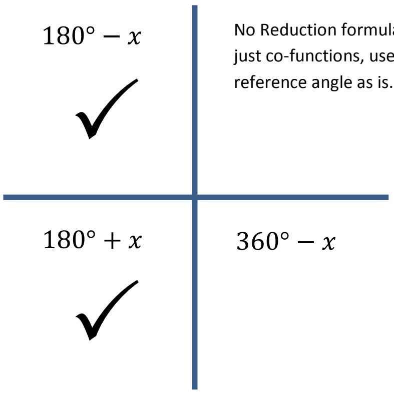
when you type 

, you will get your original Reference angle, here's how you can tackle this problem:

Example

Solve for x if $\cos x = -0.47$ and $x \in [-180°; 360°]$

Answer:



Reference angle: 62°

Quadrant II

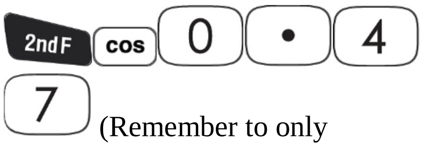
$\therefore x = 180° - 62° + k.360°$
$\therefore x = 118° + k.360°$
$\therefore x = 118°$ or

Quadrant III

$x = 180° + 62° + k.360°$
$x = 242° + k.360°$
$x = 242°; -118°$

Steps:

1. Determine in which quadrants you can work, in this case, you have a negative value for cos, therefore, you will be working in the second and third quadrant.

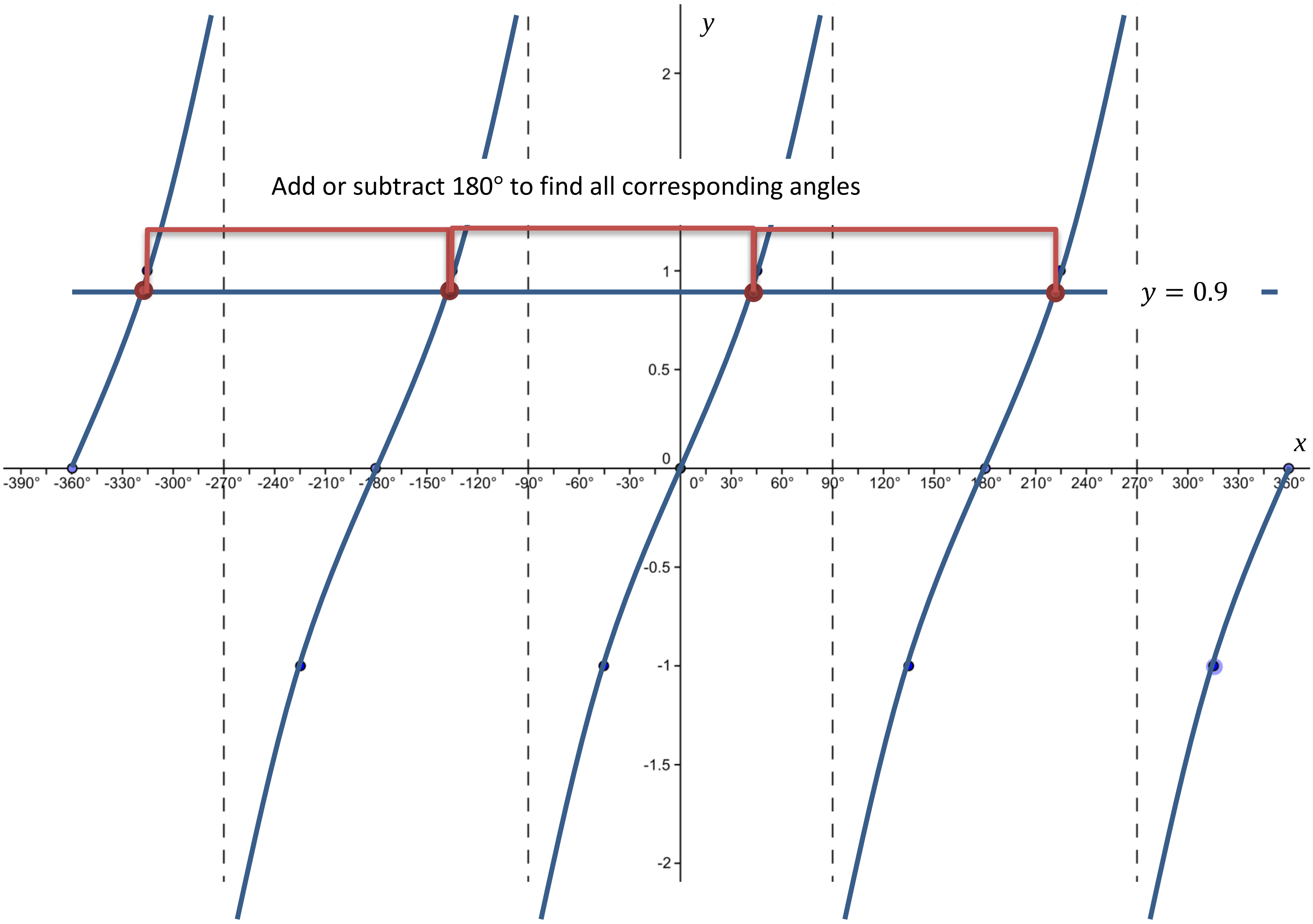
2. Find your first original Reference angle: Press 

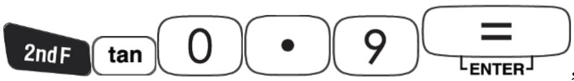
(Remember to only use positive values)

3. Use your quadrants and your reference angle to find the GENERAL SOLUTION.

4. Use your GENERAL SOLUTION to find all applicable angles within the given domain.

The GENERAL SOLUTION for TAN



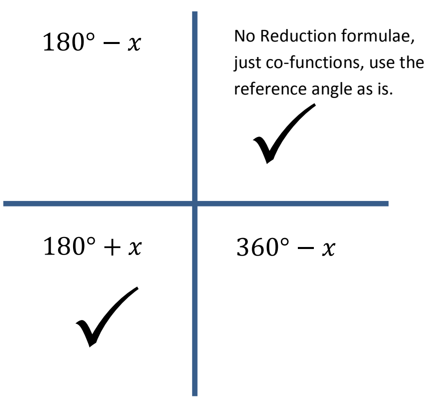
The GENERAL SOLUTION for tan is different from sin and cos. Look at the sketch, when you type 

, you will get your original Reference angle, here's how you can tackle this problem:

Example

Solve for x if $\tan x = 0.9$ and $x \in [-360°; 360°]$

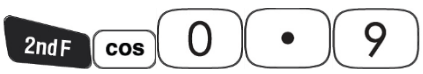
Answer:



Steps:

1. Determine in which quadrants you can work, in this case, you have a positive value for tan, therefore, you will be working in the first and third quadrant.

2. Find your first original Reference angle: Press



Reference angle: 42°

Quadrant I

$\therefore x = 42° + k.180°$

Quadrant III

$x = 180° + 42° + k.180°$

$x = 222° + k.180°$

$\therefore x = 42°; 222°; -138°; -318°$

or

$x = 222°; 42°; -138°; -318°$

3. Use your quadrants and your reference angle to find the GENERAL SOLUTION.

4. Use your GENERAL SOLUTION to find all applicable angles within the given domain.

NOTE: BOTH QUADRANTS PRODUCE THE SAME ANSWERS!! THEREFORE, YOU ONLY HAVE TO WORK IN ONE QUADRANT WHEN WORKING WITH TAN!

SUMMARY:

- When you are asked to solve an angle or find the GENERAL solution, you have to use quadrants and reduction formulae
- With sin and cos you add 360° and work in all determined quadrants
- With tan you add 180° and only work in one of the determined quadrants
- Always use a POSITIVE value when determining your original Reference Angle, determining the quadrants deals with all signs.

Examples

1. Find the General Solution for the following:

a. $2\cos x = 1.5$

b. $\sin x + 1 = 1.2$

c. $\cos(x + 20°) = -0.34$

d. $\tan 2x = 5$

e. $-\cos x = 2\sin x$

f. $\cos(x + 20°) = \sin(x - 50°)$

g. $\cos^2 x + 2\cos x \sin x - 8\sin^2 x = 0$

2. Solve for x in the above mentioned if $x \in [-180°; 520°]$.

Answers:

1.a $2\cos x = 1.5$ ⟶You have to get $\cos x$ alone first, divide by 2

$\therefore \cos x = \frac{3}{4}$

Reference Angle: 41.4° (working in quadrants 1 and 4)

QUADRANT I

$x = 41.1° + k.360°$

QUADRANT IV

$x = 360° - 41.1° + k.360°$

$x = 318.9° + k.360°$

⟶Now you are done, the question only askes for the GENERAL SOLUTION

b. $\sin x + 1 = 1.2$

$\therefore \sin x = 0.2$

Reference Angle: 11.5° (working in quadrants 1 and 2)

QUADRANT 1

$x = 11.5° + k.360°$

QUADRANT II

$x = 180° - 11.5° + k.360°$

$x = 168.5° + k.360°$

c. $\cos(x + 20°) = -0.34$ ⟶You’re angle is more complicated!

Reference Angle: 70.1° (working in quadrants 2 and 3)

QUADRANT II

$x + 20° = 180° - 70.1° + k.360°$

$x = 89.9° + k.360°$

QUADRANT III

$x + 20° = 180° + 70.1° + k.360°$

$x = 230.1° + k.360°$

NOTE: When you are working with more complicated angles, always write down the FULL general solution before you simplify *x*.

d. $\tan 2x = 5$

Reference Angle: 78.7° (working in quadrants 1 and 3)

QUADRANT I

$2x = 78.7° + k.180°$

$\therefore x = 39.4° + k.90°$

QUADRANT III

$2x = 180° + 78.7° + k.360°$

$x = 129.4° + k.90°$

e. $-\cos x = 2\sin x$ → when sin and cos appear in the same equation, two things can happen, the first is the situation where they both have the same angle, in this case, divide both sides by cos, you will end up with tan and then carry on as usual.

$\therefore \frac{\cos x}{\cos x} = \frac{2\sin x}{\cos x}$

$\therefore -1 = 2\tan x$

$\therefore -\frac{1}{2} = \tan x$

Reference Angle: 26.6° (working in quadrants 2 and 4)

QUADRANT II

$x = 180° - 26.6° + k.180°$

$x = 153.4° + k.180°$

QUADRANT 4

$x = 360° - 26.6° + k.180°$

$x = 333.4° + k.180°$

f. $\cos(x + 20°) = \sin(x - 50°)$ $\longrightarrow$ In this case, you'll have to use co-functions to change the left hand side and right hand side to the same function, after that, you assume that the value you are working with would be positive and work in those quadrants.

$\therefore \cos(x + 20°) = \cos\big(90° - (x - 50°)\big)$

$\therefore \cos(x + 20°) = \cos(90° - x + 50°)$

$\therefore \cos(x + 20°) = \cos(140° - x)$ $\rightarrow$ Now you can equal the left hand side to the right hand side.

Use as "unknown angle"   Use as Reference Angle

QUADRANT I

$x + 20° = 140° - x + k.360°$

$2x = 120° + k.360°$

$x = 60° + k.180°$

QUADRANT IV

$x + 20° = 360° - (140° - x) + k.360°$

$x + 20° = 360° - 140° + x + k.360°$

No solution

g. $\cos^2 x + 2\cos x \sin x - 8\sin^2 x = 0$ $\longrightarrow$ This is a quadratic equation, factorise and see what happens.

$(\cos x - 2\sin x)(\cos x + 4\sin x) = 0$

$\therefore \cos x = 2\sin x$ or $\cos x = -4\sin x$

$\therefore \frac{\cos x}{\cos x} = \frac{2\sin x}{\cos x}$ $\therefore \frac{\cos x}{\cos x} = -\frac{4\sin x}{\cos x}$

$\therefore 1 = 2\tan x$ $\therefore 1 = -4\tan x$

$\therefore \frac{1}{2} = \tan x$ $\therefore -\frac{1}{4} = \tan x$

For $\tan x = \frac{1}{2}$:

Reference Angle: 26.6° (working in quadrants I and III)

QUADRANT I

$x = 26.6° + k.180°$

QUADRANT III

$x = 180° - 26.6° + k.180°$

$x = 153.4° + k.180°$

For $\tan x = -\frac{1}{4}$

Reference Angle: 14° (working in quadrants II and IV)

QUADRANT II

$x = 180° - 14° + k.180°$

$\therefore x = 166° + k.180°$

QUADRANT IV

$x = 360° - 14° + k.180°$

$x = 346° + k.180°$

2.a $x = 41.1° + k.360°$ or $x = 318.9° + k.360°$

$\therefore x = 41.1°; 401.1°; 318.9°;\ -41.1°$

2.b $x = 11.5° + k.360°$ or $x = 168.5° + k.360°$

$\therefore x = 11.5°; 371.5°; 168.5°$

2.c $x = 89.9° + k.360°$ or $x = 230.1° + k.360°$

$\therefore x = 89.9°; 449.9°; 230.1°;\ -129.9°$

2.d $x = 39.4° + k.90°$ or $x = 129.4° + k.90°$

$\therefore x = 39.4°; 129.4°; 219.4°; 309.4°; 399.4°; 489.4°;\ -50.6°;\ -140.6°$

2.e $x = 153.4° + k.180°$ or $x = 333.4° + k.180°$

$\therefore x = 153.4°; 333.4°; 513.4°; -26.6°$

2.f $x = 60° + k.180°$

$\therefore x = 60°; 240°; 420°;\ -120°$

2.g $x = 26.6° + k.180°$ $x = 153.4° + k.180°$

$\therefore x = 26.6°; 206.6°; 386.6°;\ -153.4°$ or $x = 153.4°; 333.4°; 513.4°;\ -26.6°$

$x = 166° + k.180°$
$\therefore x = 166°; 346°;\ -14°$

Activity 5

5.1 Solve the following equations for $x \in [0°; 360°]$

a. $\sin x = \frac{\sqrt{2}}{2}$ b. $2\cos x = -\frac{1}{2}$ c. $\tan(x - 20°) = 3$

d. $2\cos x + 3 = 2$ e. $2\tan 3x - 2 = 0$ f. $\sin x + 2 = \sqrt{5}$

5.2 Find the general solution of the following:

a. $\sin^2 x - \cos^2 x = 0$ b. $3\sin x = 6\cos x$ c. $8\sin^2 x - 4 = 0$

d. $7\cos^2 x - 5\cos x = 0$ e. $\sin(x + 87°) = \cos 120°$ f. $\frac{1}{\sin x} - 4 = 0$

1.2.5 TRIG FUNCTIONS/GRAPHS - CHANGING THE PERIOD AND SHIFTING LEFT OR RIGHT:

$$y = a\sin b(x - p) + q$$

Amplitude

Changes Period

Shifts the graph left when $(x+?)$, and right when $(x-?)$.

Shifts the graph up when q>0, down when q<0

CHANGING THE PERIOD

As you've seen in the previous section, sine and cos reach the same value after each 360°, therefore the period (Degrees within which you would sketch a whole sine or cos graph) is 360°. This however can change:

Interpretation of the equation: e.g. $y = \sin x$ → One full sine graph within 360°

$y = \sin 2x$ → Two full sine graphs within 360°

Determining the new PERIOD: e.g. $y = \sin 2x$ → If you have two full sine graphs within 360°, you'll have one full sine graph within 180°.

$\therefore y = \sin bx$ → Period = $\frac{360°}{b}$

Example

Sketch: $y = \sin 3x$, $y = \cos 2x$ and $y = \tan 2x$ on separate systems of axes, while $x \in [0°; 360°]$

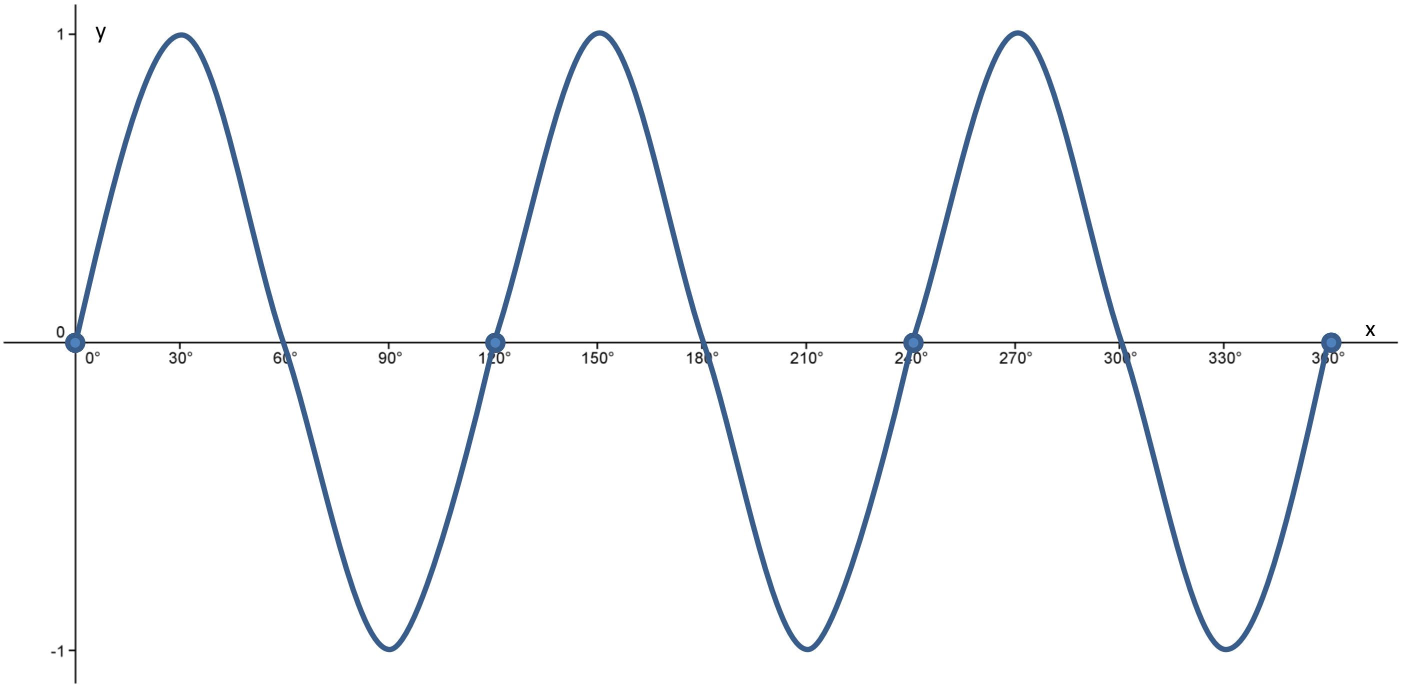
Answer:

$y = \sin 3x$

$\frac{360°}{3} = 120°$ → Now you need the significant points

$\frac{120°}{4} = 30°$ →You divide by 4 because there are 4 significant points equally distributed throughout the sine graph. Therefore every 30° you will have either a turning point or an x – intercept.

The result:

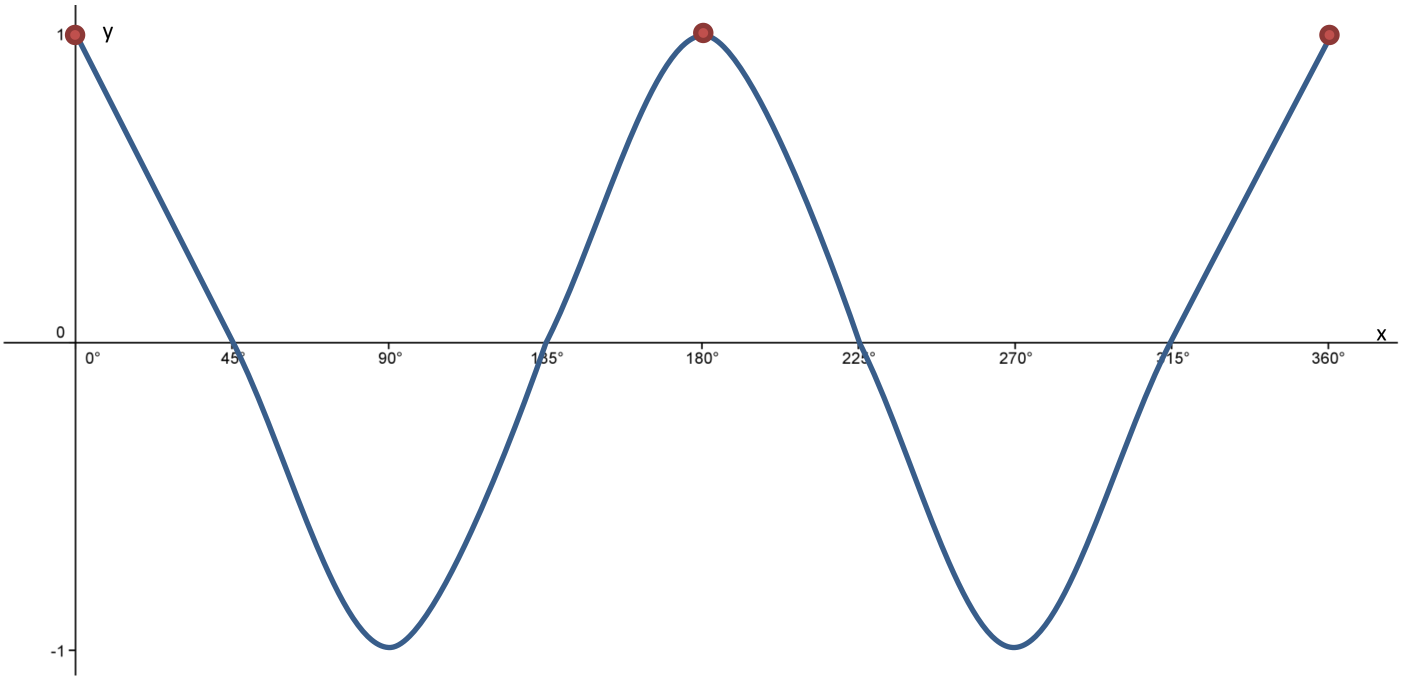


$y = \cos 2x$

$\frac{360°}{2} = 180°$

$\frac{180°}{4} = 45°$

The Result:



$y = \tan 2x$

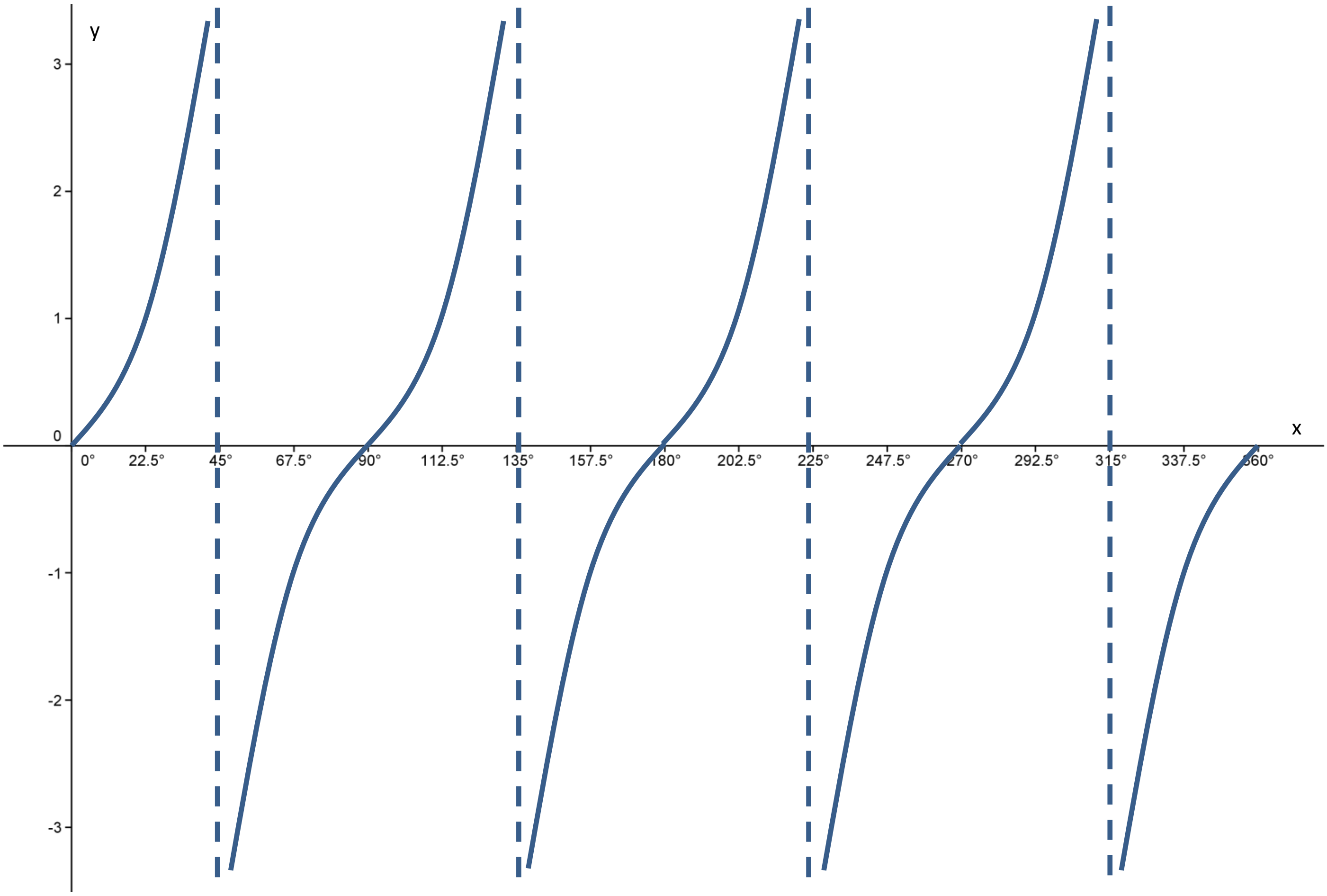
NOTE: The Tan graph completes a full cycle within 180°, it happens inbetween the asymptotes, here's how you would handle this situation:

$\frac{180°}{2} = 90°$ →Distance between asymptotes

$\frac{90°}{2} = 45°$ →Where you would find your first asymptote

$\frac{45°}{2} = 22.5°$ →Corresponding with your amplitude

The Result:

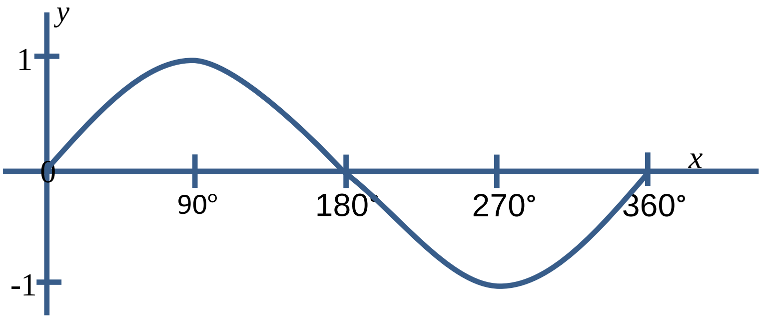


SHIFTING LEFT OR RIGHT:

Example

Sketch $y = \sin(x - 30°)\,; x \in [0°; 360°]$

Answer:

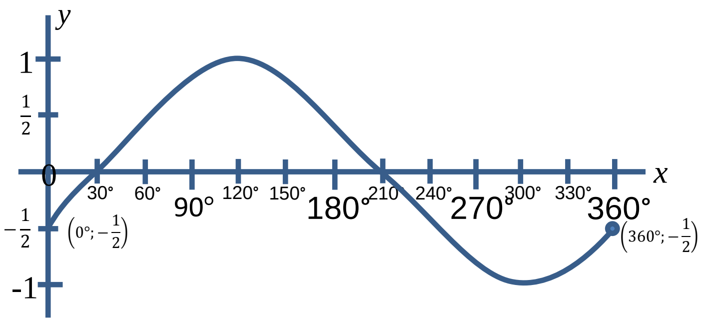


0° - Start at 0 - 30°
90° - Turn at 1 - 120°
180° - Cut through 0 - 210°
270° - Turn at -1 - 300°
360° - Stop at 0 - 390°

Discard, beyond the domain given

Start at $\left(0°; -\frac{1}{2}\right)$

Stop at $\left(360°; -\frac{1}{2}\right)$



Steps:

1. Sketch the "normal" sin graph before any shifting needs to take place, i.e. $y = \sin x$ in rough
2. Write down all significant points with a short description of each.
3. Add/Subtract the degrees that the graph is shifted by (When the shifted degrees are subtracted in the equation, you are actually moving the graph to the right, therefore ADD, when the shifted degrees are added in the equation, you are actually moving the graph to the left, therefore SUBTRACT.)
4. Discard any points that go beyond the domain asked and calculate the "start at 0°" and "stop at 360°" with your calculator.
5. Prepare the plane with the correct intervals, in this case 30° intervals.
6. Sketch starting with your summary above.

Example

Sketch $y = -2\sin(2x + 90°)$ if $x \in [0°; 180°]$

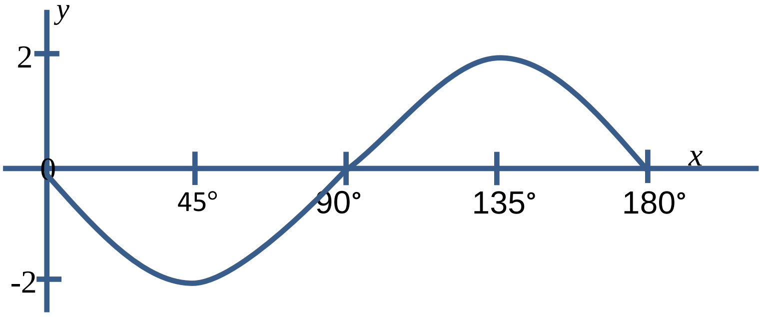
Answer:

$y = -2\sin(2x + 90°) \rightarrow x$ has to be alone in the brackets! Take out a "2" as a common factor.

$y = -2\sin 2(x + 45°)$ $\rightarrow$ Now follow all the steps

Period will change: $\frac{360°}{2} = 180°; \frac{180°}{4} = 45°$

$y = -2\sin 2x$:



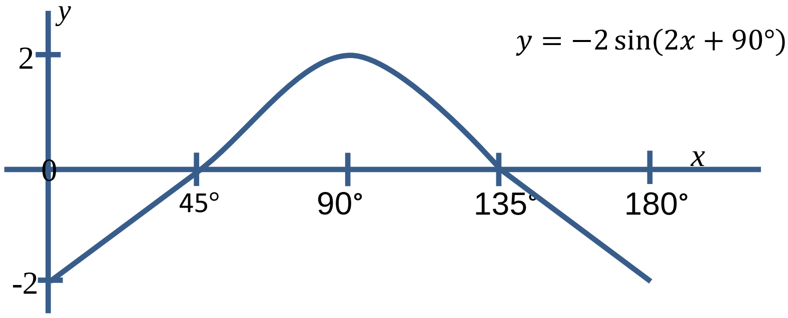
0°	-	Start	$\rightarrow -45°$
45°	-	Turn at -2	$\rightarrow 0°$
90°	-	Cut x axis	$\rightarrow 45°$
135°	-	Turn at 2	$\rightarrow 90°$
180°	-	Stop on 0	$\rightarrow 135°$

out of bounds!

not far enough, you HAVE to do the stop and starting points!

Start: $y = -2\sin(2(0°) + 90°) = -2$

Stop: $y = -2\sin(2(180°) + 90°) = -2$

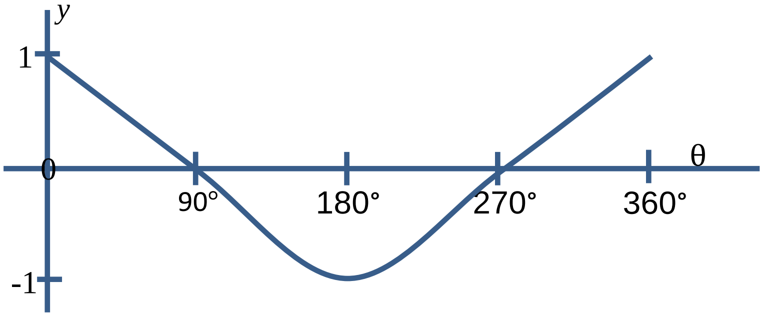


Example

Sketch: $y = \cos(\theta + 20°)\,;\ \theta \in [0°; 360°]$

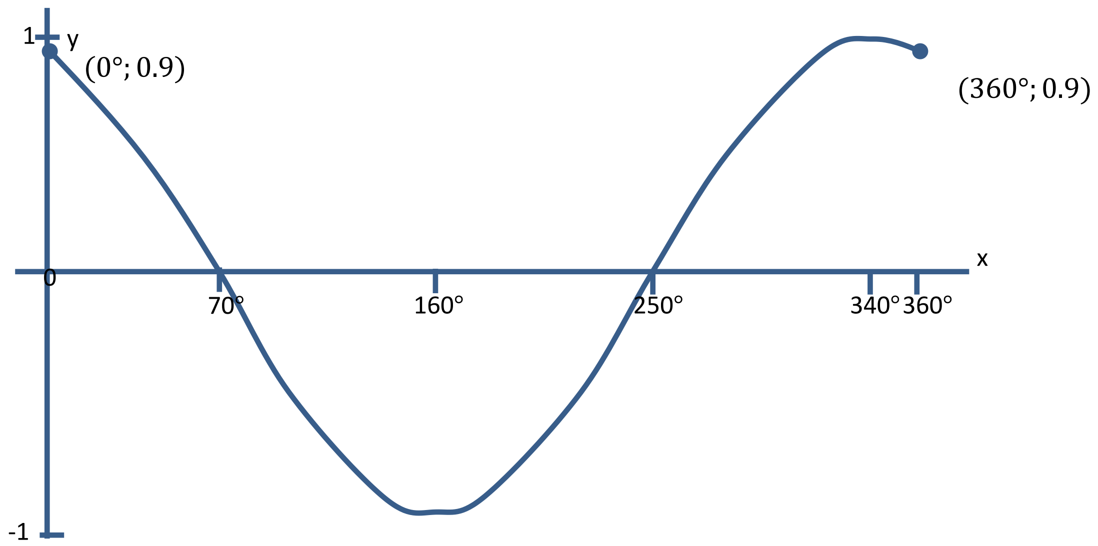
Answer:

$y = \cos\theta$



0°	-	Start at 1	→-20°
90°	-	Cut	→70°
180°	-	Turn at -1	→160°
270°	-	Cut	→250°
360°	-	Stop at 1	→340°

Start: $y = \cos(0° + 20°) = 0.9$
Stop: $y = \cos(360° + 20°) = 0.9$



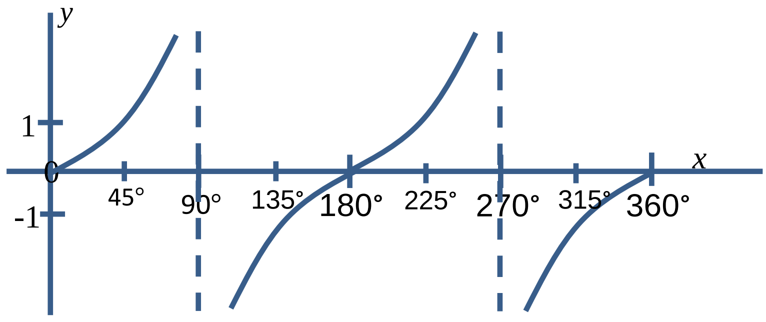
Example

Sketch: $y = \tan(x + 30°)\,; x \in [0°; 360°]$

Answer:

$y = \tan(x + 30°)$ $\rightarrow$ same steps, just remember that the asymptotes would also have to be shifted.

$y = \tan x$:



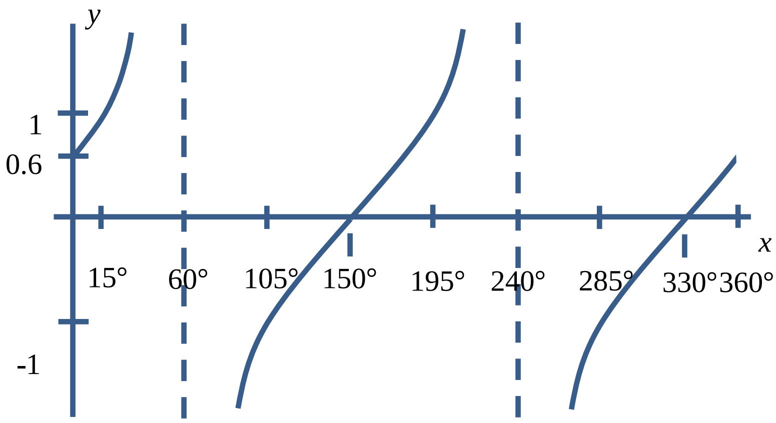
Significant points:

0°	- Start at 0	- -30°
45°	- Correspond with 1	- 15°
90°	- Asymptote	- 60°
135°	- Correspond with -1	- 105°
180°	- Cut x axis	- 150°
225°	- Correspond with 1	- 195°
270°	- Asymptote	- 240°
315°	- Correspond with -1	- 285°
360°	- Stop at 0	- 330°

Start: $y = \tan(0° + 30°) = 0.6$

Stop: $y = \tan(360° + 30°) = 0.6$

The result:



All of these functions/graphs can be drawn easily by using your calculator:

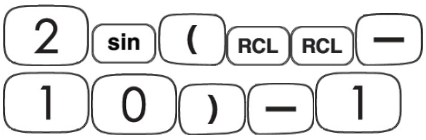
Example

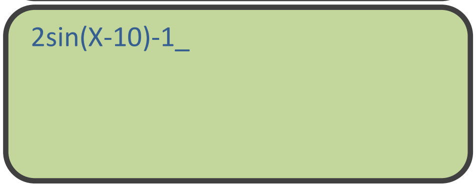
Sketch: $y = 2\sin(x - 10°) - 1$ for $x \in [0°; 360°]$

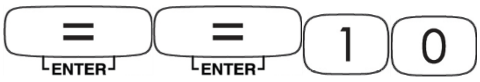
Answer:

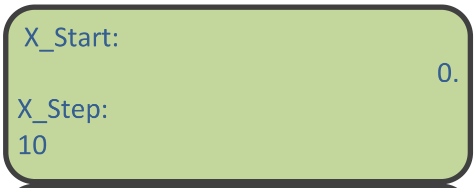
Buttons to press on Calculator (EL-535):

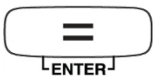
On screen:

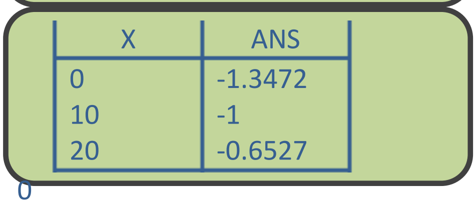












Now you have coordinates to plot and connect!

An example of an exam question:

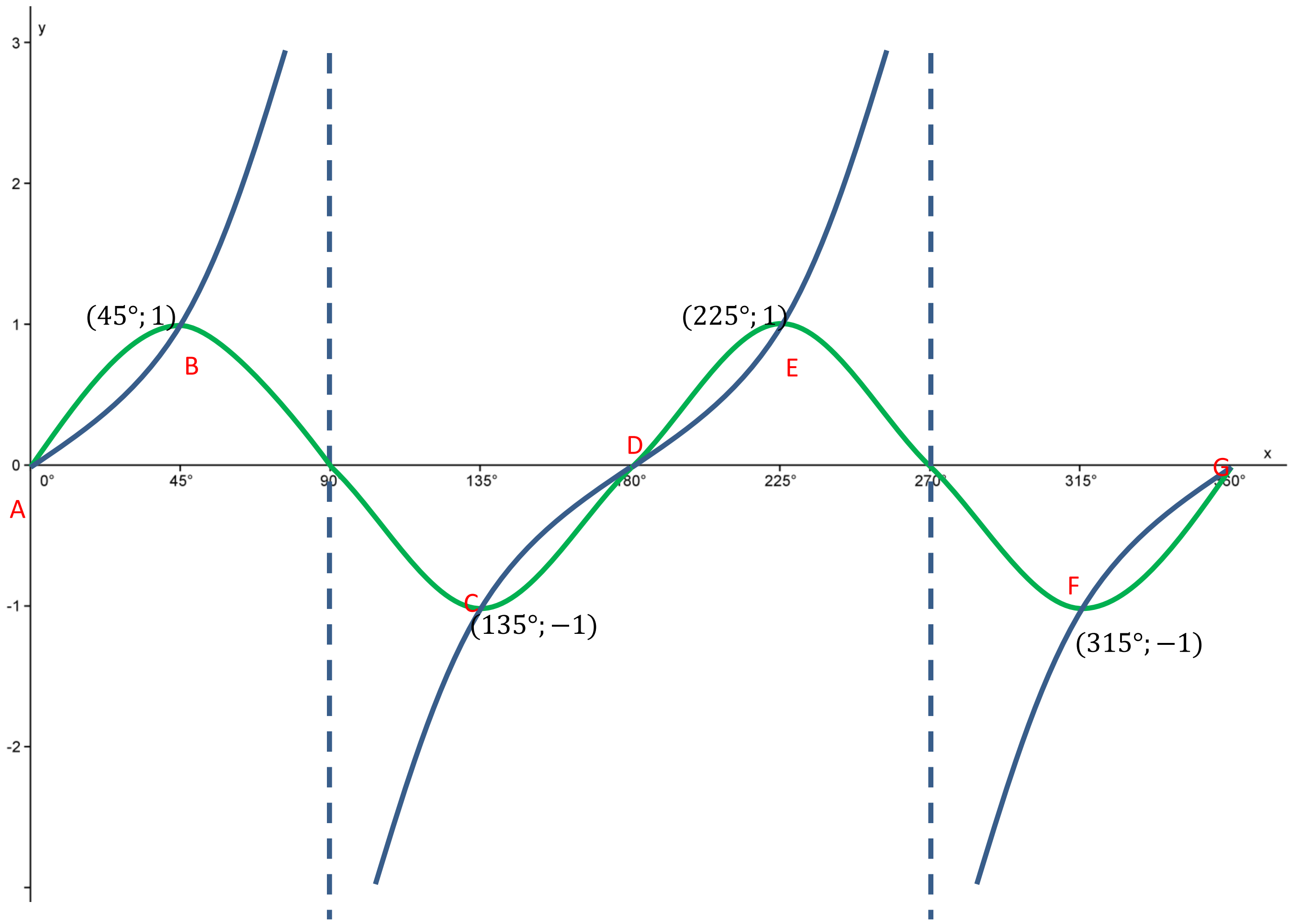
Example

Given: $f(x) = \sin 2x$ and $g(x) = \tan x$ for $0° \leq x \leq 360°$

1. Sketch both $f(x)$ and $g(x)$ on the same set of axes, showing all intercepts with the axes and all turning points.
2. Indicate on the graph with A, B, C, etc where $\sin 2x - \tan x = 0$?
3. Using the graph, for which values of x is $\sin 2x \geq \tan x$?

Answers:

1.



2. $\sin 2x - \tan x = 0$
$\therefore \sin 2x = \tan x$
Answers indicated in Red on Graph

3. $\sin 2x \geq \tan x$
in other words – Green above blue!
$x \in [0°; 45°]; x \in (90°; 135°]; x \in [180°; 225°]; x \in (270°; 315°]$

Activity 6

6.1 Sketch the following:

a. $y = 2\sin 3x\,;x \in [-90°;180°]$

b. $y = -\tan 2x\,;x \in [-45°;270°]$

c. $y = 5\cos x - 1;x \in [0°;360°]$

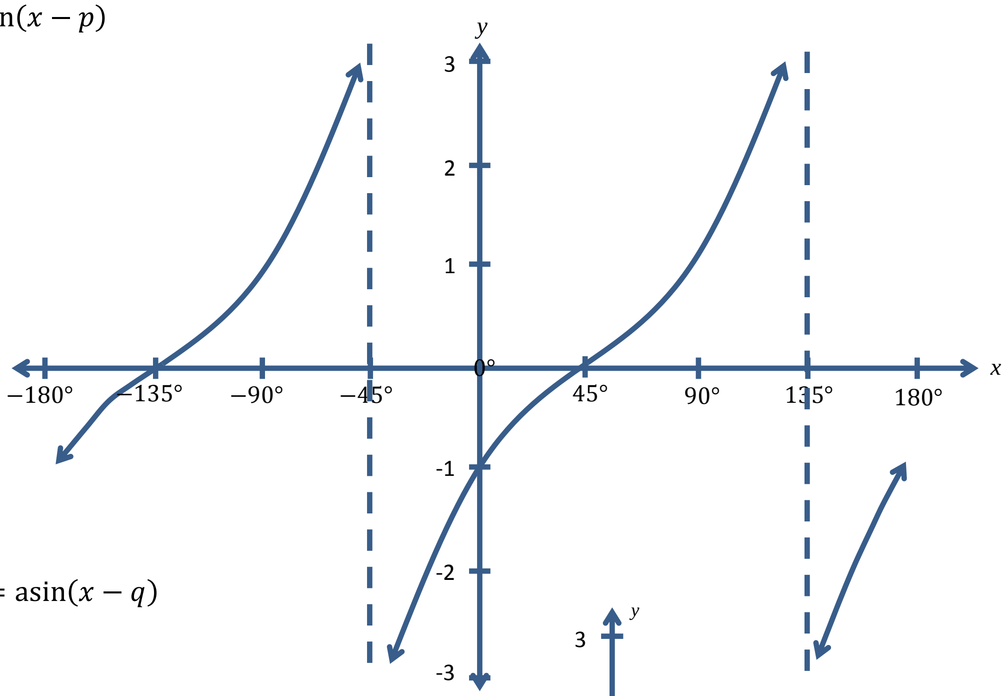
d. $y = -\sin x + 2;x \in [-90°;90°]$

e. $y = \sin\frac{1}{2}x + 1;x \in [0°;180°]$

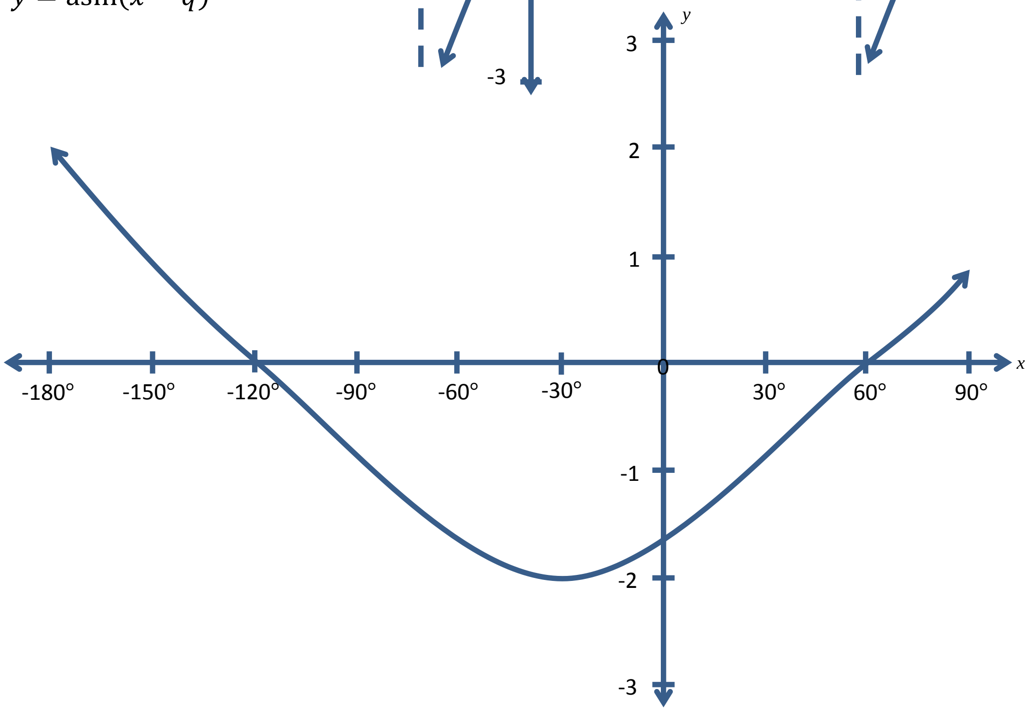
f. $y = 2\cos\frac{3}{4}x\,;x \in [0°;180°]$

6.2 Find the equation of the following graphs:

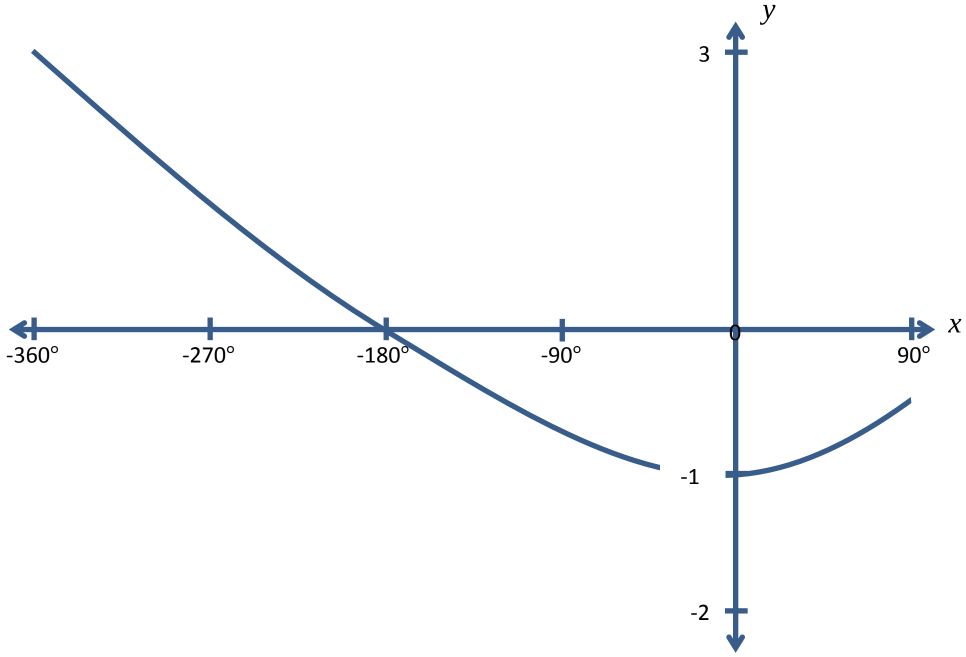
a. $y = \tan(x - p)$



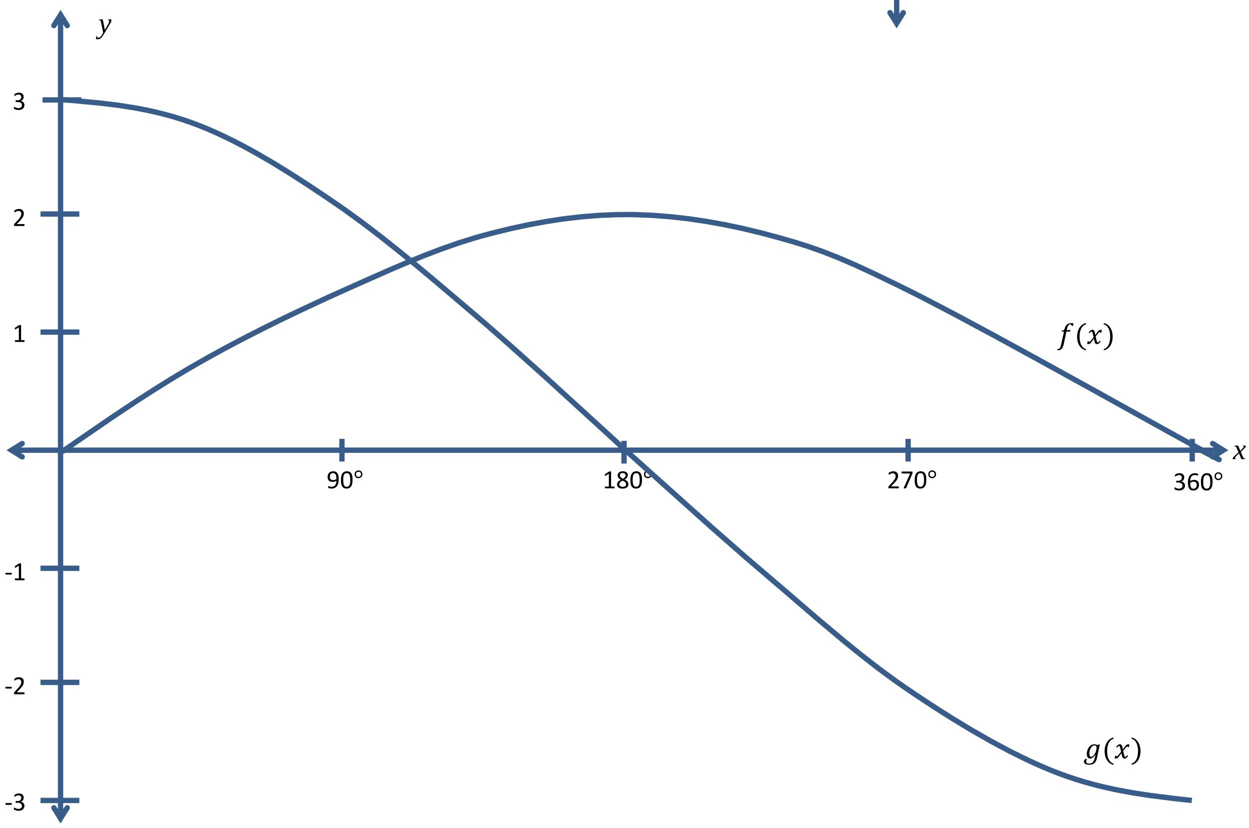
b. $y = a\sin(x - q)$



c. $y = a\cos bx + c$



6.3



Shown above, the graphs of $f(x) = a\sin bx$ and $g(x) = c\cos dx$, answer the questions that follow:

a. Find the values of a, b, c and d.

b. Calculate the point of intersection of $f(x)$ and $g(x)$ for $x \in [0°; 360°]$.

c. For which value(s) of x is $f(x) > g(x)$?

d. For which value(s) of x is $2 < f(x) - g(x) < 3$?

6.4.a Draw sketch graphs of the following: $h(x) = 2\tan x + 1$ and $g(x) = \cos 2x + 1$, where $x \in [0°; 360°]$.

b. Write down the period of $g(x)$.

c. Write down the amplitude of $g(x)$.

d. Indicate on your graph where $h(x) = g(x)$ with A,B, C, etc.

e. Use your answers in 6.4.d and find where $g(x) > h(x)$.

6.5 Given: $f(x) = \sin\frac{1}{4}x$ and $g(x) = 2\tan 2x$

a. Sketch the graphs of f and g on the same set of axes for $x \in [-180°; 90°]$.

b. Write down the equations of the asymptotes of $g(x)$ for $x \in [-180°; 90°]$.

c. What is the period of $g(x)$?

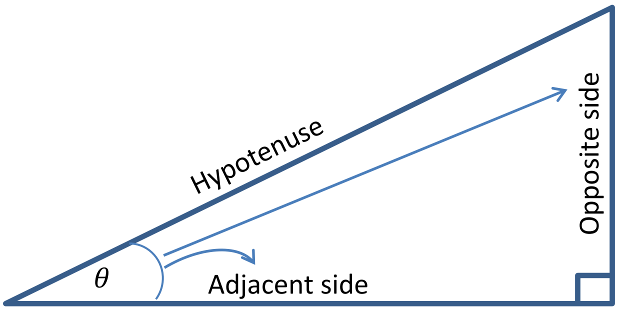
d. for which value(s) of x will $\sin\frac{1}{4}x \, . \tan x \leq 0$?

1.2.6 SOLVING 2D TRIANGLES AND REAL LIFE PROBLEMS

*Remember: $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$

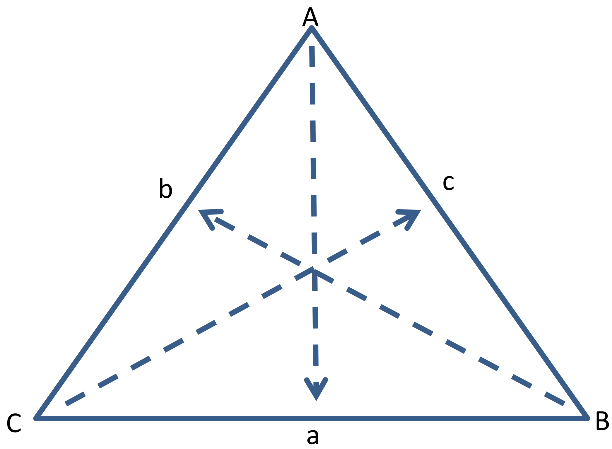
$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$

$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$



⇒What if the triangle that you're working with is not a right-angled triangle?

Solving Triangles that do not have a 90° angle:



Two rules apply when you want to find the value of an angle or a side:

1. Sine Rule: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ or $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

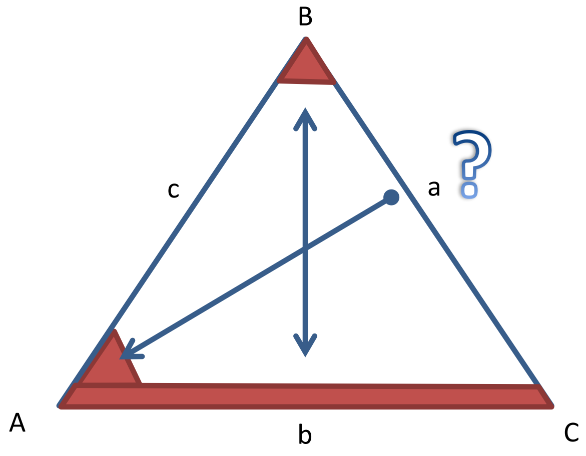
2. Cosine Rule: $a^2 = b^2 + c^2 - 2bc\cos A$ or
$b^2 = a^2 + c^2 - 2ac\cos B$ or
$c^2 = a^2 + b^2 - 2ab\cos C$ or
$\cos A = \frac{b^2+c^2-a^2}{2bc}$ or
$\cos B = \frac{a^2+c^2-b^2}{2ac}$ or
$\cos C = \frac{a^2+b^2-c^2}{2ab}$

1. SINE RULE:

When to use the Sine Rule:

- A side and it’s opposite angle is given, AND
- If you want to calculate a side, you have to have it’s opposite angle, OR
- If you want to calculate an angle, you have to have it’s opposite side.

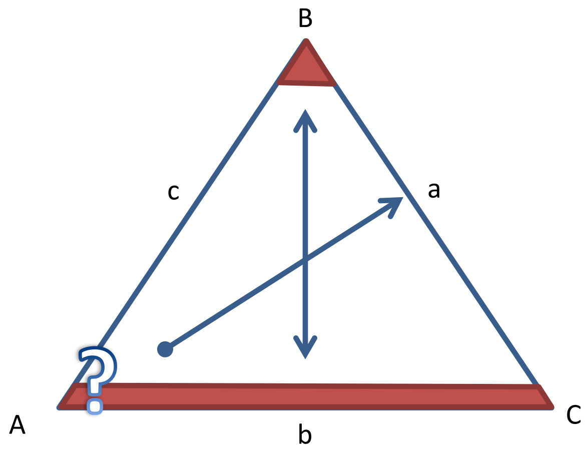
TO FIND A SIDE:



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

You only need this pair to solve this particular problem…

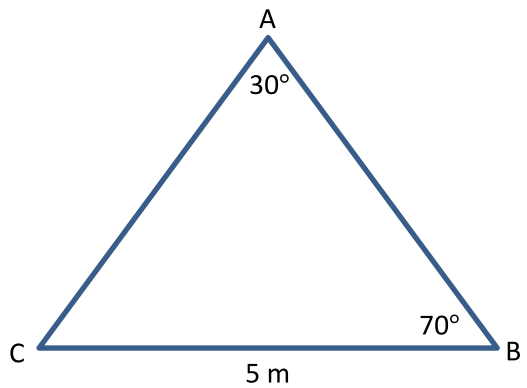
TO FIND AN ANGLE:



$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Example

Solve the triangle below:



Answer:

Finding Angle C

$$\hat{C} = 180° - 30° - 70°$$
$$\therefore \hat{C} = 80°$$

Finding side AB (c)

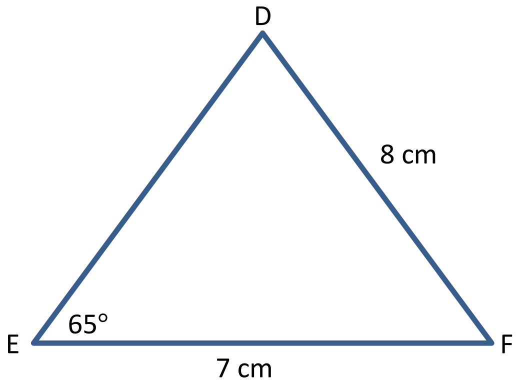
$$\frac{c}{\sin C} = \frac{a}{\sin A}$$
$$\therefore c = \frac{a\sin C}{\sin A}$$
$$\therefore c = \frac{5\sin 80°}{\sin 30°}$$
$$\therefore c = 9.8\text{ m}$$

Finding side AC (b)

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$
$$\therefore b = \frac{a\sin B}{\sin A}$$
$$\therefore b = \frac{5\sin 70°}{\sin 30°}$$
$$\therefore b = 9.4\text{ m}$$

Example

Solve the triangle below:



Answer:

To find angle D:

$$\frac{\sin D}{d} = \frac{\sin E}{e}$$
$$\therefore \sin D = \frac{d\sin E}{e}$$
$$\therefore \sin D = \frac{7\sin 65°}{8}$$
$$\therefore \hat{D} = 52.5°$$

To find angle F:

$$\hat{F} = 180° - 65° - 52.5°$$
$$\hat{F} = 62.5°$$

To find side ED (f):

$$\frac{f}{\sin F} = \frac{e}{\sin E}$$
$$\therefore f = \frac{e\sin F}{\sin E}$$
$$\therefore f = \frac{8\sin 62.5°}{\sin 65°}$$
$$\therefore f = 7.8\text{ cm}$$

WATCH OUT FOR THE AMBIGUOUS CASE:

The ambiguous case is when you have more than one solution to the triangle, this can happen when **two angles are unknown** (e.g. you do not know whether these angles are acute or obtuse)

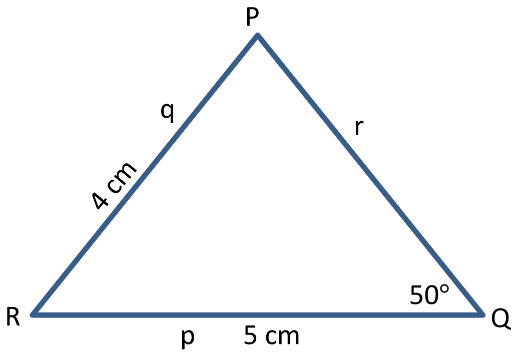
Here's an example on how to handle this situation:

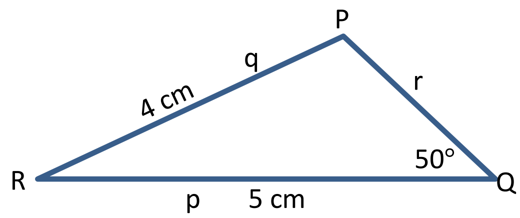
Example

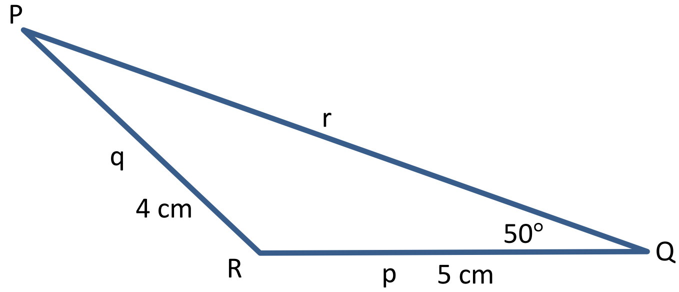
Solve ΔPQR with $\hat{Q} = 50°$, $PR = 4$ cm and $QR = 5$ cm.

NOTE: Always draw the triangle when it is not originally drawn, so you know which side goes with which angle.

Answer:







NOTE: Any one of these sketches can be used to show the triangle, this is why you have an ambiguous case.

1. Calculate an angle using the sine rule:

$$\frac{\sin P}{p} = \frac{\sin Q}{q}$$

$$\therefore \sin P = \frac{p \sin Q}{q}$$

$$\therefore \sin P = \frac{5 \sin 50°}{4}$$ →Remember trig equations

Reference angle: 73.2° (Working in first and second quadrant)

Quadrant I

$\hat{P} = 73.2° + k.360°$

Quadrant II

$\hat{P} = 180° - 73.2° + k.360°$

$\hat{P} = 106.8° + k.360°$

$\therefore \hat{P} = 73.2°$ or $\hat{P} = 106.8°$

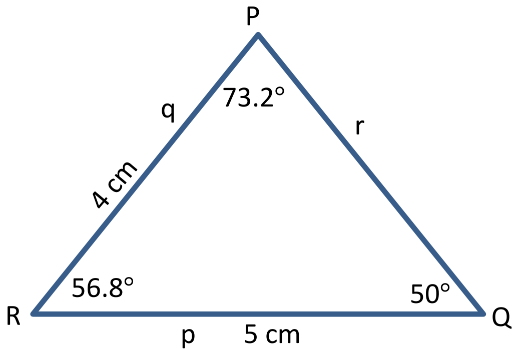
2. Calculate the other unknown angle by using "interior angles of a triangle"

$\therefore \hat{R} = 180° - 73.2° - 50°$ or $\hat{R} = 180° - 106.8° - 50°$

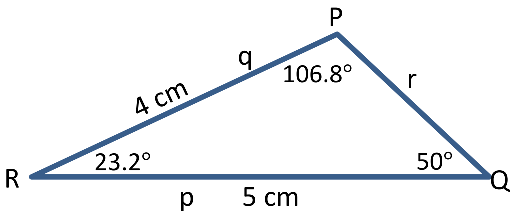
$\therefore \hat{R} = 56.8°$ $\therefore \hat{R} = 23.2°$

NOTE: If you get a negative answer in the second step, it means that the ambiguous case does NOT apply and you use the answer that "works"

3. Now solve the other sides:



OR



$$\frac{r}{\sin R} = \frac{q}{\sin Q}$$

$$\therefore r = \frac{q \sin R}{\sin Q}$$

$\therefore r = \frac{4 \sin 56.8°}{\sin 50°}$
$\therefore r = 4.4$ cm

OR

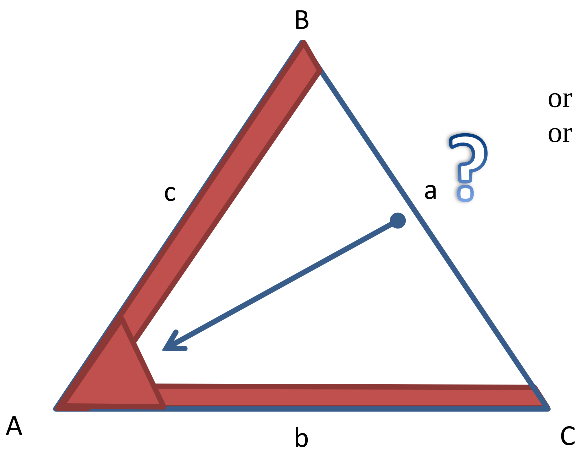
$\therefore r = \frac{4 \sin 23.2°}{\sin 50°}$
$\therefore r = 2.1$ cm

2. COSINE RULE:

When to use the Cos Rule:

- If you want to calculate a side, you have to have its opposite angle and the other two sides (two sides and an enclosed angle), OR
- If you want to calculate an angle, you have to have all three sides.

TO FIND A SIDE:



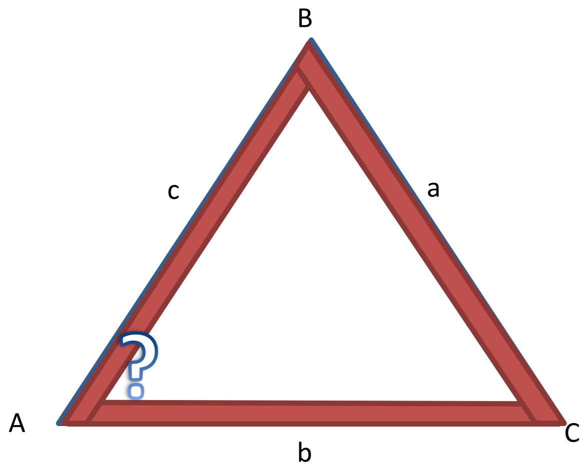
$$a^2 = b^2 + c^2 - 2bc \cos A$$

or $$b^2 = a^2 + c^2 - 2ac \cos B$$

or $$c^2 = a^2 + b^2 - 2ab \cos C$$

You only need these variables to solve this particular problem…

TO FIND AN ANGLE:



$$\frac{b^2+c^2-a^2}{2bc} = \cos A$$

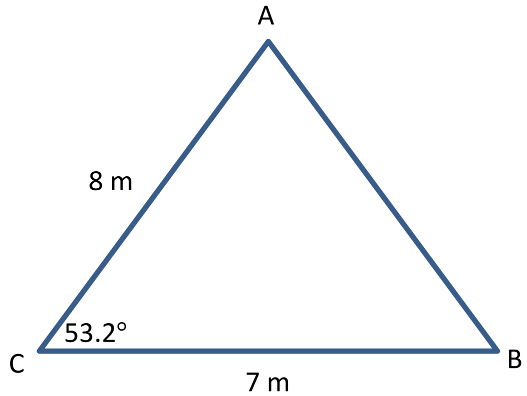
OR $$\frac{c^2+a^2-b^2}{2ac} = \cos B$$

OR $$\frac{a^2+b^2-c^2}{2ab} = \cos C$$

You only need these variables to solve this particular problem…

Example

Find AB:

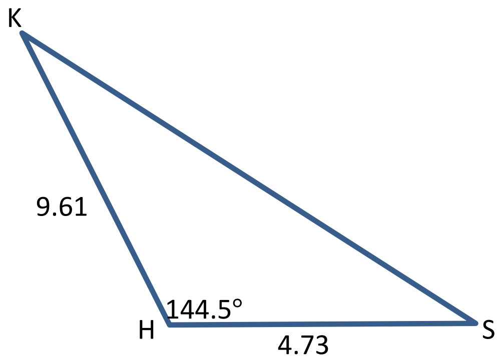


Answer:

$c^2 = a^2 + b^2 - 2ab\cos C$
$\therefore c = \sqrt{a^2 + b^2 - 2ab\cos C}$
$\therefore c = \sqrt{7^2 + 8^2 - 2(7)(8)\cos 53.2°}$
$\therefore c = 6.8 \text{ m}$

Example

Solve the triangle below:



Answer:

To find the length of KS:

$$h^2 = k^2 + s^2 - 2ks\cos H$$
$$\therefore h = \sqrt{k^2 + s^2 - 2ks\cos H}$$
$$\therefore h = \sqrt{(4.73)^2 + (9.61)^2 - 2(4.73)(9.61)\cos 144.5°}$$
$$\therefore h = 13.74 \text{ units}$$

To find the angles $\widehat{\mathrm{K}}$ and $\hat{\mathrm{S}}$: (Two methods can be used)

Method 1:

Using Cosine Rule

$$\cos K = \frac{h^2+s^2-k^2}{2(h)(s)}$$
$$\therefore \cos K = \frac{(13.74)^2+(9.61)^2-(4.73)^2}{2(13.74)(9.61)}$$
$$\therefore \widehat{K} = 11.53°$$

Method 2:

Using Sine Rule

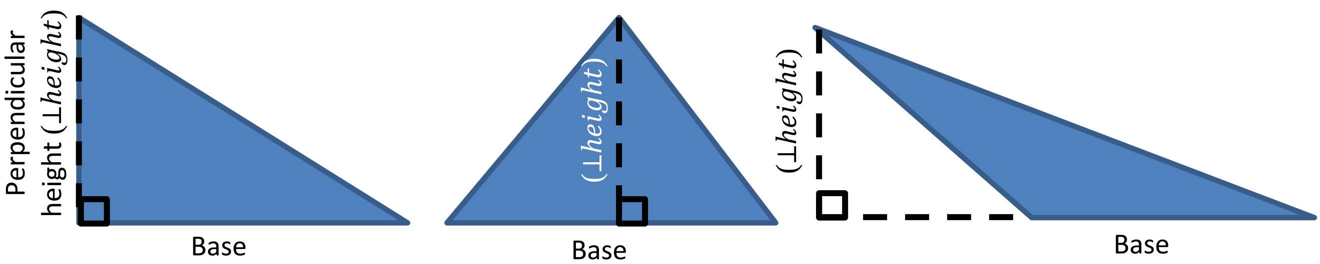
$$\frac{\sin K}{k} = \frac{\sin H}{h}$$
$$\therefore \sin K = \frac{k\sin H}{h}$$
$$\therefore \sin K = \frac{4.73\sin 144.5}{13.74}$$
$$\therefore \widehat{K} = 11.53°$$

Now find $\hat{S}$ using "interior angles of a triangle"

$$\therefore \hat{S} = 180° - 144.5° - 11.53°$$
$$\therefore \hat{S} = 23.97°$$

Finding the area of a triangle:

Previously, you had to find the area of a triangle through the following formula:
Area of $\Delta = \frac{1}{2}.\, base. \perp height.$

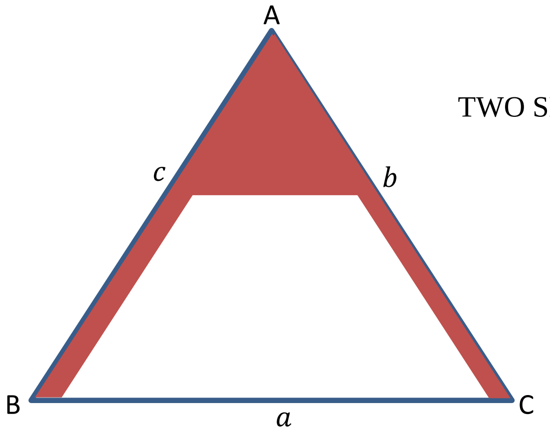


The problem here is that you need a perpendicular height! What if you do not have a perpendicular height?

Solution: Area Rule: Area $= \frac{1}{2}ab\sin C$

or Area $= \frac{1}{2}ac\sin B$

or Area $= \frac{1}{2}bc\sin A$



What you need is the following:

TWO SIDES AND AN ENCLOSED ANGLE GIVEN.

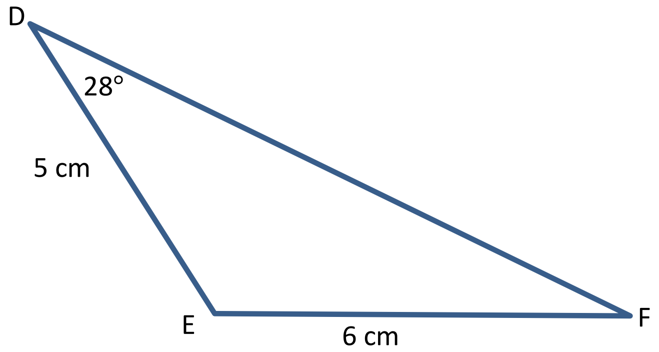
In this sketch, you need the following formula:
Area of $\triangle$ABC = $\frac{1}{2}bc\sin A$

Example

1. Find the Area of the triangle given:

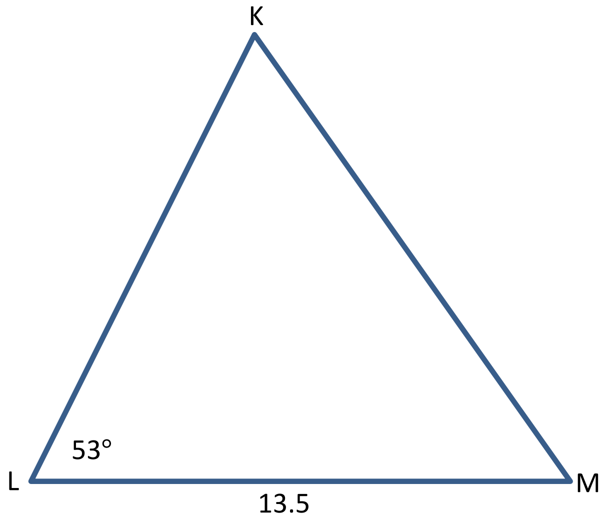
   a. Triangle ABC with BC = 5.4 m, AC = 6.2 m and $\hat{C}$ = 42°.

   b.



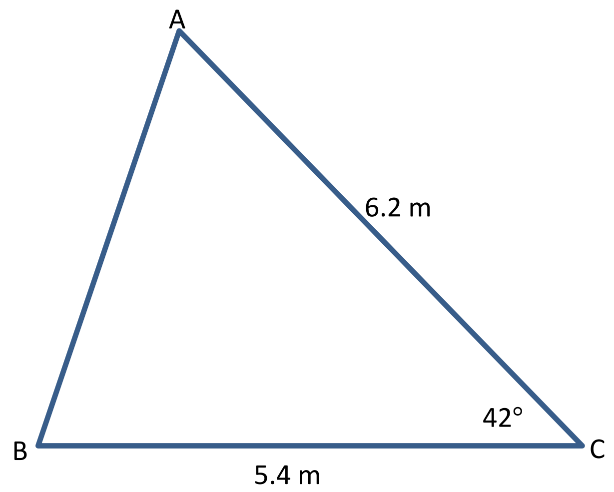
   c. Triangle GHJ with GH = 7.2 units, HJ = 5.3 units and GJ = 6.1 units.

2. Solve the following triangle if the Area of the triangle is 89.23 m^2.



Answers:

1.a



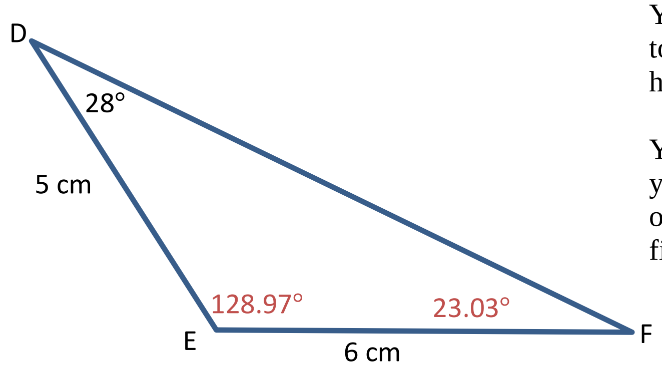
Always sketch the triangle, it is easier when you have it in front of you!

Area = $\frac{1}{2}ab\sin C$

∴Area = $\frac{1}{2}(5.4)(6.2)\sin 42°$

∴Area = 11.2 m

1.b



You do not have enough information to directly calculate the Area, so you have to FIND what you NEED!

You already have two sides, so all you need is the enclosed angle, use one of the rules, i.e. sine or cos to find this angle.

According to the info given, you are going to have to find angle F, using the sine rule and then interior angles of a triangle to find angle E, only then can you actually find the Area of this triangle!

Finding angle F:

$$\frac{\sin F}{f} = \frac{\sin D}{d}$$

$$\therefore \sin F = \frac{f\sin D}{d}$$

$$\therefore \sin F = \frac{5\sin 28°}{6}$$

$$\therefore \hat{F} = 23.03°$$

Finding angle E:

$$\hat{E} = 180° - 28° - 23.03°$$

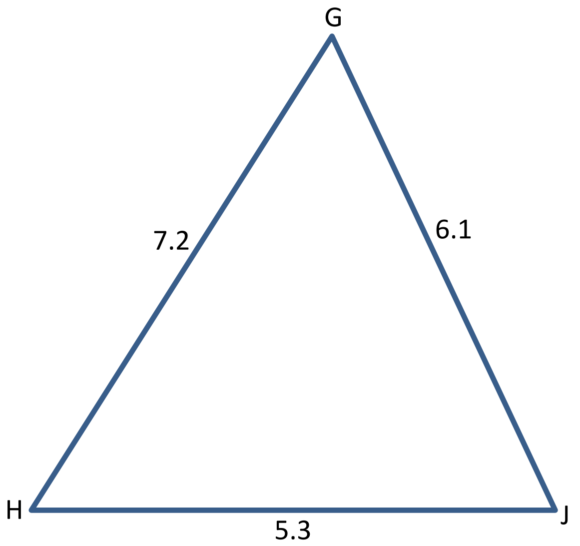
$$\therefore \hat{E} = 128.97°$$

Now to answer the question: Area of ΔDEF $= \frac{1}{2}df\sin E$

$= \frac{1}{2}(6)(5)\sin 128.97°$

$= 11.66\ \text{cm}^2$

1.c



Here again, you cannot find the Area directly, you'll have to find the value of one of the angles and then use the Area rule to do the rest.

Finding angle G:

$$\cos G = \frac{h^2+j^2-g^2}{2hj}$$

$$\therefore \cos G = \frac{(6.1)^2+(7.2)^2-(5.3)^2}{2(6.1)(7.2)}$$

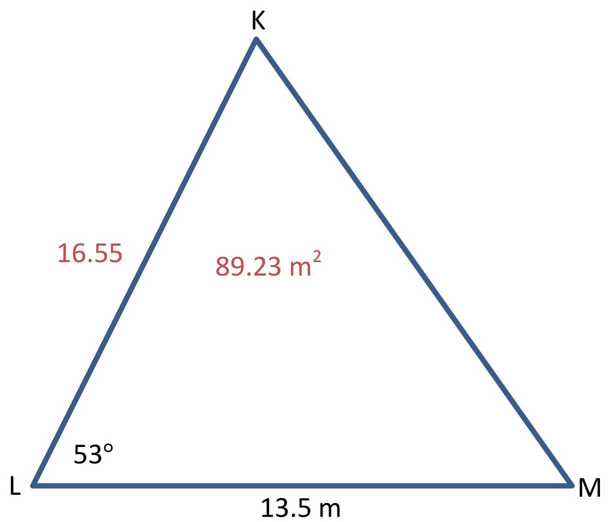
$$\therefore \hat{G} = 46.05°$$

Now to answer the question: Area of ΔGHJ $= \frac{1}{2} jh \sin G$

$$= \frac{1}{2}(7.2)(6.1)\sin 46.05°$$

$$= 15.8 \text{ units}^2$$

2.



Use the Area rule to find side KL, from there you can use either the Sine or Cosine rule to find all the other unknowns.

Finding side KL: Area of Δ KLM $= \frac{1}{2} km \sin L$

$$\therefore 89.23 = \frac{1}{2}(13.5)(m)\sin 53°$$

$$\therefore \frac{89.23 \times 2}{13.5 \sin 53°} = m$$

$$\therefore m = 16.55 \text{ m}$$

Finding side KM: $l^2 = k^2 + m^2 - 2km\cos L$

$$\therefore l = \sqrt{(13.5)^2 + (16.55)^2 - 2(13.5)(16.55)\cos 53°}$$

$$\therefore l = 13.7 \text{ m}$$

Now find one of the other angles: Finding angle K:

Two methods

Using the sine Rule

$\frac{\sin K}{k} = \frac{\sin L}{l}$

$\therefore \sin K = \frac{k \sin L}{l}$

$\therefore \sin K = \frac{13.5 \sin 53°}{13.7}$

$\therefore \hat{K} = 51.9°$

Using the Cosine Rule

$\cos K = \frac{l^2+m^2-k^2}{2lm}$

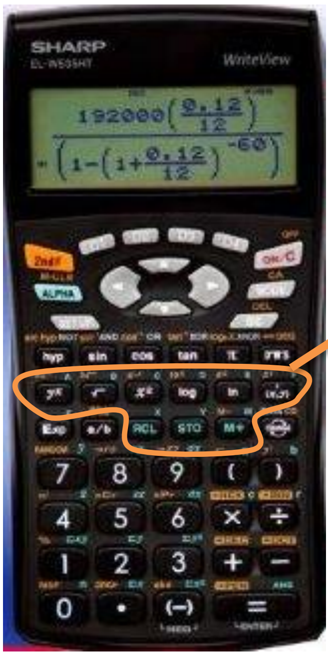
$\therefore \cos K = \frac{(13.7)^2+(16.55)^2-(13.5)^2}{2(13.7)(16.55)}$

$\therefore \hat{K} = 51.9°$

Now find the other angle:

Finding angle M:
$\hat{M} = 180° - 51.9° - 53°$
$\therefore \hat{M} = 75.1°$

NOTE: Because of all the rounding you need to do while working with further calculations, it would actually be better to STORE any answers you may need to be as accurate as possible, example 2 will be done again by using the MEMORY function in your EL – 535 calculator:



Memory buttons!

2. Finding side KL:

Area of Δ KLM $= \frac{1}{2} km \sin L$

$\therefore 89.23 = \frac{1}{2}(13.5)(m) \sin 53°$

$\therefore \frac{89.23 \times 2}{13.5 \sin 53°} = m$

$m = 16.55230589 \ldots$ - press STO y^x, on your screen you'll see: ANS⇒A meaning that your full answer is now stored in Memory "A"

Finding side KM: $l^2 = k^2 + m^2 - 2km\cos L$

$\therefore l = \sqrt{(13.5)^2 + (\text{A})^2 - 2(13.5)(\text{A})\cos 53°}$

Press ALPHA y^x to access “A”

$\therefore l = 13.6846746 \ldots$- press STO $\sqrt{\ }$, on your screen you'll see: ANS⇒B meaning that your full answer is now stored in Memory “B”

Now find one of the other angles: Finding angle K:

Two methods

Using the sine Rule

$\frac{\sin K}{k} = \frac{\sin L}{l}$

$\therefore \sin K = \frac{k \sin L}{l}$

$\therefore \sin K = \frac{13.5 \sin 53°}{\text{B}}$

$\therefore \widehat{K} = 51.98577982° \ldots$- press STO x^2

Using the Cosine Rule

$\cos K = \frac{l^2+m^2-k^2}{2lm}$

$\therefore \cos K = \frac{(\text{B})^2+(\text{A})^2-(13.5)^2}{2(\text{B})(\text{A})}$

$\therefore \widehat{K} = 51.98577982° \ldots$

Now find the other angle:

Finding angle M:

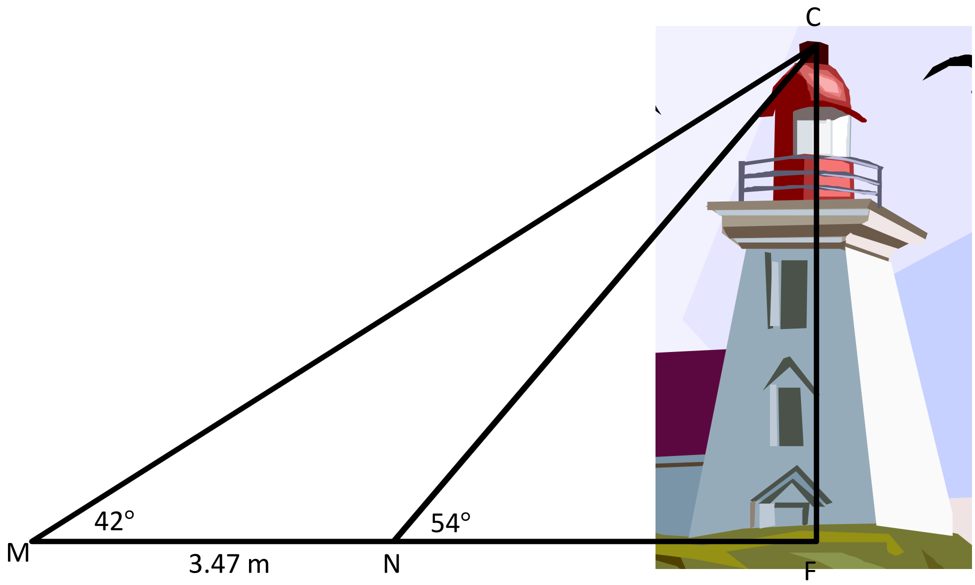
$\widehat{M} = 180° - \text{C} - 53°$

$\therefore \widehat{M} = 75.01422018°$

Real life problems using Sine, Cosine and Area Rules:

Example

The angles of elevation to the top of a Lighthouse CF, from two points N and M on the same horizontal plane as the foot of the Lighthouse F, are 42° and 54° respectively. The distance MN = 3.47 m.

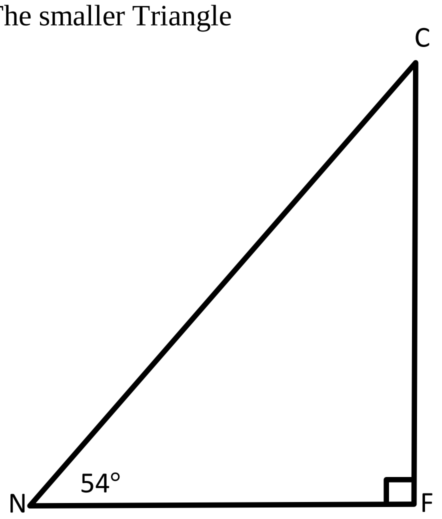


Find the height of the Lighthouse.

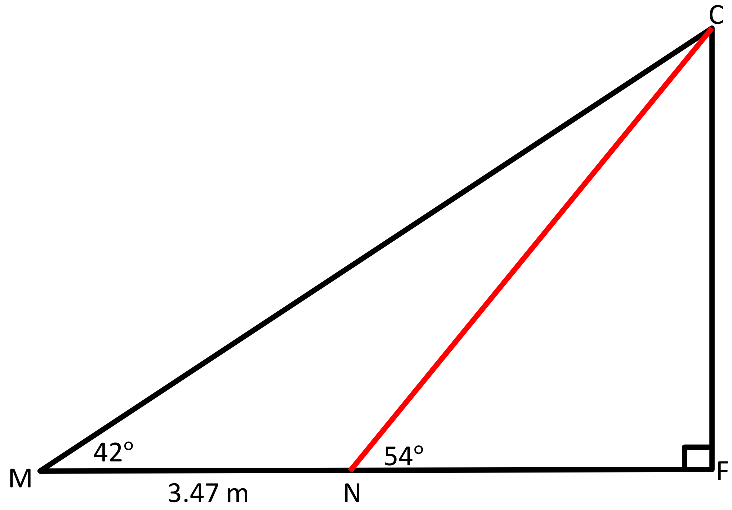
Answer:

To find the height of the Lighthouse, you need to analyse what you can use to find this height: The Lighthouse occurs in two triangles:

The smaller Triangle

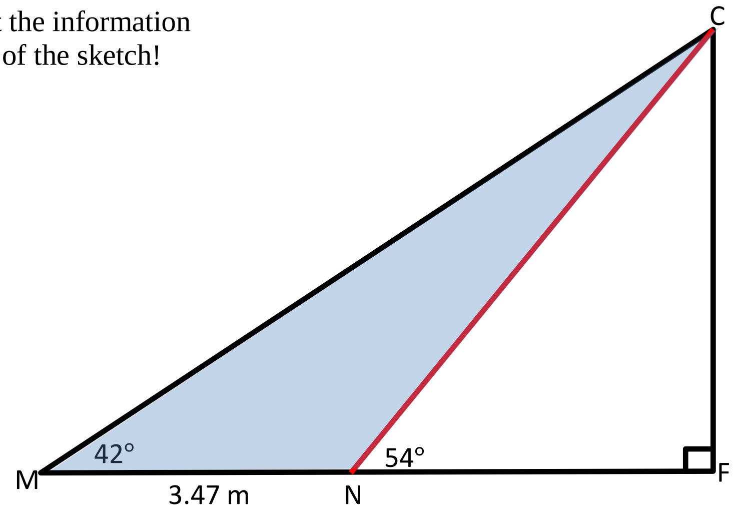


The greater triangle

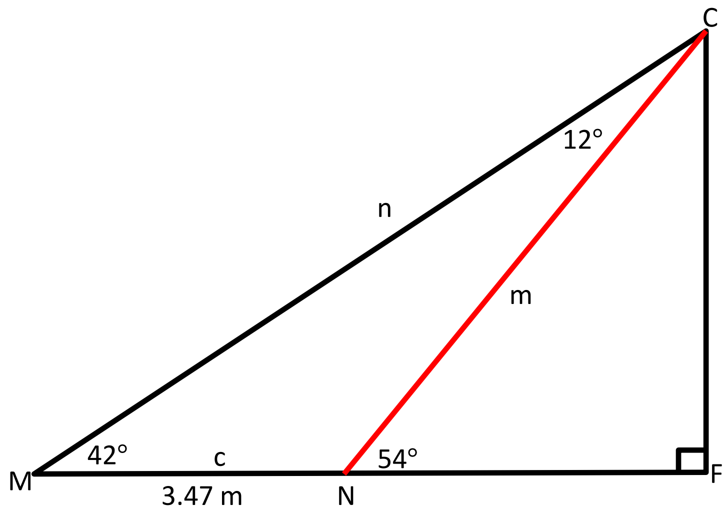


To work in the smaller triangle, you need at least an angle and a known side (that is to use the trig ratios from grade 10). You do not have enough information for this. Now try connect the information given in the rest of the sketch!

CN is a common side, if you can find a way to calculate CN's length, you would have enough information to use trig ratios and calculate CF...Use triangle MNC and the Sine rule to find the size of CN.



First find angle $M\hat{C}N$: $54° - 42° = 12°$ (Exterior angle of Δ = sum of opp int.∠s)



Now use the Sine rule: $\frac{m}{\sin M} = \frac{c}{\sin C}$

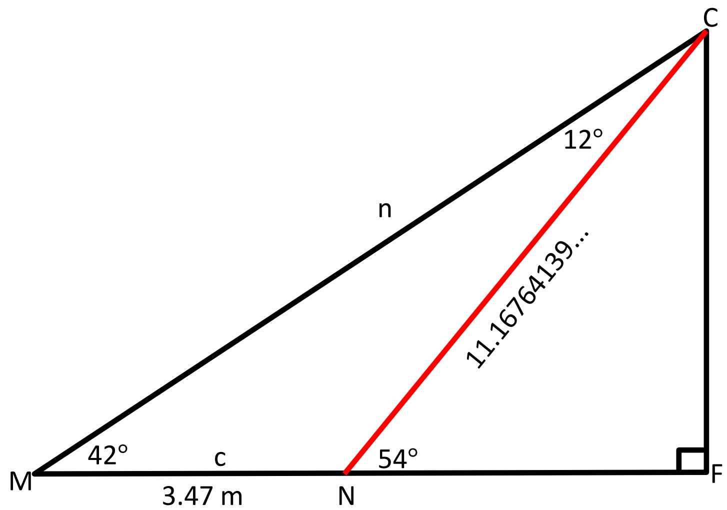
$$\therefore m = \frac{c \sin M}{\sin C}$$

$$\therefore m = \frac{3.47 \sin 42°}{\sin 12°}$$

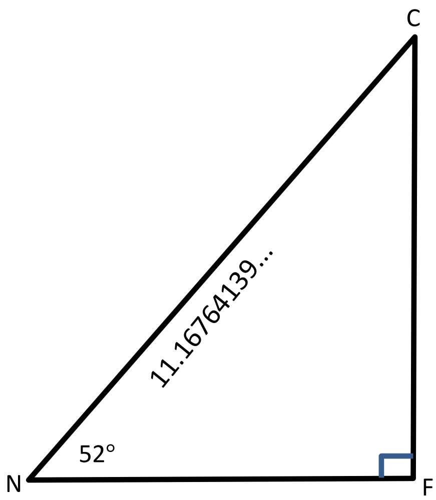
$$\therefore m = 11.16764139 \ldots$$

If you want, you can store this answer into memory for an accurate final answer, here's how:

Press: STO y^x

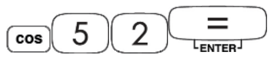


Now you have enough information to use your trig ratios to complete the question!



$$\sin N = \frac{CF}{CN}$$
$$\therefore CF = CN \sin N$$
$$\therefore CF = 11.16764139 \ldots \sin 52°$$
$$\therefore CF = 8.800221508 \ldots$$

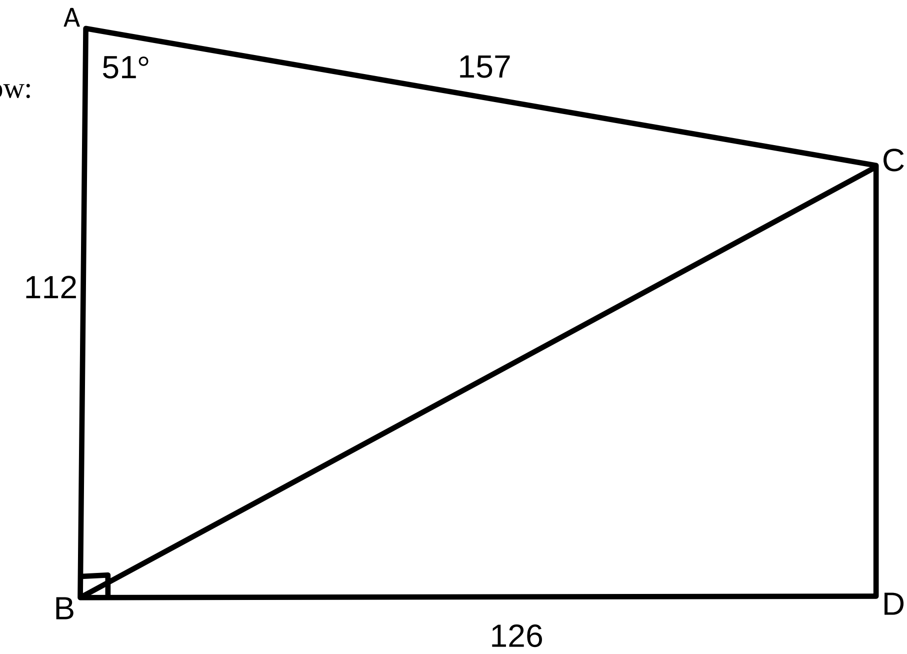
Using the memory on your calculator, you press: 



Therefore the height of the Lighthouse is 8.8 metres (Rounded to one decimal digit)

Activity 7

7.1 Refer to the diagram below:



a. Calculate $A\hat{B}C$.

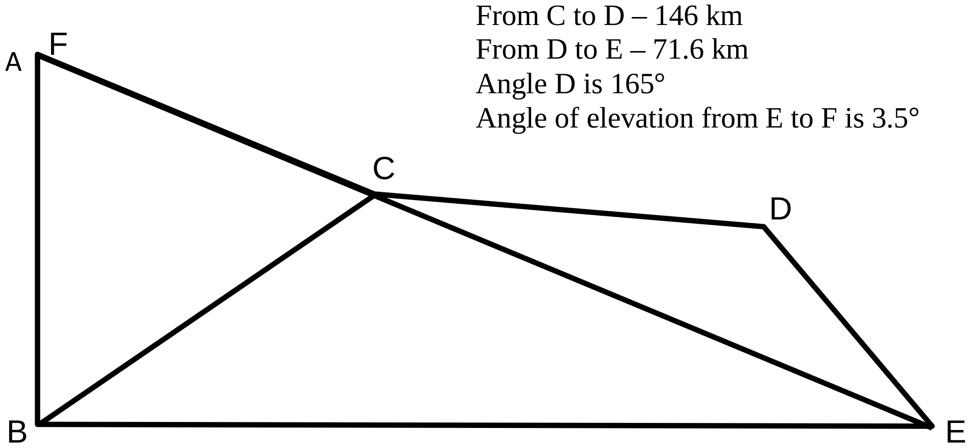
b. Calculate the Area of ΔBCD

7.2 Below is a sketch of a traveller’s plan of action:

He plans on Travelling from A to E and then straight back to his starting point, F, but needs a little help with the distance, EF.

The following is known to him:

From A to B – 69.8 km
From B to C – 146 km
From C to D – 146 km
From D to E – 71.6 km
Angle D is 165°
Angle of elevation from E to F is 3.5°



Please help the traveller calculate the distance of his last route, EF. Round all answers to the nearest integer.

1.3 GRADE 12 work:

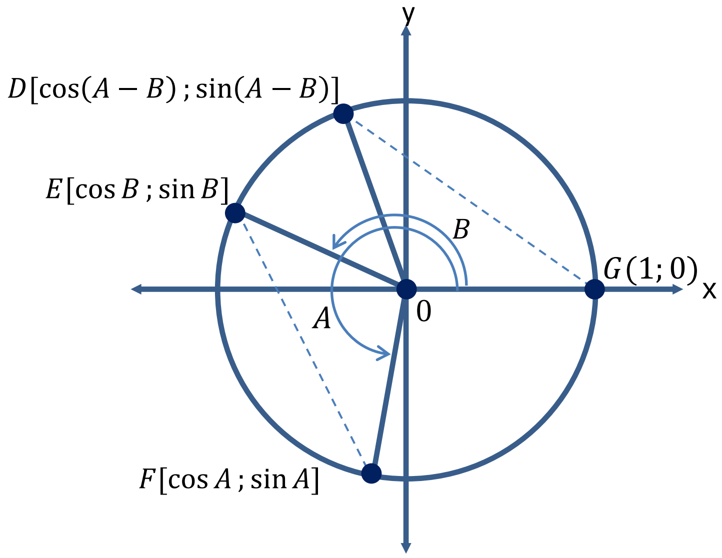
1.3.1 COMPOUND ANGLES

This is where Matric identities really get exciting! First, let's get over the nitty gritty stuff…

Proving these formulae…

We'll start with proving the formula $\mathbf{cos}(\boldsymbol{A}-\boldsymbol{B}) = \mathbf{cos}\,\boldsymbol{A}\,.\,\mathbf{cos}\,\boldsymbol{B} + \mathbf{sin}\,\boldsymbol{A}\,.\,\mathbf{sin}\,\boldsymbol{B}$:

Consider a circle with a radius of 1 unit:



Given any angles A and B. In a sketch of the unit circle, draw radii OF and OE to form angles A and B. Hence $\hat{\mathrm{FOE}} = A - B$.

Draw radii OD and OG so that $\hat{\mathrm{DOG}} = A - B$, and OG lies on the x-axis.

Since $A - B = \hat{\mathrm{FOE}} = \hat{\mathrm{DOG}}$, chord FE = chord DG.

$(\Delta EOF \equiv \Delta DOG)$

According to the distance formula:

$$\mathrm{FE}^2 = (x_\mathrm{F} - x_\mathrm{E})^2 + (y_\mathrm{F} - y_\mathrm{E})^2$$

$$\therefore \quad = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$$

Remember:
$\sin^2 B + \cos^2 B = 1$

$$\therefore \quad = \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B$$

$$\therefore \quad = 2 - 2(\cos A\cos B + \sin A\sin B)$$

and $\quad \mathrm{DG}^2 = (x_\mathrm{D} - x_\mathrm{G})^2 + (y_\mathrm{D} - y_\mathrm{G})^2$

$$\therefore \quad = [\cos(A-B) - 1]^2 + [\sin(A-B) - 0]^2$$

Remember:

$\sin^2(A-B) + \cos^2(A-B) = 1$

$$\therefore \quad = \cos^2(A-B) - 2\cos(A-B) + 1 + \sin^2(A-B)$$

$$\therefore \quad = 2 - 2\cos(A-B)$$

But DG = FE: $\quad \therefore 2 - 2\cos(A-B) = 2 - 2(\cos A\cos B + \sin A\sin B)$

$$\therefore \mathbf{cos}(\boldsymbol{A}-\boldsymbol{B}) = \mathbf{cos}\,\boldsymbol{A}\,\mathbf{cos}\,\boldsymbol{B} + \mathbf{sin}\,\boldsymbol{A}\,\mathbf{sin}\,\boldsymbol{B}$$

Using this known formula, all the other compound angle formulae can be deduced:

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

⇨ To find $\boldsymbol{\cos(A + B)}$:

$\cos[A - (-B)] = \cos A \cos(-B) + \sin A \sin(-B)$

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

⇨ To find $\boldsymbol{\sin(A - B)}$:

$\cos[90° - (A - B)] = \cos[90° - A + B] = \cos[(90° - A) + B]$

$\therefore \cos[(90° - A) + B] = \cos(90° - A)\cos B - \sin(90° - A)\sin B$

$\therefore \sin(A - B) = \sin A \cos B - \cos A \sin B$

⇨ To find $\boldsymbol{\sin(A + B)}$:

$\cos[90° - (A + B)] = \cos\,[90° - A - B] = \cos\,[(90° - A) - B]$

$\therefore \cos[(90° - A) - B] = \cos(90° - A)\cos B + \sin(90° - A)\sin B$

$\therefore \sin(A + B) = \sin A \cos B + \cos A \sin B$

So where do we use these things? Well, we can use them as identities, to simplify expressions, to solve equations (still to come), to solve 2D and 3D real life problems, etc.

Example

7.1 Simplify the following:

a. $\cos 5x \cos 2x + \sin 5x \sin 2x$ b. $\sin 5° \cos 40° + \cos 5° \sin 40°$

c. $\sin x \cos y - \sin y \cos x$ d. $\sin 20° \cos 10° + \cos 20° \sin 10°$

7.2 Expand the following:

a. $\sin(90° - \theta)$ b. $\cos(90° - \theta)$ c. $\sin(180° - \theta)$

7.3 If $\sin A = \frac{2}{3}$; $\cos A < 0$ and $\cos B = \frac{7}{9}$; $B \in [0°; 90°]$, determine $\cos(A + B)$ without the use of a calculator.

Answers:

7.1.a $\cos 5x \cos 2x + \sin 5x \sin 2x$

$= \cos(5x - 2x)$

$= \cos 3x$

b. $\sin 5° \cos 40° + \cos 5° \sin 40°$

$= \sin(5° + 40°)$

$= \sin 45°$

$= \frac{\sqrt{2}}{2}$

c. $\sin x \cos y - \sin y \cos x$

$= \sin x \cos y - \cos x \sin y$

$= \sin(x - y)$

d. $\sin 20° \cos 10° + \cos 20° \sin 10°$

$= \sin(20° + 10°)$

$= \sin 30°$

$= \frac{1}{2}$

7.2.a $\sin(90° - \theta)$

$= \sin 90° \cos\theta - \cos 90° \sin\theta$

$= 1.\cos\theta - 0.\sin\theta$

$= \cos\theta$

b. $\cos(90° - \theta)$

$= \cos 90° \cos\theta + \sin 90° \sin\theta$

$= 0.\cos\theta + 1.\sin\theta$

$= \sin\theta$

c. $\sin(180° - \theta)$

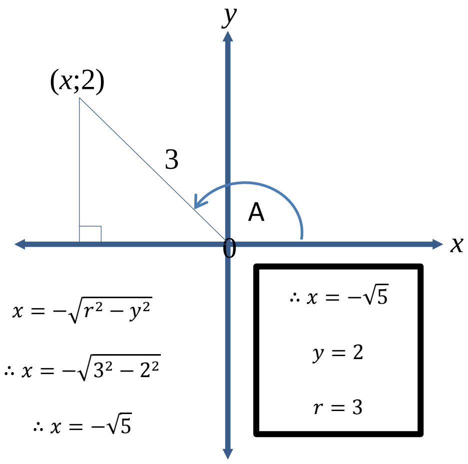
$= \sin 180° \cos\theta - \cos 180° \sin\theta$

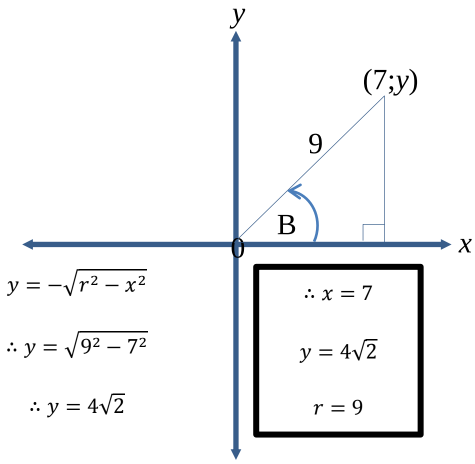
$= 0.\cos\theta - (-1).\sin\theta$

$= \sin\theta$

7.3 $\sin A = \frac{2}{3}; \cos A < 0$ and $\cos B = \frac{7}{9}; A \in [0°; 90°]$, $\cos(A + B)$

Use sketches and Pythagoras to find x, y and r in both angles.





$\cos(A + B)$ →Expand

$= \cos A \cos B - \sin A \sin B$ →Substitute

$= \left(-\frac{\sqrt{5}}{3}\right).\left(\frac{7}{9}\right) - \left(\frac{2}{3}\right).\left(\frac{4\sqrt{2}}{9}\right)$

$= -\frac{7\sqrt{5}}{27} - \frac{8\sqrt{2}}{27}$

$= \frac{-7\sqrt{5} - 8\sqrt{2}}{27}$

Activity 8

8.1 Simplify the following without the use of a calculator:

a. $\cos 3A \cos 6A - \sin 3A \sin 6A$

b. $\sin 2B \cos 4B + \cos 2B \sin 4B$

c. $\cos 5D \cos D + \sin 5D \sin D$

d. $\sin 4F \cos F - \sin F \cos 4F$

e. $\cos 31° \cos 89° - \sin 31° \sin 89°$

f. $\sin 5° \cos 25° + \cos 5° \sin 25°$

g. $\cos 149° \cos 104° + \sin 149° \sin 104°$

h. $\sin 114° \sin 54° + \cos 114° \sin 36°$

8.2 Expand the following:

a. $\cos(180° - \theta)$

b. $\tan(180° - \theta)$

c. $\sin(180° + \theta)$

d. $\cos(180° + \theta)$

e. $\sin(360° - \theta)$

f. $\cos(360° - \theta)$

g. $\sin(270° - \theta)$

h. $\cos(270° - \theta)$

i. $\tan(A - B)$

8.3 Prove the following identities:

a. $\cos(x - y)\sin(x + y) = \cos x \sin x + \cos y \sin y$

b. $\sin 7x + \sin 5x = 2 \sin 6x \cos x$

c. $\cos 2x = 1 - 2\sin^2 x$ (Hint: $\cos(2x) = \cos(x + x)$)

d. $\sin 2x = 2 \sin x \cos x$

8.4 If $\tan\theta = -\frac{4}{3}; 0° \leq \theta \leq 180°$ and $\tan\alpha = 2; \sin\alpha < 0$, determine without a calculator:

a. $\sin(\theta + \alpha)$

b. $\cos(\alpha + \theta)$

c. $\tan(\alpha + \theta)$

1.3.2 DOUBLE ANGLES

Like Compound Angles, you can use Double angles to simplify, solve, prove, etc.

You've already proved some of the double angle formulae, here are all of the double angle identities:

$$\begin{aligned}\mathbf{\cos 2A} &= \cos(A+A)\\ &= \cos A\cos A - \sin A\sin A\\ &= \mathbf{\cos^2 A - \sin^2 A}\end{aligned}$$

Remember that $\cos^2 A + \sin^2 A = 1$:

$\therefore \cos^2 A - \sin^2 A$
$= 1 - \sin^2 A - \sin^2 A$
$= \mathbf{1 - 2\sin^2 A}$

Also:

$\therefore \cos^2 A - \sin^2 A$
$= \cos^2 A - (1 - \cos^2 A)$
$= \cos^2 A - 1 + \cos^2 A$
$= \mathbf{2\cos^2 A - 1}$

$$\begin{aligned}\mathbf{\sin 2A} &= \sin(A+A)\\ &= \sin A\cos A + \cos A\sin A\\ &= \mathbf{2\sin A\cos A}\end{aligned}$$

$$\begin{aligned}\mathbf{\tan 2A} &= \tan(A+A)\\ &= \frac{\tan A+\tan A}{1-\tan A\tan A}\\ &= \mathbf{\frac{2\tan A}{1-\tan^2 A}}\end{aligned}$$

So to summarize:

$\mathbf{\cos 2A = \cos^2 A - \sin^2 A}$

$\mathbf{\cos 2A = 1 - 2\sin^2 A}$

$\mathbf{\cos 2A = 2\cos^2 A - 1}$

$\mathbf{\sin 2A = 2\sin A\cos A}$

$\mathbf{\tan 2A = \frac{2\tan A}{1-\tan^2 A}}$

Example

1. If $\cos A = \frac{2}{3}$ and $\sin A > 0$, determine the value of $\tan 2A$ without the use of a calculator.

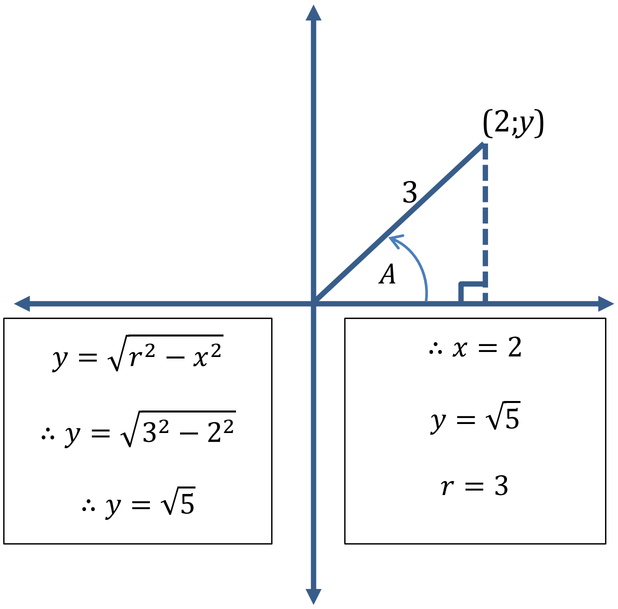
2. Prove that $\sin 2A\cos 2A = 2\sin A\cos^3 A - 2\sin^3 A\cos A$

3. Show that $\sin\frac{1}{2}B = \sqrt{\frac{1-\cos B}{2}}$

4. Prove that $\sin 3A = \sin A\,(3 - 4\sin^2 A)$

Answers:

1. $\cos A = \frac{2}{3}; \sin A > 0$



$\tan 2A$

$= \frac{2\tan A}{1-\tan^2 A}$

$= \frac{2\left(\frac{\sqrt{5}}{2}\right)}{1-\left(\frac{\sqrt{5}}{2}\right)^2}$

$= -4\sqrt{5}$

2. LHS: $\sin 2A \cos 2A$

$= 2\sin A\cos A\,(\cos^2 A - \sin^2 A)$

$= 2\sin A\cos^3 A - 2\sin^3 A\cos A$

$=$ RHS

3. $\cos B = 1 - 2\sin^2\frac{1}{2}B$ $\qquad (\cos 2B = 1 - 2\sin^2 B)$

$2\sin^2\frac{1}{2}B = 1 - \cos B$

$\sin^2\frac{1}{2}B = \frac{1-\cos B}{2}$

$\sin\frac{1}{2}B = \sqrt{\frac{1-\cos B}{2}}$

4. LHS: $\sin 3A$

$= \sin(2A + A)$

$= \sin 2A\cos A + \cos 2A\sin A$

$= 2\sin A\cos^2 A + (\cos^2 A - \sin^2 A)\sin A$

$= 2\sin A\cos^2 A + \sin A\cos^2 A - \sin^3 A$

$= 3\sin A\cos^2 A - \sin^3 A$

$= 3\sin A\,(1 - \sin^2 A) - \sin^3 A$

$= 3\sin A - 3\sin^3 A - \sin^3 A$

$= 3\sin A - 4\sin^3 A$

$= \sin A\,(3 - 4\sin^3 A)$

$=$ RHS

Activity 9:

9.1 Evaluate each of the following using double angle formulae, without the use of a calculator:

a. $2\sin 22.5°\cos °22.5°$ $\qquad$ b. $\cos^2 75° - \sin^2 75°$

c. $\sin 15°$ $\qquad$ d. $1 - 2\sin^2 105°$

9.2 Simplify the following into a single trigonometric ratio:

a. $\frac{1+\tan^2 A}{\tan A}$

b. $\frac{\cos 2x+\cos 4x}{\sin 4x-\sin 2x}$ (hint: use a substitute for $2x$)

c. $\left(\cos^2\frac{x}{4}-\sin^2\frac{x}{4}\right)\left(\cos^2\frac{x}{4}+\sin^2\frac{x}{4}\right)$

9.3 Prove the following identities:

a. $\frac{1}{\cos 2B}+\tan 2B=\frac{\sin B+\cos B}{\cos B-\sin B}$

b. $\cos 4A+8\sin^4 A=(\cos^2 A-3\sin^2 A)^2$

c. $\frac{\cos\theta}{\tan 2\theta}=\frac{\cos 2\theta}{2\sin\theta}$

d. $\frac{2\sin 2A}{\cos 2A+\sin^2 A}=4\tan A$

1.3.3 SOLVING 3D TRIANGLES

The worst part about solving 3D triangles is to distinguish between the different spaces in 3D. One of the easier things to do is to imagine you are inside the figure and to use the walls and corners of the room you are actually in to help you see what they want you to calculate.

For you to do this, you need all the formulae you have learnt to solve triangles:

In a right angled triangle:

$$\sin\theta = \frac{opposite\ side}{hypotinuse}$$

$$\cos\theta = \frac{adjacent\ side}{hypotinuse}$$

$$\tan\theta = \frac{opposite\ side}{adjacent\ side}$$

$$\text{AREA} = \frac{1}{2} base \times \perp height$$

In any other triangle: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$ *or* $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

$a^2 = b^2 + c^2 - 2bc\cos A$ $\qquad$ $\cos A = \frac{b^2+c^2-a^2}{2bc}$

$b^2 = a^2 + c^2 - 2ac\cos B$ $\qquad$ or $\qquad$ $\cos B = \frac{a^2+c^2-b^2}{2ac}$

$c^2 = a^2 + b^2 - 2ab\cos C$ $\qquad$ $\cos C = \frac{a^2+b^2-c^2}{2ab}$

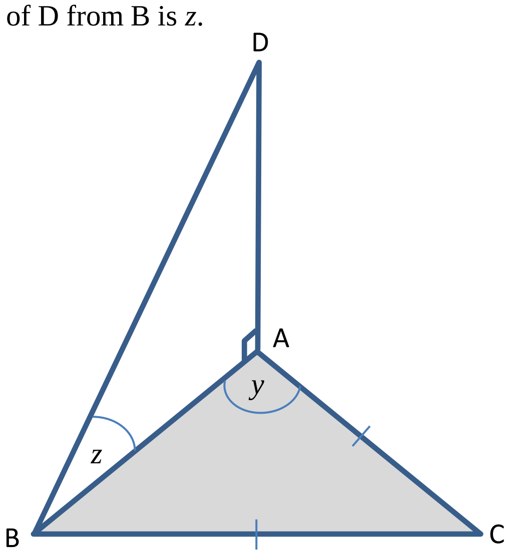
AREA = $\frac{1}{2}ab\sin C$ or $\frac{1}{2}ac\sin B$ or $\frac{1}{2}bc\sin A$

Example

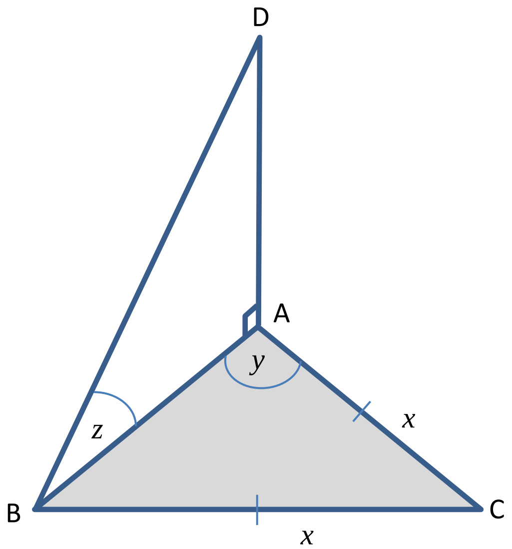
A, B and C are three points in the same horizontal plane such that AC = BC = x and B$\hat{A}$C = y. AD is perpendicular to the plane and the angle of elevation of D from B is z.

a. Prove: DA = $2x.\cos y.\tan z$

b. Given that $x = 60$ m; $y = 70°$ and $z = 25°$, calculate BD.



Answer:



Steps:

1. Reread the description of the sketch given and label anything that has not been labelled yet.

2. Imagine you are in the middle of a room, looking into a corner of the room, draw a line from the top corner you are facing to the bottom corner on your left, this line is now BD. Obviously from the bottom corner you are facing to the bottom corner on your right would be called AC.

a. This question is asking you to find a "Value" for AD. To do this, you need to understand the triangle AD appears in ΔABD, which is a right angled triangle. To find the value of a side in a right angled triangle, you need an angle given (B is given as *z*) and a side given, which is missing!

Now you need to find a side somehow... You can do this by finding a line that is connected to both ΔABD and ΔABC: line AB. So what you now need to find is the value for line AB, so that you can use it in the right angle triangle.

$\widehat{\mathrm{B}} = y$ (AC=BC, Isosceles triangle)

$\hat{C} = 180° - 2y$ (Interior angles of a triangle are supplementary)

3. Find the value of angle C:

$c^2 = a^2 + b^2 - 2ab\cos C$

4. Use the cos rule to find the value of AB:

$\therefore c = \sqrt{x^2 + x^2 - 2x^2\cos(180° - 2y)}$

$\therefore \mathrm{AB} = \sqrt{2x^2 + 2x^2\cos 2y}$

$\therefore \mathrm{AB} = \sqrt{2x^2 + 2x^2(2\cos^2 y - 1)}$

$\therefore AB = \sqrt{2x^2(1 + 2\cos^2 y - 1)}$

$\therefore AB = \sqrt{4x^2\cos^2 y}$

$\therefore AB = 2x\cos y$

$\therefore \tan B = \frac{DA}{AB}$

$\therefore DA = AB\tan B$

$\therefore DA = 2x\cos y\tan z$

5. Use right angled triangle ratios to find DA and simplify until it looks like the question.

To answer the second question, we need to get an expression for BD first, then we can substitute the values given:

$\cos B = \frac{AB}{BD}$

$\therefore BD = \frac{AB}{\cos B}$

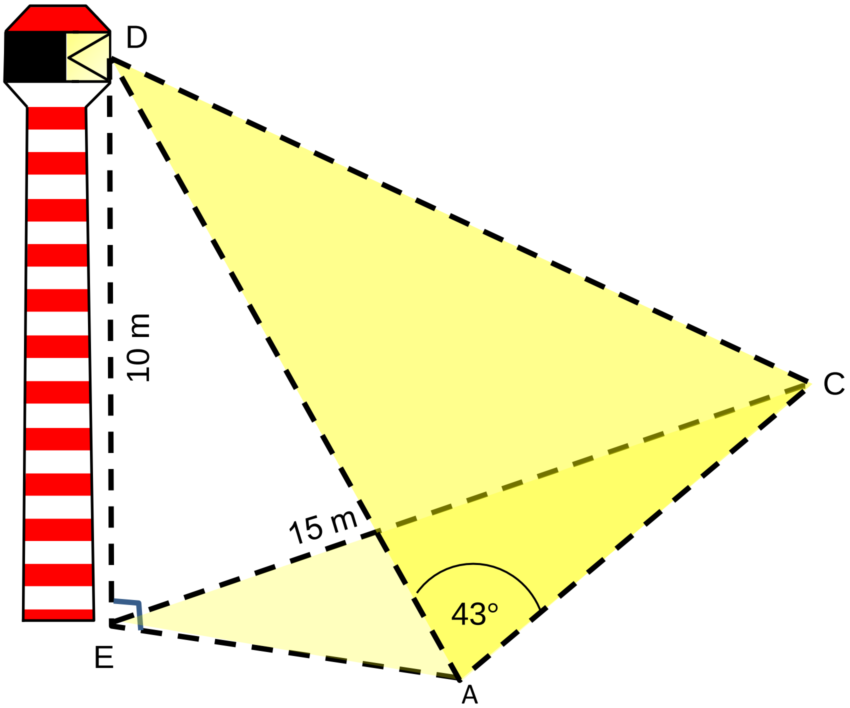
$\therefore BD = \frac{2x\cos y}{\cos z}$

$\therefore BD = \frac{2(60)\cos(70°)}{\cos(25°)}$

$\therefore BD = 45.29\ \mathrm{m}$

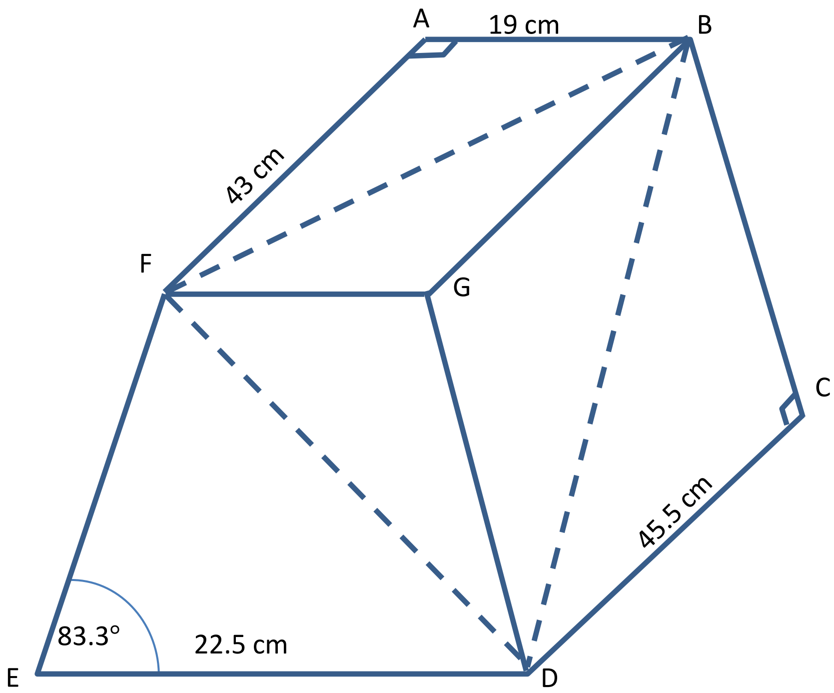
6. Use right angled triangle ratios to find BD and simplify by substituting the values given.

Activity 10



10.1 The height of a Lighthouse is 10 m, and shines from point A to point C. The distance from the foot of the Lighthouse E to C is 15 m. D$\hat{A}$C is 43°. Calculate the area the light of the Lighthouse covers on the sea, i.e. the area of EAC. (Remember that a Lighthouse shines it's light in a circular design)

10.2

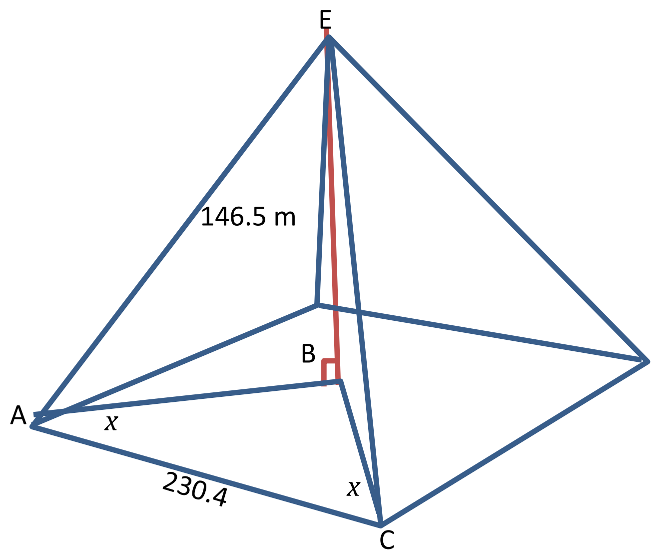


Shown above is an isosceles trapezoid prism with FE = GD = BC and FBD sliced off:

a. Calculate sides FB, FD and BD

b. Calculate The Area of FDB

10.3



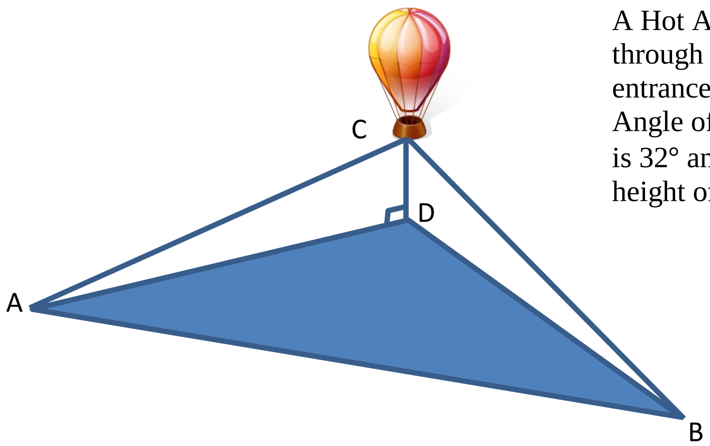
The Great Pyramid of Giza has a base of 230.4 metres and a height of 146.5 metres. The sketch represents this Pyramid.

a. Show that $AB = \frac{115.2}{\cos x}$.

b. The Great Pyramid of Giza is a square pyramid, with this knowledge, calculate the Area of ΔAEC.

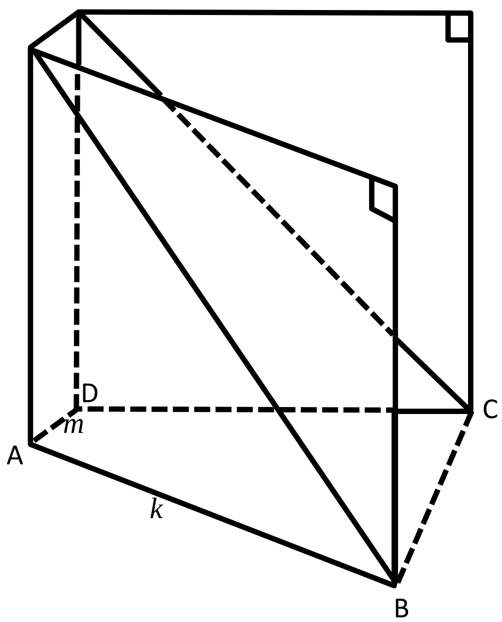
c. If the total number of lime stones used to cover the exterior of the pyramid (sides) was estimated at 2.3 million, calculate the size of the cover stones.

10.4



A Hot Air Balloon remains stationary in the air through two anchors in the ground, A and B. The entrance to the Balloon area, AB, is 24 meters long. Angle of elevation from A to C and B to C respectively is 32° and 37°. Angle $A\hat{D}B$ is 120°. Calculate the height of the balloon.

10.5



An open file is displayed on the left.
AB = k units and the back of the file has breadth m units. The sides of the file are identical.

a. Show that the angle the file will close at (i.e. the angle that will form when B and C is brought together) is $\cos^{-1}\left(1 - \frac{1}{2}\left(\frac{m}{k}\right)^2\right)$.

b. If the ratio of the height of the file to the length of the file is $1{:}\frac{1}{2}$, determine the height of the file in terms of k.

c. Calculate the **minimum** storage space needed to put this file away in terms of k and m. Name this storage space S.

d. If $k = 30$ cm and $m = 10$ cm, calculate S.