Technical Mathematics

Grade 11

Learner's Book

This book was developed with the participation of the Department of Basic Education of South Africa
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Technical Maths Grade 11 Learner's Book

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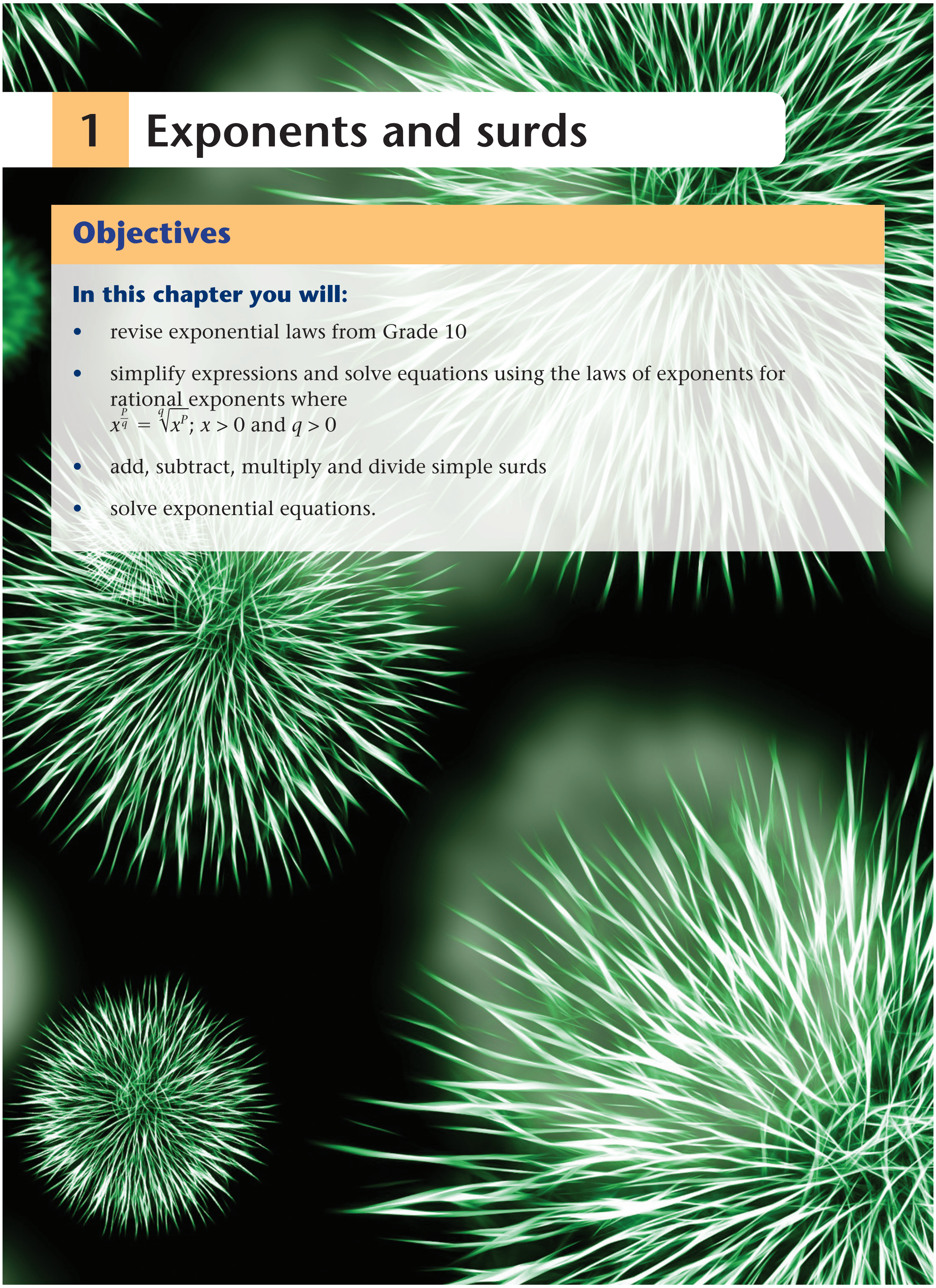

Objectives

In this chapter you will:

- revise exponential laws from Grade 10
- simplify expressions and solve equations using the laws of exponents for rational exponents where $x^{\frac{p}{q}} = \sqrt[q]{x^p}$; $x > 0$ and $q > 0$
- add, subtract, multiply and divide simple surds
- solve exponential equations.

Biologists use exponents to monitor how viruses and bacteria multiply and spread

1.1 Laws of exponents and exponential equation

The expression, 2^5 is read as two to the power of five. In this expression, two is called the base and five is called the exponent. Some people call the exponent an index; but both terms refer to the same thing.

Laws and properties of exponents

It is important for you to remember the laws of exponents. When you work with exponents, you will use these laws to solve various expressions and equations.

There are also two special exponential laws that you need to remember.

The first states that if any number a is raised to the power of zero, it will always equal one. For example, $9^0 = 1$. It does not matter how big or small the base number is, if it has an exponent of zero, the answer will always be one.

The second law states that if any number a is raised to the power of one, it will always equal a. For example, $12^1 = 12$.

The following laws or principles are important for you to know:

Law or principle or property	Illustration
1. For any non-zero number $\boldsymbol{a}$ and any real number $\boldsymbol{m}$ and $\boldsymbol{n}$: $a^m \times a^n = a^{m+n}$	$3^{-2} \times 3^5 = 3^{-2+5} = 3^3$
2. For any non-zero number $\boldsymbol{a}$ and any real numbers $\boldsymbol{m}$ and $\boldsymbol{n}$: $\frac{a^m}{a^n} = a^{m-n}$	$\frac{3^{-2}}{3^5} = 3^{-2-5} = 3^{-7}$
3. For any real numbers $\boldsymbol{m}$ and $\boldsymbol{n}$: $(a^m)^n = a^{mn}$	$(5^3)^2 = 5^{3 \times 2}$
4. For any real number $\boldsymbol{n}$: $(ab)^n = a^n b^n$	$(3 \times 7)^2 = 3^2 \times 7^2$
5. For any real number $\boldsymbol{n}$: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	$\left(\frac{3}{5}\right)^2 = \frac{3^2}{5^2}$
6. For any non-zero real number $\boldsymbol{a}$: $a^0 = 1$	$2^0 = 1$
7. For any real number a, and any real number $\boldsymbol{n}$: $a^{-n} = \frac{1}{a^n}$	$6^{-3} = \frac{1}{6^3}$

Worked example 1.1

Simplify $2^3 \times 2^7$.

Solution

$2^3 \times 2^7$
$= 2^{3+7}$
$= 2^{10}$
$= 1\ 024$

Worked example 1.2

Simplify $2^5 \div 2^3$.

Solutions

$2^5 \div 2^3$
$= 2^{5-3}$
$= 2^2$
$= 4$

Worked example 1.3

Simplify $(4^4)^4$.

Solution

$(4^4)^4$
$= 4^{4 \times 4}$
$= 4^{16}$
$= 42\,949\,672\,916$

Worked example 1.4

Simplify $(6 \times 7)^3$.

Solution

$(6 \times 7)^3$
$= 6^3 \times 7^3$
$= 74\,088$

Worked example 1.5

Simplify $\left(\frac{4}{2}\right)^3$

Solution

$\left(\frac{4}{2}\right)^3$
$= \frac{4^3}{2^3}$
$= \frac{64}{8}$
$= 8$

Worked example 1.6

Simplify 12^{-4}.

Solution

12^{-4}
$= \frac{1}{12^4}$

EXERCISE 1.1

Simplify the following.

a) 2×2^5

b) $2^3 \times 2 \times 2^3$

c) $2^4 \times 3^4$

d) $2^5 \times 2^3$

e) $\frac{2^{11}}{2^4}$

f) $\frac{2^7}{2^{11}}$

g) $2 \div 2^2$

h) $3^{-2}.2,5^{-1}$

i) $3^{-2}(3^2.2,5)$

j) $6 - 2^3$

k) $\frac{10^8}{20^2.25^3.8}$

l) $3^{-2}(2.5)^{-1}$

m) $(1\,000^{100} \times 6\,013^{-1})^0$

n) $\frac{9 \times 12^5}{4^3 \times 6^4}$

o) $3^3 \div 4^{-3}$

p) $2^3 \times 2^{-3}$

q) $5 + 5^{-1}$

r) $3^3 - 2^2$

s) $\frac{4^4 \times 18^3}{9^2 \times 6^7 \times 2^{-1}}$

t) $\frac{45^2.12}{36\,000} \times \frac{20^4}{30^4}$

u) $6^{-2}.36 + \frac{3}{4} \times 2^3$

1.2 Simplifying expressions whose bases are variables

Sometimes you will have to work with expression that have variables as their bases. A variable is usually a letter, like a, b, c, … x, y, z, and it represents an unknown.

The same laws that you worked with in the previous section also apply to these expressions.

Worked example 1.7

Simplify $\dfrac{(ab^4)^5 \times a^3b^2}{(a^2b^3)^5}$

Solution

$$\begin{aligned}\frac{(ab^4)^5 \times a^3b^2}{(a^2b^3)^5} &= \frac{a^5b^{20} \times a^3b^2}{a^{10}b^{15}}\\ &= \frac{a^8b^{22}}{a^{10}b^{15}}\\ &= \frac{b^7}{a^2}\end{aligned}$$

EXERCISE 1.2

Simplify the following:

a) $x^2y \times x^4y^5$

b) $x^2y^{-3} \times x^2y^3$

c) $\dfrac{2}{3}a^{-2} - \left(\dfrac{3}{a^{-2}}\right)^{-1}$

d) $2^{3x} \times 8^{y-2x}$

e) $(3m^3)^2 + 3(2m^3)^3$

f) $\dfrac{1}{2}\left(\dfrac{2^{2h+10}}{16.4^h}\right)$

g) $\dfrac{3^{5x-1}.81^{2x+1}}{3^{12x+3}}$

h) $\dfrac{16^{2y+2} \times 4^{1-y}}{64^{y+2}}$

i) $\dfrac{a^{2x-1}.a^{-6x+1}}{a^{-4x}}$

j) $\dfrac{49a^5b^2c}{14(a^ybc)^2}$

k) $2^x.10^{2x-1}.25^{1-x}.8^{-3x}$

l) $\dfrac{ab \times (2b^2) \times 6a^9b^7}{(2b^6)^2 \times 8(a^5b^{-2})^2}$

1.3 Solving exponential equations

So far we have simplified expressions. Remember that an expression has no equality or inequality sign, whereas an equation has an equality or inequality sign.

When solving equations, look at the base of the term with the exponent in it. Determine whether you are able to convert the other term to a term with the same base. If you are able to do so, the equation should be simple to solve.

Worked example1.8

Solve the equation, $2^x = 32$.

Solution

$2^x = 32$

$2^x = 2^5$

$x = 5$

Note:
To check if your answer is correct, simply substitute it back into the original equation.

EXERCISE 1.3

Solve the equations given below:

a) $2^x = 128$

b) $2^{n-1} = 8$

c) $7^{x-1} = 1$

d) $(b^k)^2 = b^2$

e) $8\left(\frac{7}{2}\right)^r = 343$

f) $2.3^x = 18$

g) $2^{n+1} = 64$

h) $\frac{5^{2k-1}}{5} = 25$

i) $b^{k+2} = b^4$

j) $\frac{64^{2m-1}}{16^{m+1}}$ 0= 4

k) $2^{\frac{a}{3}} = 8$

l) $(3^n)^3 = 27$

m) $6^x = \frac{1}{216}$

n) $b^k.b^2 = b^4$

o) $2^n.4^{n+1} = 16^n$

1.4 Rational (fractional) exponents

In this section we extend the laws of exponents to include cases where the exponents are rational numbers. Rational numbers include integers, natural numbers, whole numbers, fractions and decimals that can be represented as fractions.

The principal square root

Consider the following: What does the square root symbol represent? For example, what does $\sqrt{100}$ represent? Is the answer to $\sqrt{100}$ equal to 10 or is the answer equal to ±10?

In Mathematics, we define the square root of a number r to be a number x, such that $x^2 = r$. For example, the square root of a number 100 is 10, such that $10^2 = 100$. So we say 10 is the square root of 100.

But what about –10? Is it not true that $(-10)^2 = 100$?

By convention, when we use the square root symbol $(\sqrt{\ })$ in Mathematics, we are referring to the **positive square root** only. This is called the **principal square root**. Thus, the $\sqrt{100} = 10$, because 10 is the positive square root of 100.

EXERCISE 1.4

Calculate the following (without the use of a calculator).

a) $\sqrt{16 + 9}$ b) $\sqrt{100 - 36}$ c) $\sqrt{16} + \sqrt{9}$ d) $\sqrt{100} - \sqrt{36}$

Changing from surd form to exponential form

Calculating surds and converting surds to exponents are relatively common operations in algebra. To convert from surd form to exponential form, use a fraction in the power form to indicate that it stands for a root or a radical. We will explore this in the next section.

The square root

It is common practice to write the square root of a number using the root symbol $(\sqrt{\ })$.

However, another way of writing the square root of a number, is by using rational exponents to represent the taking of a root. For example, $\sqrt{100}$ can also be written as $100^{\frac{1}{2}}$. Sometimes expressing the square root like this makes the number easier to work with – it is easier working with exponents, than it is to work with roots.

The table below gives a few examples of the square root of a number expressed in exponential form.

Surd form	Exponential form	Integer
$\sqrt{49}$	$49^{\frac{1}{2}}$	7
$\sqrt{64}$	$64^{\frac{1}{2}}$	8
$\sqrt{81}$	$81^{\frac{1}{2}}$	9

EXERCISE 1.5

1. Calculate the following.
   a) $\sqrt{225}$
   b) $\sqrt{289}$
   c) $\sqrt{625}$
   d) $\sqrt{676} - \sqrt{144}$
   e) $\sqrt{169} + \sqrt{400}$
2. Calculate the value of each of the following by using the appropriate laws of exponents.
   a) $9^{\frac{1}{2}}$
   b) $-16^{\frac{1}{2}}$
   c) $(-16)^{\frac{1}{2}}$
   d) $\left(\frac{1}{9}\right)^{\frac{1}{2}}$
   e) $\left(\frac{9}{16}\right)^{\frac{1}{2}}$
   f) $(0,01)^{\frac{1}{2}}$
   g) $-\left(25^{\frac{1}{2}}\right)$
   h) $(0,25)^{\frac{1}{2}}$
3. Write each number below in surd form and then simplify.
   a) $9^{\frac{1}{2}}$
   b) $-16^{\frac{1}{2}}$
   c) $(-16)^{\frac{1}{2}}$
   d) $\left(\frac{1}{9}\right)^{\frac{1}{2}}$
   e) $\left(\frac{9}{16}\right)^{\frac{1}{2}}$
   f) $(0,01)^{\frac{1}{2}}$
   g) $-\left(25^{\frac{1}{2}}\right)$
   h) $(0,25)^{\frac{1}{2}}$

Note:

When changing from surd form to exponential form, remember:

- The nth root of a can be written as a fractional exponent with a raised to the reciprocal of that power, that is $\sqrt[n]{a} = a^{\frac{1}{n}}$.
- When the nth root of a^m is taken, it is raised to the power, $\frac{1}{n}$. When a power is raised to another power, you multiply the powers together, and so the m $\left(\text{otherwise written as } \frac{m}{1}\right)$ and the $\frac{1}{n}$ are multiplied together, that is $\sqrt[n]{a^m} = a^{\frac{m}{n}}$.

1.5 The cube root

The cube of 2 is 8. Why? Because cubing a number means to raise it to the power of three. So when we want to find the cube root of a number, we 'undo' what we did to the number when we raised it to the power of 3.

We can also convert a cube root to its exponential form by using a fraction in the power to indicate that it stands for a root or a radical. Thus, we write the cube root of a number by using rational exponents to represent the taking of a root, for example, $\sqrt[3]{27}$ can also be written as $27^{\frac{1}{3}}$.

The table below gives a few examples of the cube root of a number expressed in exponential form.

Surd form	Exponential form	Integer
$\sqrt[3]{27}$	$27^{\frac{1}{3}}$	3
$\sqrt[3]{64}$	$64^{\frac{1}{3}}$	4
$\sqrt[3]{-27}$	$-27^{\frac{1}{3}}$	-3

Worked example 1.9

Use the appropriate laws of exponents to calculate the value of $64^{\frac{1}{3}}$.

Solution

$$64^{\frac{1}{3}} = \left(4^3\right)^{\frac{1}{3}}$$
$$= 4^{3 \times \frac{1}{3}}$$
$$= 4$$

Worked example 1.10

Use the appropriate laws of exponents to calculate the value of $8^{\frac{1}{3}}$.

Solution

$$8^{\frac{1}{3}} = \left(2^3\right)^{\frac{1}{3}}$$
$$= 2$$

EXERCISE 1.6

1. Use the appropriate laws of exponents to calculate the value of the following numbers.
   a) $125^{\frac{1}{3}}$
   b) $-125^{\frac{1}{3}}$
   c) $216^{\frac{1}{3}}$
   d) $-216^{\frac{1}{3}}$
   e) $729^{\frac{1}{3}}$
   f) $-729^{\frac{1}{3}}$
   g) $(0,001)^{\frac{1}{3}}$
   h) $(-0,001)^{\frac{1}{3}}$
   i) $\left(\frac{8}{27}\right)^{\frac{1}{3}}$
2. Express the following to a surd form and simplify.
   a) $-27^{\frac{1}{3}}$
   b) $\left(\frac{64}{27}\right)^{\frac{1}{3}}$
   c) $\left(\frac{125}{216}\right)^{\frac{1}{3}}$
   d) $4\ 096^{\frac{1}{3}}$
   e) $\left(\frac{8x^3}{343}\right)^{\frac{1}{3}}$
   f) $(-512)^{\frac{1}{3}}$

In general, we have the following:

Surd form	Exponential form
$\sqrt{a}$	$a^{\frac{1}{2}}$
$\sqrt[3]{a}$	$a^{\frac{1}{3}}$
$\sqrt[4]{a}$	$a^{\frac{1}{4}}$
$\sqrt[5]{a}$	$a^{\frac{1}{5}}$
$\sqrt[n]{a}$	$a^{\frac{1}{n}}$

Worked example 1.11

Use the laws of exponents to calculate the value of $256^{\frac{1}{4}}$.

Solution

$$256^{\frac{1}{4}} = \left(4^4\right)^{\frac{1}{4}}$$
$$= 4^{4 \times \frac{1}{4}}$$
$$= 4$$

Worked example 1.12

Use the laws of exponents to calculate the value of $-243^{\frac{1}{5}}$.

Solution

$$-243^{\frac{1}{5}} = \left(-3^5\right)^{\frac{1}{5}}$$
$$= -3^{5 \times \frac{1}{5}}$$
$$= -3$$

EXERCISE 1.7

1. Use laws of exponents to calculate the value of each of the following numbers.
   a) $32^{\frac{1}{5}}$
   b) $-32^{\frac{1}{5}}$
   c) $128^{\frac{1}{7}}$
   d) $-128^{\frac{1}{7}}$
   e) $64^{\frac{1}{2}}$
   f) $-64^{\frac{1}{2}}$
   g) $16^{\frac{1}{4}}$
   h) $64^{\frac{1}{6}}$
   i) $256^{\frac{1}{8}}$

2. Use the laws of exponents to calculate the value of each of the following numbers.

a) $(625)^{\frac{1}{4}}$

b) $\left(\frac{16}{81}\right)^{\frac{1}{4}}$

c) $\left(\frac{1\ 296}{625}\right)^{\frac{1}{4}}$

d) $-32^{\frac{1}{5}}$

e) $(1\ 296)^{\frac{1}{5}}$

f) $\left(\frac{16\ 807}{243}\right)^{\frac{1}{4}}$

Note:

- In the real number system, we can calculate:
  - even roots of only non-negative numbers: for example, $\sqrt{16}$; $\sqrt[4]{16}$; $\sqrt[6]{64}$
  - odd roots of any real number: for example, $\sqrt[3]{-27}$; $\sqrt[3]{27}$; $\sqrt[5]{32}$; $\sqrt[7]{-128}$

1.6 Expressions of the form $a^{\frac{1}{n}}$

The expression $a^{\frac{m}{n}}$ is equivalent to $\left(\sqrt[n]{a}\right)^m$.

Worked example 1.13

Express $4^{\frac{3}{2}}$ in a surd form and simplify.

Solution

$4^{\frac{3}{2}}$

$= \left(\sqrt{4}\right)^3$

$= (2)^3$

$= 8$

Worked example 1.14

Express $(-27)^{\frac{2}{3}}$ in a surd form and simplify.

Solution

$(-27)^{\frac{2}{3}}$

$= \left(\sqrt[3]{-27}\right)^2$

$= (-3)^2$

$= 9$

Worked example 1.15

Express $64^{\frac{2}{3}}$ in a surd form and simplify.

Solution

$64^{\frac{2}{3}}$

$= \left(\sqrt[3]{64}\right)^2$

$= (4)^2$

$= 16$

EXERCISE 1.8

1. Write the following in a surd form and simplify.

a) $8^{\frac{2}{3}}$
b) $27^{\frac{2}{3}}$
c) $-64^{\frac{2}{3}}$
d) $125^{\frac{2}{3}}$
e) $216^{\frac{2}{3}}$
f) $343^{\frac{1}{3}}$
g) $512^{\frac{2}{3}}$
h) $1\ 000^{\frac{2}{3}}$
i) $16^{\frac{5}{4}}$
j) $9^{\frac{3}{2}}$
k) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$
l) $\left(\frac{27}{64}\right)^{\frac{1}{3}}$
m) $\left(\frac{27}{64}\right)^{-\frac{1}{3}}$
n) $\left(\frac{64}{125}\right)^{-\frac{2}{3}}$
o) $8^{-\frac{2}{3}}$
p) $64^{-\frac{2}{3}}$

1. Use the appropriate laws of exponents to calculate the values of each of the following numbers.

a) $8^{\frac{2}{3}}$
b) $27^{\frac{2}{3}}$
c) $-64^{\frac{2}{3}}$
d) $125^{\frac{2}{3}}$
e) $216^{\frac{2}{3}}$
f) $343^{\frac{1}{3}}$
g) $512^{\frac{2}{3}}$
h) $1\ 000^{\frac{2}{3}}$
i) $16^{\frac{5}{4}}$
j) $9^{\frac{3}{2}}$
k) $\left(\frac{8}{27}\right)^{\frac{2}{3}}$
l) $\left(\frac{27}{64}\right)^{\frac{1}{3}}$
m) $\left(\frac{27}{64}\right)^{-\frac{1}{3}}$
n) $\left(\frac{64}{125}\right)^{-\frac{2}{3}}$
o) $8^{-\frac{2}{3}}$
p) $64^{-\frac{2}{3}}$

1.7 Simplifying expressions that contain surds

We can simplify expressions that contain surds by rewriting them as equivalent expressions with rational exponents, then applying the laws of exponents.

Worked example 1.16

Simplify $\sqrt[4]{16a^{16}}$; $a \in \mathbb{R}$; $a \geq 0$

Solution

$\sqrt[4]{16a^{16}}$; $a \in \mathbb{R}$; $a \geq 0$

$= \sqrt[4]{2^4 a^{16}}$

$= 2a^4$

Check: $(2a^4)^4$

$= 2^4 a^{4 \times 4}$

$= 16a^{16}$

Or

$\sqrt[4]{16a^{16}}$; $a \in \mathbb{R}$; $a \geq 0$

$= (16a^{16})^{\frac{1}{4}}$

$= 2^{4 \times \frac{1}{4}}\ a^{16 \times \frac{1}{4}}$

$= 2a^4$

Check: $(2a^4)^4$

$= 2^4 a^{4 \times 4}$

$= 16a^{16}$

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EXERCISE 1.9

Simplify.

a) $\sqrt{16x^2}$

b) $\sqrt[4]{64x^4y^8z^{12}}$

c) $\sqrt[6]{729m^{12}n^{18}}$

d) $\sqrt[7]{128a^{14}}$

e) $\sqrt[3]{8x^6}$

f) $\sqrt[3]{27y^9}$

For the expressions assume all variables to be positive real numbers.

1.8 Solving exponential equations that contain rational exponents

To solve an exponential equation, we can use a method that is fairly simple, but requires a very special form of the exponential equation, that is, if $b^x = b^y$ then $x = y$.

Note:
Solving equations in this way, requires the base in both exponentials to be the same.

Let's look at an example.

Worked example 1.17

Solve for x.

$x^{\frac{2}{3}} = 81$

Solution

$$\begin{aligned} x^{\frac{2}{3}} &= 81 \\ x^{\frac{2}{3}\times\frac{3}{2}} &= 3^{4\times\frac{3}{2}} \\ x &= 3^6 \\ x &= 729 \end{aligned}$$

Check

$729^{\frac{2}{3}}$

$= 9^{3\times\frac{2}{3}}$

$= 9^2$

$= 81$

EXERCISE 1.10

1. Solve for x:

a) $4x^{\frac{5}{2}} = 128$

b) $25^{2x} = 5^{x-3}$

c) $x^{\frac{1}{3}} = 27$

d) $125^{x+1} = 5$

e) $3x^{\frac{3}{7}} = 81$

f) $6^{x-1} = 36^{-x}$

g) $3x^{\frac{3}{4}} = 375$

h) $7x^{\frac{2}{3}} = 343$

i) $64^{x-1} = 16^x$

2. In a crate, each container weighs 63 kg. What is the weight of 65 containers?
3. An asteroid is 10^{12} kilometers away from the Earth and it is travelling at a speed of 10^5 kilometers per day. How many days will it take the comet to reach the Earth?
4. The weight of a newborn baby panda is $2{,}2 \times 10^{-1}$ kg. If an adult panda weighs $1{,}2 \times 10^3$ times more than a newborn panda, how much does an adult panda weigh?

1.9 Surds

The roots of some numbers can be written as rational numbers, for example:

a) $\sqrt{4} = 2$ b) $\sqrt[3]{27 = 3}$ c) $\sqrt{\frac{9}{25}} = \frac{3}{5}$

What about $\sqrt{8}$? Can we write the square root of 8 either as a whole number or as a fraction? The answer is no. Roots of some numbers cannot be written either as whole numbers or as fractions. We will refer to such roots as **surds**.

EXERCISE 1.11

1. Copy and complete the table in your exercise book.

Number	1	2	3	4	5	6	7	8	9	10
Perfect square of the number										

2. Without using a calculator, write down the value of each of the following. Then use your calculator to check your answer:

a) $\sqrt{9}$ b) $\sqrt{900}$ c) $\sqrt{16}$ d) $\sqrt{1\ 600}$

3. Copy and complete this table giving the multiples of some of the perfect squares. Do not to use your calculator.

$\times$	2	3	4	5	6	7	8	9	10	100
4										
16				80						
25										
36										
49							392			
64										
81										
100								900		

4. Say whether the following statements are true or false.

a) $\sqrt{900} = \sqrt{9 \times 100} = \sqrt{9} \times \sqrt{100} = 30$

b) $\sqrt{1\ 600} = \sqrt{16 \times 100} = \sqrt{16} \times \sqrt{100} = 40$

c) $\sqrt{2\ 500} = \sqrt{25 \times 100} = \sqrt{25} \times \sqrt{100} = 50$

d) $\sqrt{4\ 900} = \sqrt{49 \times 100} = \sqrt{49} \times \sqrt{100} = 70$

e) $\sqrt{75} = \sqrt{3 \times 25} = \sqrt{3} \times \sqrt{25} = 5\sqrt{3}$

f) $\sqrt{486} = \sqrt{6 \times 81} = \sqrt{6} \times \sqrt{81} = 9\sqrt{6}$

Properties of surds

The following properties will be useful in simplifying expressions involving surds.

Surd form	Exponential equivalent	Restriction on the variables
$\sqrt{x} \times \sqrt{x} = x$	$x^{\frac{1}{2}}x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x$	$x, y \in \mathbb{R}\,; x \geq 0$
$\sqrt{xy} = \sqrt{x} \times \sqrt{y}$	$(xy)^{\frac{1}{2}} = x^{\frac{1}{2}}y^{\frac{1}{2}}$	$x, y \in \mathbb{R}\,; x, y \geq 0$
$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$	$\left(\frac{x}{y}\right)^{\frac{1}{2}} = \frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}}$	$x, y \in \mathbb{R}\,; y \neq 0$

Worked example 1.18

Calculate $\sqrt{4} \times \sqrt{4}$.

Solution

$\sqrt{4} \times \sqrt{4}$

$= 2 \times 2$

$= 4$

Worked example 1.19

Calculate $\sqrt{16} \times \sqrt{16}$.

Solution

$\sqrt{16} \times \sqrt{16}$

$= 4 \times 4$

$= 16$

Worked example 1.20

Calculate $\sqrt{100} \times \sqrt{100}$.

Solution

$\sqrt{100} \times \sqrt{100}$

$= 10 \times 10$

$= 100$

Worked example 1.21

Simplify the expression $\sqrt{300} \times \sqrt{16}$.

Solution

$\sqrt{300} \times \sqrt{16}$

$= \sqrt{3 \times 100} \times 4$

$= \sqrt{3} \times \sqrt{100} \times 4$

$= 10\sqrt{3} \times 4$

$= 4 \times 10\sqrt{3}$

$= 40\sqrt{3}$

EXERCISE 1.12

1. Simplify: (Leave the answer in surd form.)

a) $\sqrt{392}$
b) $\sqrt{600}$
c) $\sqrt{343}$
d) $\sqrt{810}$
e) $\sqrt{128}$
f) $\sqrt{12}$
g) $\sqrt{98}$
h) $\sqrt{175}$
i) $\sqrt{392} \times \sqrt{8}$
j) $3 \times \sqrt{12}$
k) $\sqrt{3} \times \sqrt{12}$
l) $\sqrt{192}$

Note:
Know perfect squares of natural numbers as well as their multiples. It will be very useful when you work with simplifying surds.

- If you know the perfect square of a number, you know its principal square root.
- Knowing the multiples of perfect squares allows you to write them as a product of:
  - a perfect square and a non-perfect square:(For example: $\sqrt{48} = \sqrt{3 \times 16}$
  - a perfect square and a perfect square: (For example: $\sqrt{144} = \sqrt{9 \times 16}$

2. Simplify and leave the answer in surd form where necessary:

a) $\frac{\sqrt{75}}{5}$
b) $\frac{\sqrt{125}}{\sqrt{5}}$
c) $\frac{\sqrt{192}}{\sqrt{108}}$
d) $\frac{\sqrt{5}}{\sqrt{125}}$
e) $\frac{1}{\sqrt{3}} \times \frac{5}{\sqrt{3}}$
f) $\frac{2}{\sqrt{7}} \times \frac{\sqrt{7}}{2}$
g) $\frac{2}{\sqrt{8}} \times \frac{\sqrt{8}}{\sqrt{8}}$
h) $\frac{\sqrt{25}}{\sqrt{75}}$

Adding and subtracting of simple surds

In earlier grades you learned that we add or subtract only like terms.

$4\sqrt{3}$ and $6\sqrt{3}$ are called like **surds**, just like $4x$ and $6x$ are called like terms.

$3\sqrt{2}$ and $7\sqrt{5}$ are called **unlike surds,** just like $3x$ and $7y$ are called unlike terms.

To decide whether the given surds are like or unlike, consider the number inside the root sign $(\sqrt{\ })$. For surds to be like, the number inside the root sign must be the same.

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Worked example 1.22

Simplify $4\sqrt{3} + 6\sqrt{3}$

Solution

$4\sqrt{3} + 6\sqrt{3}$

$= (4 + 6)\sqrt{3}$

$= 10\sqrt{3}$

EXERCISE 1.13

Simplify:

a) $7\sqrt{5} + 3\sqrt{2} + 8\sqrt{5} - 2\sqrt{2}$

b) $3\sqrt{3} - \sqrt{2} - 4\sqrt{3}$

c) $\sqrt{3} + 2\sqrt{3} + 3\sqrt{3} + 4\sqrt{3} + 5\sqrt{3} + 6\sqrt{3}$

d) $10\sqrt{7} - 3\sqrt{7} - 7\sqrt{7}$

Can we simplify: $\sqrt{27} + \sqrt{8} + \sqrt{75}$?

Note:

By just looking at the expression above, we may get the impression that there are no like terms in the expression. Be careful when working with questions like this, first check whether the given surds are written in their simplest form. If not, rewrite the numbers under the root sign such that they have no factors that are perfect squares. Then simplify the expression.

Worked example 1.23

Simplify: $\sqrt{27} + \sqrt{8} + \sqrt{75}$

Solution

$\sqrt{27} + \sqrt{8} + \sqrt{75}$

$= \sqrt{3 \times 9} + \sqrt{2 \times 4} + \sqrt{3 \times 25}$

$= 3\sqrt{3} + 2\sqrt{2} + 5\sqrt{3}$

$= 8\sqrt{3} + 2\sqrt{2}$

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EXERCISE 1.14

Simplify:

a) $\sqrt{48} - \sqrt{50} + \sqrt{5} - 4\sqrt{3}$

b) $\sqrt{18} + \sqrt{12} - 2\sqrt{2} + 7\sqrt{3}$

c) $-5\sqrt{20} + \sqrt{28} - \sqrt{5}$

d) $\sqrt{243} + \sqrt{300} + \sqrt{108}$

e) $\sqrt{700} + \sqrt{112} - \sqrt{175}$

f) $\sqrt{54} + 6\sqrt{216} + \sqrt{6} - 2\sqrt{3}$

g) $5\sqrt{180} + \sqrt{63} - 2\sqrt{112}$

h) $-\sqrt{18} + \sqrt{27} + \sqrt{147} - \sqrt{162}$

i) $\sqrt{96} + \sqrt{180} - \sqrt{24}$

j) $\sqrt{245} + \sqrt{147} + \sqrt{96}$

Multiplying simple surds

In this section we will use the rules learnt in algebra to solve simple surds. These are:

- $x^2 - y^2 = (x - y)(x + y)$
- $(x + y)^2 = x^2 + 2xy + y^2$
- $(x - y)^2 = x^2 - 2xy + y^2$
- $(a + b)(c + d) = ac + ad + bc + bd$
- $a(b + c) = ab + ac$

In multiplication we can combine and split surds.

We call this grouping. So, for example, $2\sqrt{5} \times 3\sqrt{7}$, can be rewritten as $2 \times \sqrt{5} \times 3 \times \sqrt{7}$.

We can then simplify this by multiplying the numbers together in one group and the surds together in another. Thus, this becomes $2 \times 3 \times \sqrt{5} \times \sqrt{7} = 6 \times \sqrt{35}$ or $6\sqrt{35}$.

Worked example 1.24

Simplify the expression $(3 + \sqrt{2})(3 - \sqrt{2})$

Solution

$(3 + \sqrt{2})(3 - \sqrt{2})$

$= (3)^2 - (\sqrt{2})^2$

$= 9 - 2$

$= 7$

Note:
$x^2 - y^2 = (x - y)(x + y)$ **is useful in simplifying surds.**

$(x + y)^2 = x^2 + 2xy + y^2$

$(x - y)^2 = x^2 - 2xy + y^2$

EXERCISE 1.15

Simplify the following expressions:

a) $(\sqrt{3} + 3)(\sqrt{3} - 3)$

b) $(3 + \sqrt{3})(3 - \sqrt{3})$

c) $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$

d) $(a - \sqrt{b})(a + \sqrt{b})$

e) $(\sqrt{7} - x)(\sqrt{7} + x)$

f) $(1 + \sqrt{2})(1 - \sqrt{2})$

Worked example 1.25

Simplify $(3 + \sqrt{2})^2$

Solution

$(3 + \sqrt{2})^2$

$= (3 + \sqrt{2})(3 + \sqrt{2})$

$= (3)^2 + 2 \times 3 \times \sqrt{2} + (\sqrt{2})^2$

$= 9 + 6\sqrt{2} + 2$

$= 11 + 6\sqrt{2}$

EXERCISE 1.16

Simplify

a) $(1 - \sqrt{3})^2$

b) $(\sqrt{3} - 1)^2 \quad (1 - \sqrt{3})^2$

c) $(2\sqrt{3} + 1)^2$

d) $(\sqrt{3} + \sqrt{2})^2$

e) $(-1 - \sqrt{5})^2$

e) $(3 - 3\sqrt{7})^2$

Note:

$(a + b)(c + d) = ac + ad + bc + bd$

$a(b + c) = ab + ac$

Worked example 1.26

Simplify $(3 - \sqrt{2})(1 + \sqrt{3})$

Solution

$(3 - \sqrt{2})(1 + \sqrt{3})$

$= 3 \times 1 + 3\sqrt{3} - 1\sqrt{2} - \sqrt{2}.\sqrt{3}$

$= 3 + 3\sqrt{3} - \sqrt{2} - \sqrt{6}$

Worked example 1.27

Simplify $2(1 + \sqrt{3})$

Solution

$2(1 + \sqrt{3})$

$= 2 + 2\sqrt{3}$

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EXERCISE 1.17

Simplify.

a) $\sqrt{3}(5 + \sqrt{3})$

b) $\sqrt{2}(\sqrt{2} - 7\sqrt{3})$

c) $-5(1 - \sqrt{125})$

d) $\sqrt{6}(\sqrt{2} + 1)$

e) $2\sqrt{7}(1 - 2\sqrt{7})$

f) $9(\sqrt{27} + 1)$

Dividing simple surds

We can also use grouping for division. For example, $6\sqrt{10} \div 3\sqrt{2}$ can be written as a fraction, thus $6\sqrt{10} \div 3\sqrt{2} = \frac{6}{3} \times \frac{\sqrt{10}}{\sqrt{2}} = 2 \times \sqrt{5} = 2\sqrt{5}$.

We can also first simplify the surds that we are given. Simplifying surds can help us to solve the expressions quicker. So, for example, $9\sqrt{28} \div 3\sqrt{7}$ can be solved by grouping straight away, but let's try and simplify the surds first. $\sqrt{28}$ can be simplified to $2\sqrt{7}$, but $\sqrt{7}$ cannot be simplified.

Our question then becomes $9 \times 2\sqrt{7} \div 3\sqrt{7} = 18\sqrt{7} \div 3\sqrt{7} = 6 \div 1 = 6$ (the $\sqrt{7}$ cancels out!).

Worked example 1.28

Simplify $\frac{\sqrt{21}}{3\sqrt{3}}$.

Solution

$$\frac{\sqrt{21}}{3\sqrt{3}}$$

$$= \frac{\sqrt{3 \times 7}}{3\sqrt{3}}$$

$$= \frac{\sqrt{3} \times \sqrt{7}}{3\sqrt{3}}$$

$$= \frac{\sqrt{7}}{3}$$

EXERCISE 1.18

Simplify where necessary:

a) $\frac{3\sqrt{2}}{\sqrt{6}}$

b) $\frac{\sqrt{15} + \sqrt{3}}{\sqrt{3}}$

c) $\frac{\sqrt{27}}{3}$

d) $\frac{(\sqrt{3} - 1)(\sqrt{3} + 1)}{\sqrt{3}}$

e) $\frac{\sqrt{3}}{\sqrt{2}} \div \sqrt{15}$

f) $6\frac{\sqrt{7}}{\sqrt{14}}$

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CONSOLIDATION EXERCISE

1. Simplify.
   a) $81^{\frac{1}{4}}$
   b) $81^{\frac{1}{2}}$
   c) $81^{\frac{3}{4}}$
   d) $\left(\frac{5}{6}\right)^{3}$
   e) $\left(\frac{125}{216}\right)^{\frac{1}{3}}$
   f) $\left(\frac{16}{81}\right)^{\frac{3}{4}}$

2. Solve for x.
   a) $2^{x+1} = 8$
   b) $\left(\frac{2}{3}\right)^{x} = \frac{8}{27}$
   c) $3^{3x} = 27$

3. State whether the numbers given below are surds or not. Explain your answer.
   a) $\frac{4\sqrt{16}}{8}$
   b) $5\sqrt{3}$
   c) $3 + \sqrt[3]{64}$

4. Simplify.
   a) $\sqrt{\frac{45}{25}}$
   b) $3\sqrt[4]{16}$
   c) $(2\sqrt{5})^{2}$

5. What is the side length, in simplest form, of a square with the given area? Leave the answer in surd form where necessary.
   a) 45 m^2
   b) 112 cm^2
   c) 552 cm^2

6. Which of the following pairs are like terms? Give an explanation why you say so.
   a) $\sqrt{75}$ and $\sqrt{25}$
   b) $3\sqrt{72}$ and $\sqrt{378}$
   c) $-3\sqrt{363}$ and $7\sqrt{847}$

7. Simplify.
   a) $\sqrt{7}(1 + 2\sqrt{7})$
   b) $\sqrt{15} \times \sqrt{3}$
   c) $(1 - \sqrt{11})(1 + \sqrt{11})$

8. Simplify.
   a) $\sqrt{2} + 3\sqrt{8} - 4\sqrt{12} + 10\sqrt{3}$
   b) $1 + \sqrt[3]{125} + \sqrt{363}$
   c) $\sqrt{20} - 2\sqrt{32} + 8\sqrt{18}$

9. Simplify.
   a) $\frac{\sqrt{108}}{3}$
   b) $\frac{\sqrt{75}}{5\sqrt{3}}$
   c) $\left(\frac{16}{81}\right)^{\frac{3}{4}} \times \frac{\sqrt{15}}{2\sqrt{5}}$

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Summary

- For any value of a, except zero, $a^0 = 1$, for example, $(-101)^0 = 1$
- For any number a, $a^{-n} = \frac{1}{a^n}$, for example, $5^{-2} = \frac{1}{5^2} = \frac{1}{25}$
- Exponents can be fractions, in general:
  - $a^{\frac{m}{n}} = \sqrt[n]{a}$, for example, $a^{\frac{5}{3}} = \sqrt[3]{a^5}$;
  - $a^{\frac{1}{n}} = \sqrt[n]{a}$, for example, $a^{\frac{1}{2}} = \sqrt{a}$ or $100^{\frac{1}{2}} = \sqrt{100} = 10$;
  - $a^{\frac{1}{3}} = \sqrt[3]{a}$, for example, $a^{\frac{1}{3}} = \sqrt[3]{a}$ or $125^{\frac{1}{3}} = \sqrt[3]{125} = 5$
- If $ax = ay$ then $x = y$, for example:
  - Solve for x: $3^x = 81$

$$3^x = 3^4$$
$$x = 4$$

- A number written exactly using square roots is called a surd.
- The two laws below can be used to simplify surds:
  - $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
  - $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- When simplifying surds, one of the highest factors should be a perfect square, for example:
  - Simplify $\sqrt{20}$

$$\begin{aligned}\sqrt{20} &= \sqrt{4 \times 5}\\ &= \sqrt{4} \times \sqrt{5}\\ &= 2\sqrt{5}\end{aligned}$$

  - Simplify $\frac{4\sqrt{27}}{\sqrt{3}}$

$$\begin{aligned}\frac{4\sqrt{27}}{\sqrt{3}} &= 4\sqrt{\frac{27}{3}} &\quad \text{or} \quad& \frac{4\sqrt{27}}{\sqrt{3}} = \frac{4\sqrt{9 \times 3}}{\sqrt{3}}\\ &= 4\sqrt{9} &\quad \text{or} \quad& = \frac{4\sqrt{9} \times \sqrt{3}}{\sqrt{3}}\\ &= 4 \times 3 &\quad \text{or} \quad& = 4\sqrt{9}\\ &= 12 &\quad \text{or} \quad& = 12\end{aligned}$$

- Add or subtract like surds only, for example:
  - $2\sqrt{3} + 8\sqrt{3} = 10\sqrt{3}$
  - $-2\sqrt{5} + \sqrt{7} + 2\sqrt{5} = \sqrt{7}$
  - $3\sqrt{8} - 4\sqrt{2} = 3\sqrt{4 \times 2} - 4\sqrt{2}$

$$\begin{aligned}&= 3\sqrt{4} \times \sqrt{2} - 4\sqrt{2}\\ &= 6\sqrt{2} - 4\sqrt{2}\\ &= 2\sqrt{2}\end{aligned}$$

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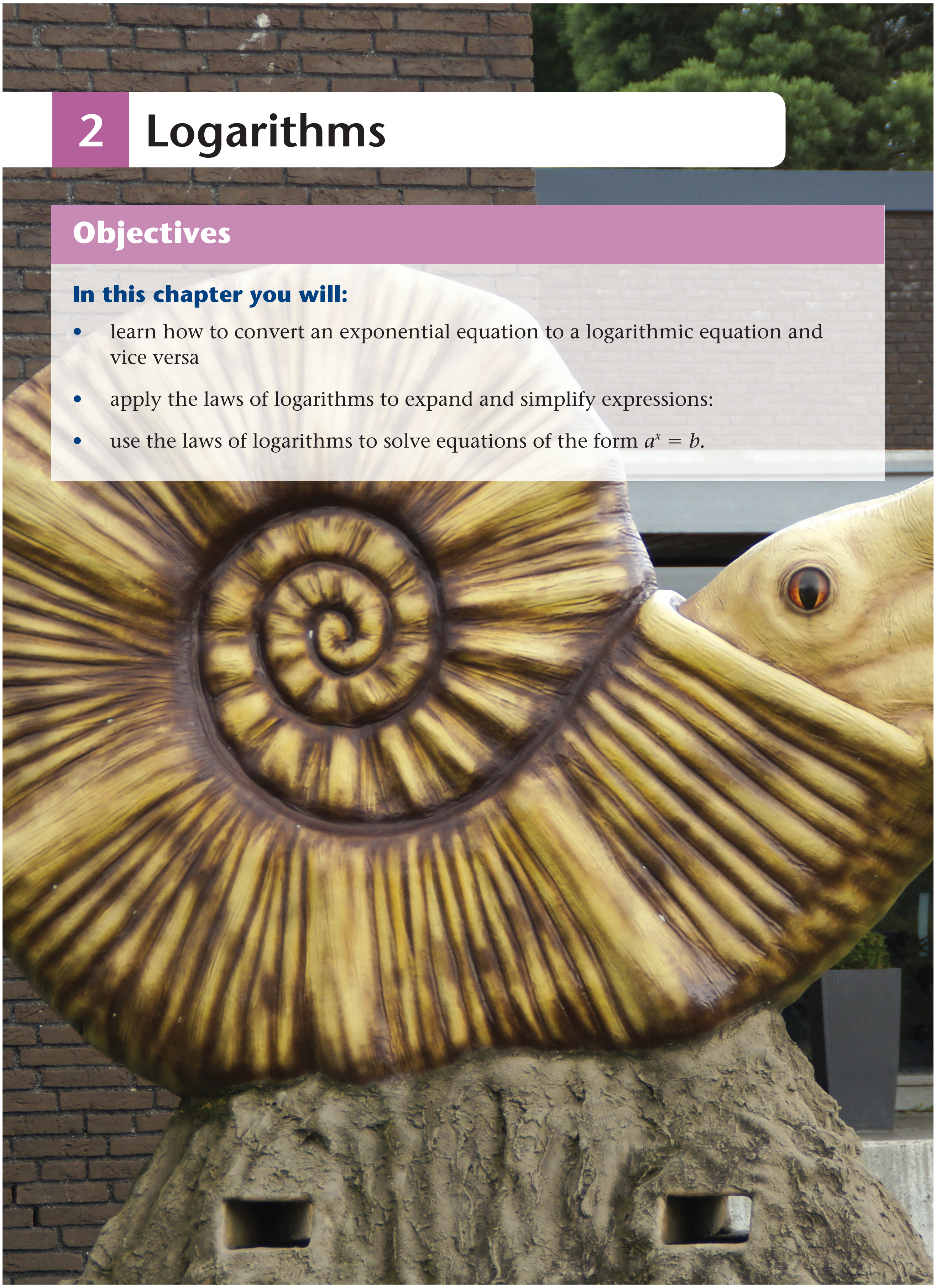
2 Logarithms

Objectives

In this chapter you will:

- learn how to convert an exponential equation to a logarithmic equation and vice versa
- apply the laws of logarithms to expand and simplify expressions:
- use the laws of logarithms to solve equations of the form $a^x = b$.



A nautilus shell with a logarithmic spiral

2.1 The relationship between logarithms and exponents

A logarithm is the exponent to which a number must be raised to get another number. For example, the base 10 logarithm of 100 is 2, because $\log_{10}100 = 2$ or $10^2 = 100$. We call it a base 10 logarithm because ten is the number that is raised to an exponent. So:

Exponential form	Log form
$10^2 = 100$	$\text{Log}_{10}100 = 2$

Note:
The logarithm of the number 100 to base 10 is 2 because $100 = 10^2$. We write the statement 'the logarithm of the number 100 to base 10 is 2'. Using mathematical notation as $\log_{10}100 = 2$

We can use different base units when working with logarithms. For example, you could use two as a base unit. For instance, the base two logarithm of eight is three, because $2^3 = 8$, and $\log_2 8 = 3$.

In general, we write 'log' followed by the base number as a subscript.

Work through the following exercise, which considers the numbers given in the table below.

EXERCISE 2.1

1. Complete the table below.

	Number	The exponent/index to which 10 must be raised to give the number	Number written as a power of base 10
Example	1	0	100
Example	0,1	–1	10^{-1}
a)	0,000001		
b)	0,00001		
c)	0,0001		
d)	0,001		
e)	0,01		
f)	10		
g)	100		
h)	1 000		
i)	10 000		
j)	1 00 000		
k)	1 000 000		

2. Write each of the following statements using mathematical notation:
   a) The logarithm of the number 1 000 to base 10 is 3.
   b) The logarithm of the number 1 000 000 to base 10 is 6.
   c) The logarithm of the number 10 to base 10 is 1.
   d) The logarithm of the number 1 to base 10 is 0.
   e) The logarithm of the number 10 000 to base 10 is 4.
   f) The logarithm of the number 0,01 to base 10 is –2.
   g) The logarithm of the number 0,0001 to base 10 is –4.
   h) The logarithm of the number 0,1 to base 10 is –1.

Note:
Logarithms to base 10 are commonly used perhaps because we use a decimal (base 10) system of counting.

However, bases other than 10 are also used in many calculations in science, engineering and related fields.

3. What is the logarithm of:
   a) 32 to base 2?
   b) 64 to base 2?
   c) 64 to base 4?
   d) 64 to base 8?
   e) 81 to base 3?
   f) 81 to base 9?
   g) 7 to base 49?
   h) 7 to base 7?
   i) 125 to base 5?
   j) 25 to base 5?
   k) 625 to base 5?
   l) 27 to base 3?

In general, the equation $a = b^x$ is equivalent to $\log_b a = x$. The only restriction we put on the base is that it must be a positive number excluding 1.

Worked example 2.1

Write the equation $8 = 2^3$ as a logarithmic equation.

Solution

Exponential equation	Logarithmic equation
$8 = 2^3$	$\text{Log}_2 8 = 3$

EXERCISE 2.2

Copy the table below in your exercise book. Write the equivalent form of the given equation in the appropriate space.

	Exponential equation		Logarithmic equation
	$729 = 9^3$	a)	
	$512 = 8^3$	b)	
c)			$\log_4 1024 = 5$
d)			$\log_6 216 = 3$
	$1 = 9^0$	e)	
f)			$\log_{12} 1 = 0$
	$3^5 = 243$	g)	
h)			$2 = \log_{25} 625$
	$12^2 = 144$	i)	
j)			$2 = \log_{17} 289$
k)			$1 = \log_{100} 100$

Consider the number $\frac{1}{100}$.

We can write this number in different equivalent forms as shown in the table below:

$\frac{1}{100} =$		
$0{,}01$	$\frac{1}{10^2}$	10^{-2}

Using mathematical notation, we write: $\frac{1}{100} = 10^{-2}$, which is equivalent to $\log_{10}\frac{1}{100} = -2$ or $0{,}01 = \frac{1}{10^2} = 10^{-2}$, which is equivalent to $\log_{10} 0{,}01 = -2$.

EXERCISE 2.3

1. Copy the table below in your exercise book. Write the equivalent form of the given equation in the appropriate space.

	Exponential form		Logarithmic form
	$\frac{1}{729} = 9^{-3}$	a)	
	$\frac{1}{512} = 8^{-3}$	b)	
c)			$\log_4 \frac{1}{1024} = -5$
d)			$\log_6 \frac{1}{216} = -3$

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Exponential form			Logarithmic form
	$\frac{1}{9} = 3^{-2}$	e)	
f)			$\log_{12}\frac{1}{12} = -1$
	$3^{-5} = \frac{1}{243}$	g)	
h)			$-2 = \log_{25}\frac{1}{625}$
	$12^{-2} = \frac{1}{144}$	i)	
j)			$-2 = \log_{17}\frac{1}{289}$
	$0{,}1 = \frac{1}{10} = 10^{-1}$	k)	
l)			$\log 100{,}001 = -3$
	$0{,}000 = 10^{-4}$	m)	

2. Rewrite the equations given below in exponential form:
   a) $\log_3 x = 81$
   b) $\log_2 128 = x$
   c) $\log_4 x = 4$
   d) $\log_2 y = 10$
   e) $\log_{10} y = 2$
   f) $\log_4 4 = x$
3. Solve for the unknown in each of the cases in question 2 a) to f).
4. Solve the following by inspection:
   a) $y = 4^3$
   b) $625 = 5^x$
   c) $216 = x^3$
   d) $a = 13^2$
   e) $14^x = 196$
   f) $1\,000 = x^3$
5. Write the equations in question 4 a) to f) in logarithmic form.
6. Solve the unknown value:
   a) $\log_4 64 = y$
   b) $\log_5 625 = x$
   c) $\log_x 216 = 3$
   d) $\log_{13} a = 2$
   e) $\log_{14} 196 = x$
   f) $\log_x 1\,000 = 3$

2.2 Laws of logarithms

The laws of logarithms allow expressions involving logarithms to be rewritten in a variety of different ways. The laws apply to logarithms of any base, but the same base must be used throughout a calculation.

There are basic laws that we will work with. These are the:

- logarithm of a product of numbers: $\log_a xy = \log_a x + \log_a y$
- logarithm of a quotient number: $\log_a \frac{x}{y} = \log_a x - \log_a y$
- logarithm of a power: $\log_a x^n = n\log_a x$
- base of a logarithm rule: $\log_a b = \frac{\log_c b}{\log_c a}$

Note:
The logarithm of 1 to any base is always 0, and the logarithm of a number to the same base is always 1. In particular, $\log_{10} 10 = 1$, and $\log_e e = 1$

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The logarithm of a product of numbers: $\log_a xy = \log_a x + \log_a y$

When exponential expressions with the same base are multiplied, the result is an exponential expression with the same base and an exponent equal to the sum of the exponents. That is, $a^m \times a^n = a^{m+n}$ or $a^m a^n = a^{m+n}$

In section 2.1 we described an exponent as a logarithm. Thus, when we use laws governing how we operate on logarithms, we are actually thinking about what we are supposed to do with the exponents when we have either a product or a quotient.

Consider the expression: $\log_2(8 \times 64)$.

According to the property $a^m \times a^n = a^{m+n}$, when we multiply two numbers that can be expressed in terms of the same base, we add the exponent.

Thus, when applying this rule to logarithms and using mathematical notation, we write:

$$\begin{aligned}\log_2(8 \times 64) &= \log_2 8 + \log_2 64\\ &= 3 + 6\\ &= 9\end{aligned}$$

In general, for any positive real numbers x, and y; and a not equal to 1; $\log_a xy = \log_a x + \log_a y$.

Note:
$8 = 2^3$ and $64 = 2^6$

Worked example 2.2

Write $\log_x abc$ in expanded form.

Solution

$\log_x abc$

$= \log_x a + \log_x b + \log_x c$

Worked example 2.3

Evaluate $\log_6(216 \times 36)$ without using the calculator.

Solution

$\log_6(216 \times 36)$

$= \log_6 216 + \log_6 36$

$= \log_6 6^3 + \log_6 6^2$

$= 3\log_6 6 + 2\log_6 6$

$= 3 + 2$

$= 5$

EXERCISE 2.4

Expand and evaluate (where appropriate) each of the following expressions without using a calculator:

a) $\log_3(729 \times 9)$

b) $\log_5(625 \times 125)$

c) $\log_2 8st$

d) $\log_4(64 \times 256)$

e) $\log_2 100xyz$

f) $\log_2 25x$

The logarithm of a quotient number: $\log_a \frac{x}{y} = \log_a x - \log_a y$

Another basic property (law) of exponents states that when exponential expressions with the same base are divided, the result is an exponential expression with the same base and an exponent equal to the difference of the numerator and the denominator, that is: $\frac{a^m}{a^n} = a^{m-n}$ or $a^m \div a^n = a^{m-n}$.

Let's apply this to logarithms.

Consider the expression: $\log_2 \frac{8}{64}$.

According to the property $\frac{a^m}{a^n} = a^{m-n}$, when we divide two numbers that can be expressed in terms of the same base, we subtract the exponent of the number 64 (expressed as 2^6) from the exponent of the number 8 (expressed as 2^3).

Thus, when applying this rule to logarithms and using mathematical notation, we write:

$\log_2 \frac{8}{64}$

$= \log_2 8 - \log_2 64$

$= \log_2 2^3 - \log_2 2^6$

$= 3 - 6$

$= -3$

Note:
$8 = 2^3$ and $64 = 2^6$

Worked example 2.4

Rewrite $\log_5 \frac{25}{125}$ using the quotient property of logarithms and simplify where possible.

Solution

$\log_5 \frac{25}{125}$

$= \log_5 25 - \log_5 125$

$= \log_5 5^2 - \log_5 5^3$

$= 2 - 3$

$= -1$

EXERCISE 2.5

Simplify

a) $\log_9 \frac{729}{81}$

b) $\log_4 \frac{64}{256}$

c) $\log_5 \frac{625}{25}$

d) $\log_3 \frac{k}{27}$

e) $\log \frac{1\,000}{z}$

f) $\log_7 \frac{A}{B}$

The logarithm of a power: $\log_a x^n = n\log_a x$

The logarithm of x raised to the power of n is n times the logarithm of x. In general, we state that $\log_a x^n = n\log_a x$; where a and x are positive real numbers excluding 1 and $n \in \mathbb{R}$.

Worked example 2.5

Simplify $\log_{10}100^x$ using appropriate log properties and simplify where possible.

Solution

$\log_{10}100^x$

$= x\log_{10}100$

$= x\log_{10}10^2$

$= x \times 2\log_{10}10$

$= x \times 2$

$= 2x$

EXERCISE 2.6

1. Simplify the following without the use of a calculator:

   a) $\log_5 125$

   b) $\log_3 81$

   c) $\log_{10}10^5$

   d) $\log_2 128$

   e) $\log_8 512$

   f) $\log_9 59\,049$

2. Use your calculator to complete the table below. For example: $\log 2 = 0{,}301$ and $\log 8 = 0{,}903$.

x	1	2	3	4	5	6	7	8	9	10
$\log x$		0,301						0,903		

3. Use the results from exercises 1 and 2 to answer the following questions without using a calculator.

   a) $3\log 7$

   b) $6\log 10$

   c) $2\log 3$

   d) $3\log 2$

   e) $4\log 5$

   f) $3\log 9$

4. Now use your calculator to check whether the statements given below are true or false.

   a) $3\log 7 = \log 343$

   b) $6\log 10 = \log 1\,000\,000$

   c) $2\log 3 = \log 8$

   d) $3\log 2 = \log 9$

   e) $4\log 5 = \log 625$

   f) $3\log 9 = \log 729$

   g) $\log 2 = 3{,}279$

   h) $\log \frac{1}{2} = 0{,}03$

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EXERCISE 2.7

(Do not use a calculator for this exercise.)

1. Simplify each of the following using appropriate log properties:

   a) $\log_5 625^{10}$
   b) $\log_x x^{1\,000}$
   c) $\log_5 x^{-3}$
   d) $\log_x y^z$
   e) $\log_a a^2$
   f) $\log_4 \left(\frac{1}{4}\right)^x$

Worked example 2.6

Simplify the expression $\frac{\log 729}{\log 81}$ without the aid of a calculator.

Solution

$\frac{\log 729}{\log 81}$

$= \frac{\log 9^3}{\log 9^2}$

$= \frac{3\log 9}{2\log 9}$

$= \frac{3}{2}$

$= 1\frac{1}{2}$

2. Simplify without the aid of a calculator:

   a) $\frac{\log 2}{\log 64}$
   b) $\frac{\log 32}{\log 128}$
   c) $\frac{\log 343}{\log 49}$
   d) $\frac{\log 216}{\log 36}$
   e) $\frac{\log 625}{\log 25}$
   f) $\frac{\log 512}{\log 4}$

3. Simplify the log expressions given below:

   a) $\frac{\log a^3}{\log a}$
   b) $\frac{\log a^3}{3a}$
   c) $\frac{2\log xy}{\log x^2 y}$
   d) $\frac{\log 8abc}{3\log 2abc}$
   e) $\frac{\log 27}{3a}$
   f) $\frac{b\log_5 625}{b}$

Worked example 2.7

Write $3\log a - \log b + 4\log c$ as a single logarithm.

Solution

$3\log a - \log b + 4\log c$

$= \log a^3 - \log b + \log c^4$

$= \log \frac{a^3 c^4}{b}$

Worked example 2.8

Write $7\log x + 4\log y$ as a single logarithm.

Solution

$7\log x + 4\log y$

$= \log x^7 + \log y^4$

$= \log x^7 y^4$

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4. Simplify (without the use of a calculator).

a) $\log a^2 - \log 3 + \log 10$

b) $5\log x + 3\log y - \log w$

c) $\frac{1}{2}\log a - \frac{2}{3}\log 27$

d) $\frac{\log 4 + \log 25}{\log 0{,}001}$

e) $\log 6 + 2\log 20 - \log 3 - 3\log 2$

f) $\log_x 64 + \log_x 4 - \log_x 8$

g) $\frac{\log abc}{2\log a + \log b + \log c}$

h) $\log 2 + \log 3$

i) $\log a - \log 27$

The base of a logarithm rule: $\log_a b = \frac{\log_c b}{\log_c a}$

By convention, when the base of a logarithm is not indicated, it is assumed to be 10. Logarithms to base 10 are called common logarithms, for example, the base for each of the following logarithms is 10: log1 000, log8, log$a$, log2.

As we learnt earlier, bases other than 10 are needed for applications in science, engineering and related fields.

Note:
The log button on your calculator is used to calculate values

The rule $\log_a b = \frac{\log_c b}{\log_c a}$ allows us to rewrite a logarithm to any base other than 10. This rule is useful when we have to use the calculator to calculate the value of a logarithm of a number to a base other than 10.

Worked example 2.9

Change $\log_2 5$ to base 10.

Solution

$\log_2 5$

$= \frac{\log 5}{\log 2}$

Worked example 2.10

Change $\log_3 6$ to base 10.

Solution

$\log_3 6$

$= \frac{\log 6}{\log 3}$

EXERCISE 2.8

1. Change to base 10.

a) $\log_{11} 10$

b) $\log_{12} 11$

c) $\log_{12} 6$

d) $\log_{14} 7$

e) $\log_{11} 12$

f) $\log_{32} 64$

g) $\log_6 12$

h) $\log_4 9$

i) $\log_{25} 125$

2. Use the calculator to evaluate the following correct to 2 decimal places.

a) $\log_{11} 10$

b) $\log_{12} 11$

c) $\log_{12} 6$

d) $\log_{14} 7$

e) $\log_{11} 12$

f) $\log_{32} 64$

g) $\log_6 12$

h) $\log_4 9$

i) $\log_{25} 125$

2.3 Solving exponential equations of the form: $a^x = b$

In previous grades you solved equations similar to these: (1) $2x - 3 = 8$ and (2) $2^x = 8$.

The strategy we use to solve $2^x = 8$ is to rewrite the equation such that both sides of the equation have the same base, that is, $2^x = 2^3$. We then ask: For what value(s) of x are the two expressions equal? Since the bases are the same we can focus on the exponents. We conclude that the two sides can only be equal when the bases are raised to the same exponent, that is, when $x = 3$.

So then how do we solve an equation like $2^x = 11$?

When considering this equation, we see that it is not possible to write one number in terms of the base of the other. We have already discussed:

- that we can think of an exponent as a logarithm
- how to convert between the exponential and logarithmic forms of an equation
- how to change the base of a logarithm

We will use the above knowledge to solve the given equation.

Let's work through an example.

Worked example 2.11

Solve for x in the equation $5^x = 3$. Give your answer correct to three decimal places.

Solution

$5^x = 3$

$\log_5 3 = x$ [Rewrite the given equation in a logarithmic form]

$\frac{\log 3}{\log 5} = x$ [Change the base of the logarithm to base 10]

$0{,}683 = x$ [Calculator]

$5^{0{,}683} = 3$ [Check]

EXERCISE 2.9

Solve for x correct to two decimal places:

a) $2^x = 11$
b) $3^x = 2$
c) $3^x = 5$
d) $7^x = 5$
e) $5^x = 27$
f) $2^x = 12$
g) $10^x = 99$
h) $4^x = 52$
i) $6^x = 200$
j) $8^x = 90$
k) $9^x = 100$
l) $13^x = 170$

2.4 Solving more equations involving logarithms

Work through the examples below and use the knowledge gained in previous grades and this chapter to solve equations involving logarithms.

Worked example 2.12

Solve for x: $\log_2 8 = x$; where $x \in \mathbb{R}$.

Solution

$\log_2 8 = x$

$2^x = 8$

$2^x = 2^3$

$x = 3$

Worked example 2.13

Solve for x: $2\log x + 2 = \log 900$; where $x \in \mathbb{R}$.

Solution

$2\log x + 2 = \log 900$

$\log x^2 + 2 = \log 9 + \log 100$

$\log x^2 - \log 9 = \log 100 - 2$

$\log \frac{x^2}{9} = 2 - 2$

$\log \frac{x^2}{9} = 0$

$\frac{x^2}{9} = 10^0$

$x^2 = 9$

$x = 3$

Check

LHS: $2\log 3 + 2 = \log 3^2 + \log 100$

$= \log(3^2 \times 100)$

$= \log 900$

LHS $=$ RHS

$\therefore x = 3$ is the solution

Worked example 2.14

Solve for x: $\log_2(x + 2) + \log_2(x - 4) = 3$; where $x \in \mathbb{R}$.

Solution

$\log_2(x + 2)(x - 4) = 3$

$(x + 2)(x - 4) = 8$

$x^2 - 2x - 8 = 8$

$x^2 - 2x - 16 = 0$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-16)}}{2(1)}$$

$$x = \frac{2 + \sqrt{68}}{2} \text{ or } \frac{2 - \sqrt{68}}{2}$$

$x = 5{,}1$ or $x = 3{,}1$

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-16)}}{2(1)}$$

$$x = \frac{2 + \sqrt{68}}{2} \text{ or } \frac{2 + \sqrt{68}}{2}$$

$x = 5{,}1$ or $-3{,}1$

The valid value of x is 5,1

Check

$\log_2(5{,}1 + 2) + \log_2(5{,}1 - 4)$

$= \log_2(7{,}1)(1{,}1)$

$= \log_2(7{,}81)$

$= 2{,}97 \approx 3$

EXERCISE 2.10

1. Solve forx: $x \in \mathbb{R}$

a) $\log_3 27 = x$

b) $\log_x 16 = 4$

c) $\log_5 x = 3$

d) $\log_3 \frac{1}{27} = x$

e) $\log_2(x + 1) = 3$

f) $\log x = \log(2 - x)$

g) $2\log x = 4$

h) $\log_2 3x = \log_2 300$

i) $\log x + 3 = \log 1\,000$

j) $\frac{1}{2}\log x = \log 10$

k) $4\log_2 x - 1 = \log_2 8$

l) $\log_2 \frac{1}{8} = x$

Worked example 2.15

Solve for x; where $x \in \mathbb{Z}$.

$\log_2 x + \log_2 3 = 3$

Solution

$\log_2 x + \log_2 3 = 3$

$\log_2 3x = 3$

$3x = 2^3$

$x = \frac{8}{3}$

$= 2\frac{2}{3}$

Check

LHS $= \log_2 \frac{8}{3} + \log_2 3$

$= \log_2 8 - \log_2 3 - \log_2 3$

$= \log_2 8$

$= 3$

$x = \frac{8}{3}$ is the solution

Worked example 2.16

Solve for x; where $x \in \mathbb{R}$.

$2\log_c x = \log_c 4 + \log_c(x - 1)$

Solution

$\log_c x^2 = \log_c 4(x - 1)$

$\log_c x^2 = \log_c(4x - 4)$

$\therefore x^2 = 4x - 4$

$x^2 - 4x + 4 = 0$

$(x - 2)(x - 2) = 0$

$\therefore x = 2$

Check

$2\log_c 2 = \log_c 4 + \log_c(2 - 1)$

$\log_c 2^2 = \log_c(4(2) - 4)$

$\therefore 2^2 = 4(2) - 4$

$\therefore 4 = 4$ LHS = RHS

2. Solve for x; where $x \in \mathbb{R}$.
   a) $\log_7 x + \log_7 7 = \log_7 14$
   b) $\log_3(x - 8) + \log_3 x = 2$
   c) $\log_3 x - \log_3 3 = 2$
   d) $\log_2(x - 1) = -\log_2 x + 2$
   e) $\log x + 1 = 0$
   f) $\log_x 3 + \log_3 x = 2$
   g) $\log_x 2 - 2 = -\log_2 x$
   h) $\log_4(x) - \log_4(x - 1) = \frac{1}{2}$
   i) $\log(x - 3) + \log(x - 2) = \log(2x + 24)$
3. Prove that:
   a) $\frac{\log_a 25 - \log_a 125}{2[\log_a 5^4 - \log_a 5^6]} = \frac{1}{4}$
   b) $\log_9 81 + \log_9 1 + \log_2 16 - \log_{25} 0{,}04 = 5$
   c) $\log_8 \frac{1}{8} + \log_{49} 7^{\frac{1}{2}} - \log_6\left(\frac{1}{216}\right) - \log_a 1 = \frac{9}{4}$
4. Suppose that a rapidly growing bacteria culture contains $1\,000 \times 2^{0{,}05t}$ bacteria after t minutes.
   a) How many bacteria will be present after 20 minutes?
   b) When will the size of the culture reach 7 000?
5. One thousand Rand deposited into a savings account grows to $1\,000 \times 2^{0{,}2t}$ rand after t years.
   a) What will the balance be after 5 years?
   b) When will the balance reach R7 000?
6. The world population is currently 4 billion and will be $4 \times 10^{0{,}0087t}$ after t years. When will the population reach 6 billion?

CONSOLIDATION EXERCISE

1. Calculate the value of each of the following:
   a) $\log_3 27 + \log_2 64$
   b) $\dfrac{\log_3 27}{\log_2 64}$
   c) $\log_3 27 - \log_2 64$
   d) $\log_3 27 \times \log_2 64$
   e) $\log_2 32 + \log_4 64 - \log 100 \times \log_5 125 + \log_2 1$
2. Solve for x; where $x \in \mathbb{R}$.
   a) $2^x = 17$
   b) $7^x = 100$
   c) $10^x = 100$
3. Solve for x; where $x \in \mathbb{R}$.
   a) $3^x = 729$
   b) $5^x - 1 = 5^3$
   c) $4^{x+1} = 256$
4. Solve for x; where $x \in \mathbb{R}$.
   a) $\log_x 32 = 5$
   b) $\log_2 x = 5$
   c) $\log_2 32 = x$
   d) $\log(x + 1) = 0$
   e) $\log(x - 1) = 0$
   f) $\log x + \log(x - 1) = \log 100$
5. Calculate the value of:
   a) $\log_5 6$
   b) $\log_3 11$
   c) $\log_8 9$
6. Write each of the following expressions in expanded form:
   a) $\log \dfrac{xyz^3}{w^2}$
   b) $\log \dfrac{\sqrt{ab}}{\sqrt[3]{c}}$ $\log \dfrac{\sqrt{ab}}{\sqrt[3]{c}}$
   c) $\log 10x$
7. Write each expression as a single logarithm:
   a) $\log(x + 1) - 3 + \log(x - 1)$
   b) $\log x + 1$
   c) $2 - \log_5 3x$

Summary

- the number $\log_a y$ is the exponent that a must be raised to, in order to get the number y.
- $\log_a y = x$ implies and is implied by $ax = y$
- we read $\log_a y = x$ as 'the logarithm of y to base a' is equal to x.
- the equation $\log_a y = x$ is called the logarithmic form of the equation.
- the equation $a^x = y$ is called the exponential form of the equation.
- in the expression $\log_a y$, the number a is called the base of the logarithm. The base of a logarithm must always be positive and not equal to 1.
- in the expression $\log_a y$, y is the input and must always be a positive number.
- the laws of logarithms states:
  - $\log_a xy = \log_a x + \log_a y$
  - $\log_a \frac{x}{y} = \log_a x - \log_a y$
  - $\log_a x^y = y\log_a x$
- for the above laws to apply: $a > 0$; $a \neq 1$; $x > 0$ and $y > 0$.
- the change of base formula for logarithms states: $\log_a y = \frac{\log_b y}{\log_b a}$
- the change of base formula helps us change from any base a to any base b.
- for the change of base formula to apply, both a and b must be positive numbers not equal to 1 and y must always be a positive number.

3 Equations

Objectives

In this chapter you will:

- solve quadratic equations using:
  - factorisation
  - the quadratic formula
- solve quadratic inequalities by interpreting solutions graphically
- determine the nature of roots and determine the conditions for which roots are real, non-real, equal, unequal, rational and irrational
- solve equations in two unknowns, in which one is linear and the other quadratic
- solve word problems
- manipulate formulae (making a variable the subject of the formula).

A skateboarding ramp rises in height and we are able to determine the slope using equations.

3.1 Revising factorisation of quadratic equations

A quadratic equation is a polynomial of the form $ax^2 \pm bx \pm c$, where a, b and c are numbers. When factoring, you will find the two numbers that will not only multiply to equal the constant term c, but will also add up to equal b, the coefficient of the x-term.

Note:
A polynomial consists of more than two terms.

Factorising expressions of the form $ax^2 \pm bx \pm c$

The polynomial $ax^2 \pm bx \pm$ c can be factorised as follows:

Factors will be:	If ab is:	If $a + b$ is:	If $a > b$:
$(x + a)(x + b)$	positive	positive	not relevant
$(x - a)(x - b)$	positive	negative	not relevant
$(x + a)(x - b)$	negative	positive	$a > b$ because $(a + b) > 0$
$(x - \mathrm{a})(x + b)$	negative	negative	$a < b$ because $(a + b) < 0$

Note:
If the sign of the last term is a '+', both signs are the same. To find out what these signs are, look at the middle sign. If the middle sign is '+' both signs will be '+'. If the middle sign is '–', both signs will be '–'.

If the sign of the last term is '–', the signs are different. If the middle sign is '+', the sign in the second bracket is '+'. If the middle sign is '–', the sign in the second bracket is '–'.

The smallest number always goes in the first column!

Worked example 3.1

Factorise: $x^2 - 3x + 2$

Solution

$x^2 - 3x + 2$

$= (x - 2)(x - 1)$

Worked example 3.2

Factorise: $x^2 + x - 2$

Solution

$x^2 + x - 2$

$= (x + 2)(x - 1)$

Worked example 3.3

Factorise: $x^2 - x - 2$

Solution

$x^2 - x - 2$

$= (x + 1)(x - 2)$

EXERCISE 3.1

1. Factorise.

a) $x^2 + 7x + 12$
b) $x^2 - 7x + 12$
c) $x^2 - x - 12$
d) $x^2 + x - 12$
e) $x^2 + 5x + 6$
f) $x^2 - 5x + 6$
g) $x^2 + x - 6$
h) $x^2 - x - 6$

2. Check your answers to question 1 a) to h) above by expanding the factors to test whether you get the original expression.

Factorising trinomials of the form $ax^2 \pm bx + c^2$

When factorising a trinomial of the form $ax^2 \pm bx \pm c^2$, in which the first and the last variables are perfect squares, put the square root of the first variable at the beginning of the brackets and the square root of the last variable at the end of the bracket.

Worked example 3.4

Factorise: $x^2 + 4ax + 4a^2$

Solution

$x^2 + 4ax + 4a^2$
$= (x + 2a)(x + 2a)$
$= (x + 2a)^2$

Worked example 3.5

Factorise: $4x^2 + 36x + 81$

Solution

$4x^2 + 36x + 81$
$= (2x + 9)(2x + 9)$
$= (2x + 9)^2$

EXERCISE 3.2

Factorise.

a) $x^2 + 14x + 49$
b) $x^2 - 14x + 49$
c) $x^2 - 6x + 9$
d) $x^2 + 6x + 9$
e) $p^2 + 2pq + p^2$
f) $p^2 + 8p + 16$
g) $p^2 + 10pq + 25q^2$
h) $r^2 - 12r + 36$

Factorising trinomials of the form $ax^2 + bx + c$

When factorising a trinomial of the form $ax^2 \pm bx \pm c$, start off by checking if there is a common factor. Once you have considered this, take out the common factor and include it in your final answer. Then factorise the rest of the trinomial to find the terms.

Worked example 3.6

Factorise: $3x^2 + 10x + 3$

Solution

$3x^2 + 10x + 3$
$= 3x^2 + 9x + x + 3$
$= 3x(x + 3) + x + 3$
$= (x + 3)(3x + 1)$

Worked example 3.7

Factorise: $2x^2 - 9x + 4$

Solution

$2x^2 - 9x + 4$
$= (2x^2 - 8x - x + 4)$
$= 2x(x - 4) - (x - 4)$
$= (x - 4)(2x - 1)$
$= (2x - 1)(x - 4)$

EXERCISE 3.3

Factorise the trinomials given below. Remember to check your answers.

a) $3m^2 - 5m - 12$
b) $6m^2 - 14m - 12$
c) $15b^2 - 11b + 2$
d) $5b^2 + 27b + 10$
e) $12s^2 - 13s - 35$
f) $24s^2 - 46s - 18$
g) $6r^2 + 23r + 7$
h) $20r^2 + 19r + 3$
i) $-5x^2 + 7x - 2$
j) $-3x^2 - x + 14$
k) $-x^2 - x + 2$
l) $-6x^2 + x + 1$

Factorising trinomials of the form $a^2 - b^2$

An expression of the form $a^2 - b^2$ is called the difference of two squares. These trinomial have factors in the form $(a + b)(a - b)$.

Worked example 3.8

Factorise: $x^2 - 9$

Solution

$x^2 - 9$
$= (x - 3)(x + 3)$

Worked example 3.9

Factorise: $4x^2 - 225$

Solution

$4x^2 - 225$
$= (2x)^2 - (15)^2$
$= (2x - 15)(2x + 15)$

EXERCISE 3.4

Factorise the expressions given below.

a) $x^2 - 1$
b) $x^2 - 81$
c) $a^2 - b^2$
d) $x^2 - 4y^2$
e) $9x^2 - 1$
f) $9x^2 - 9$
g) $200 - 2m^2$
h) $64m^2 - 25n^2q^2$
i) $p^2q^2r^2 - 1$
j) $9x^{10} - 4y^8$
k) $5a^4 - 20b^2$
m) $225b^2 - 169c^2$

3.2 Solving quadratic equations using factorisation

When we solve a quadratic equation, we are calculating the value(s) of the input that makes the quadratic equation a true mathematical statement.

There are three common techniques we use to solve quadratic equations:

- by means of factorisation
- by completing a square
- by using the quadratic formula

This year we will discuss only two techniques, namely, solving quadratic equations by means of:

- factorisation
- the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Worked example 3.10

Solve for x: $x^2 - 10x - 24 = 0$

Solution

Factors of -24 that add to -10 are $+2$ and -12.
$x^2 - 10x - 24 = 0$
$(x - 12)(x + 2) = 0$
Either $x - 12 = 0$ or $x + 2 = 0$
$\therefore x = 12$ or $x = -2$ are solutions

Check

$x = 12$

$x^2 - 10x - 24$

$= (12)^2 - 10(12) - 24$

$= 144 - 120 - 24$

$= 0$

Check

$x = -2$

$x^2 - 10x - 24$

$= (2)^2 - 10(-2) - 24$

$= 4 + 20 - 24$

$= 0$

EXERCISE 3.5

Solve for x:

a) $y^2 - 4 = 10$
b) $7t^2 + 14t = 0$
c) $12y^2 + 24y + 12 = 0$
d) $y^2 + 28y = 100$
e) $y^2 - 5y + 6 = 0$
f) $y^2 + 5y - 36 = 0$
g) $-y^2 - 11y - 24 = 0$
h) $13y - 42 = y^2$
i) $\frac{y - 2}{y + 1} = \frac{2y + 1}{y - 7}$

Worked example 3.11

Solve for x: $8x^2 + 22x + 15$; $x \in \mathbb{R}$

Solution

To solve the given quadratic equation, we first have to rewrite the quadratic expression to the left of the equal sign as a product of two linear factors $(ax + b)(cx + d)$.

Quadratic expression	Possible factors: $(ax + b)(cx + d)$
$8x^2 + 22x + 15$	$(2x + 3)(4x + 5) = 2x(4x + 5) + 3(4x + 5) = 8x^2 + 10x + 12x + 15$
	$(8x + 1)(x + 15) = 8x(x + 15) + 1(x + 15) = 8x^2 + 120x + x + 15$
	$(2x + 1)(4x + 15) = 2x(4x + 15) + 1(4x + 15) = 8x^2 + 30x + 4x + 15$
	$(8x + 3)(x + 5) = 8x(x + 5) + 3(x + 5) = 8x^2 + 40x + 3x + 15$

The only factors of $8x^2 + 22x + 15$ that give the middle term $22x = 10x + 12x$ are $(2x + 3)$ and $(4x + 5)$.

So we rewrite the given quadratic equation as a product of its linear factors as shown below:

$8x^2 + 22x + 15 = 0$

$(2x + 3)(4x + 5) = 0$

$2x + 3 = 0$ or $4x + 5 = 0$

$2x = -3$ or $4x = -5$

$x = -\frac{3}{2}$ or $x = -\frac{5}{4}$

$\quad = -1\frac{1}{2}$ or $x = -1\frac{1}{4}$

Check

$$\begin{aligned} 8x^2 + 22x + 15 &= 8\left(-\tfrac{3}{2}\right)^2 + 22\left(-\tfrac{3}{2}\right) + 15 \\ &= 8\left(\tfrac{9}{4}\right) - \tfrac{66}{2} + 15 \\ &= 18 - 33 + 15 \\ &= 0 \end{aligned}$$

or

$$\begin{aligned} 8x^2 + 22x + 15 &= 8\left(-\tfrac{5}{4}\right)^2 + 22\left(-\tfrac{5}{4}\right) + 15 \\ &= 8\left(\tfrac{25}{16}\right) - 22\left(-\tfrac{5}{4}\right) + 15 \\ &= \tfrac{25}{2} - \tfrac{110}{4} + 15 \\ &= \tfrac{50 - 110 + 60}{4} \\ &= \tfrac{0}{4} \end{aligned}$$

So both $x = -1\frac{1}{2}$ or $x = -1\frac{1}{4}$ are solutions

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Worked example 3.12

Solve for x: $6x^2 + 7x + 2$; where $x \in \mathbb{R}$.

Solution

If the equation can be factorised, then it must be possible to rewrite it as a product of its linear factors. Using trial-and-error we determine that the factors are:

$(2x + 1)(3x + 2) = 0$

Then we rewrite $6x^2 + 7x + 2 = 0$ as $(2x + 1)(3x + 2) = 0$

$2x + 1 = 0$ or $3x + 2 = 0$

$2x = -1$ or $3x = -2$

$x = -\frac{1}{2}$ or $x = -\frac{2}{3}$

Check

$$\begin{aligned} 6x^2 + 7x + 2 &= 6\left(-\tfrac{1}{2}\right)^2 + 7\left(-\tfrac{1}{2}\right) + 2 \\ &= 6\left(\tfrac{1}{4}\right) - \tfrac{7}{2} + 2 \\ &= \tfrac{3}{2} - \tfrac{7}{2} + 2 \\ &= \tfrac{3 - 7 + 4}{2} \\ &= \tfrac{0}{2} \\ &= 0 \end{aligned}$$

or

$$\begin{aligned} 6x^2 + 7x + 2 &= 6\left(-\tfrac{2}{3}\right)^2 + 7\left(-\tfrac{2}{3}\right) + 2 \\ &= 6\left(\tfrac{4}{9}\right) - \tfrac{14}{3} + 2 \\ &= \tfrac{8}{3} - \tfrac{14}{3} + 2 \\ &= \tfrac{8 - 14 + 6}{3} \\ &= \tfrac{0}{3} \\ &= 0 \end{aligned}$$

So both $x = -\frac{1}{2}$ or $x = -\frac{2}{3}$ are solutions.

EXERCISE 3.6

Solve the following quadratic equations using factorisation.

a) $x^2 - 8x + 7 = 0$

b) $x^2 + 11x + 18 = 0$

c) $2x^2 - x - 3 = 0$

d) $3x^2 + 13x + 4 = 0$

e) $14x^2 - 19x - 3 = 0$

f) $2x^2 + 5x - 3 = 0$

g) $3x^2 + 19x + 20 = 0$

h) $6x^2 - 7x + 2 = 0$

i) $6x^2 - 17x - 3 = 0$

j) $x^2 + 11x + 18 = 0$

3.3 Solving quadratic equations using the quadratic formula

The perfect square

The quadratic expression $a^2x^2 + 2axb + b^2$ has two identical binomial factors $(ax + b)$. This means that we can rewrite $a^2x^2 + 2axb + b^2$ as $(ax + b)(ax + b) = (ax + b)^2$. The expression $(ax + b)^2$ is called a perfect square.

A perfect square is a quadratic expression whose factors are identical binomials. We have already come across expressions that are perfect squares. Some examples of quadratic expressions that can be written as perfect squares are given below:

$x^2 + 8x + 4 = (x + 2)(x + 2) = (x + 2)^2$

$x^2 - 10x + 25 = (x - 5)(x - 5) = (x - 5)^2$

The idea of a perfect square is very useful in solving quadratic equations. If we can write a given quadratic equation as a perfect square, then solving it becomes easier because we simply take the square roots on both sides of the equation.

Worked example 3.13

Solve for x: $(2x - 1)^2 = 16$; $x \in \mathbb{R}$.

Solution

$\sqrt{(2x - 1)^2} = \pm\sqrt{16}$

$2x - 1 = \pm 4$

$2x - 1 = 4$ or $2x - 1 = -4$

$2x = 5$ or $2x = -3$

$x = \frac{5}{2}$ or $x = -\frac{3}{2}$

$x = 2\frac{1}{2}$ or $x = -1\frac{1}{2}$

Check

$\left[2\left(\frac{5}{2}\right) - 1\right]^2$	or $\left[2\left(-\frac{3}{2}\right) - 1\right]^2$
$= [5 - 1]^2$	$= [-3 - 1]^2$
$= [4]^2$	$= [-4]^2$
$= 16$	$= 16$

EXERCISE 3.7

a) $(2x + 1)^2 = 1$

b) $\left(x - \frac{1}{2}\right)^2 = 9$

c) $(x + 3)^2 = \frac{1}{4}$

d) $\left(\frac{3}{4}x - 7\right)^2 = 5$

e) $(x + 7)^2 = 49$

f) $(x + 5)^2 = 25$

The quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

One of the challenges we face when we use factorisation as a method of solving quadratic equations is that it does not always work. Some quadratic equations are not factorable. Thus we need a method that will always work.

Mathematicians came up with a method to help us solve quadratic equations that can be used to calculate the roots of any quadratic equation. This method uses an equation called the quadratic formula, which can be used to solve any quadratic equation. The quadratic formula is derived by completing the square on the quadratic equation $ax^2 + bx + c = 0$.

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Let's work through how the quadratic formula is derived.

Consider a quadratic equation in standard form: $ax^2 + bx + c = 0$; where a $\neq$ 0.

$$\frac{ax^2}{a} + \frac{bx}{a} + \frac{c}{a} = \frac{0}{a}$$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right) = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\sqrt{\left(x + \frac{b}{2a}\right)^2} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ or $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ are the roots or the solutions of the quadratic equation $ax^2 + bx + c = 0$.

Worked example 3.14

Solve for x using quadratic formula:
$2x^2 - x - 10 = 0$

Solution

$a = 2$; $b = -1$; $c = -10$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-10)}}{2(2)}$$

$$x = \frac{1 \pm \sqrt{81}}{4}$$

$$x = \frac{1 + \sqrt{81}}{4} \quad \text{or} \quad x = \frac{1 - \sqrt{81}}{4}$$

$$x = \frac{1 + 9}{4} \quad \text{or} \quad x = \frac{1 - 9}{4}$$

$$x = \frac{10}{4} \quad \text{or} \quad x = \frac{-8}{4}$$

$$x = 2\tfrac{1}{2} \quad \text{or} \quad x = -2$$

Worked example 3.15

Solve for x: $3x^2 - 10 = 2x$.
Give your answer correct to 2 decimal places.

Solution

Rewrite the quadratic equation in Standard Form:
$3x^2 - 2x - 10 = 0$

$a = 3$; $b = -2$; $c = -10$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-10)}}{2(3)}$$

$$x = \frac{2 \pm \sqrt{124}}{6}$$

$$x = \frac{2 + \sqrt{124}}{6} \quad \text{or} \quad x = \frac{2 - \sqrt{124}}{6}$$

Use your calculator to solve:
$x = 2{,}19$ or $x = -1{,}52$

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EXERCISE 3.8

Solve for x.

Note:
Answers must be given correct to two decimal places where necessary.

a) $2x^2 - 2x - 1 = 0$
b) $x^2 - 2x = 2(x - 1)$
c) $-3x^2 + 3x + 1 = 0$
d) $6x^2 - 5x - 2 = 0$
e) $x^2 - 4x = 3$
f) $x^2 + 1 = 6x$
g) $x^2 - 4x - 21 = 0$
h) $x^2 - 3x = 0$
i) $x^2 + 6x = 10$

3.4 The discriminant: $\Delta = b^2 - 4ac$

The discriminant, $\Delta = b^2 - 4ac$, is derived from the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

By calculating the value of the discriminant we can tell: whether the roots of the equation we are solving are real or non-real (imaginary).

But before we continue with our discussion on the discriminant, let's do the exercise below.

EXERCISE 3.9

Copy and complete the table below in your exercise book. Make a cross for each category that applies to each number.

		Nature of the number					
	Number	Real	Non-Real	Rational	Irrational	Integer	Natural
a)	0						
b)	$\sqrt{3}$						
c)	$-0,7$						
d)	$1,234$						
e)	-2						
f)	π						
g)	$-\frac{\sqrt{3}}{2}$						
h)	$-\frac{3}{2}$						
i)	$\sqrt{16}$						
j)	$\frac{\sqrt{25}}{3}$						
k)	$\sqrt{-25}$						

The nature of roots

Quadratic equation can have:

- at most two real roots
- two equal roots
- no real root.

We use the value of the discriminant to determine the number and the nature of the roots of the given quadratic equation.

A. If $b^2 - 4ac = 0$, then $ax^2 + bx + c = 0$ has two real, equal, rational roots.

Worked example 3.16

Consider the equation $4x^2 + 4x + 1 = 0$.

a) Calculate the value of the discriminant.
b) What is the nature of the root(s)?
c) Solve for x; where $x \in \mathbb{R}$.

Solution

a) $b^2 - 4ac = (4)^2 - 4(4)(1)$

$= 16 - 16$

$= 0$

b) The value of the discriminant is 0. So the equation $4x^2 + 4x + 1 = 0$ has 2 equal roots that are real and rational.

c) $4x^2 + 4x + 1 = 0$

$(2x + 1)(2x + 1) = 0$

$2x + 1 = 0$ or $2x + 1 = 0$ [Repeated root]

Thus, $2x = -1$

$x = -\frac{1}{2}$

B. If $b^2 - 4ac > 0$ and is also a perfect square, then the quadratic equation has two real, rational, unequal roots.

Worked example 3.17

Consider the equation $-3x^2 + 5x + 2 = 0$.

a) What is the nature of the roots of the equation?
b) Solve for x.
c) How do you know that the values of x you have calculated above are indeed the solutions?

Solution

a) $b^2 - 4ac = (5)^2 - 4(-3)(2)$

$= 25 + 24$

$= 49$

The discriminant is greater than zero and is a perfect square, so the roots are real, unequal and rational.

b) $-3x^2 + 5x + 2 = 0$

$x = \frac{-5 \pm \sqrt{(5)^2 - 4(-3)(2)}}{2(-3)}$

$x = \frac{-5 \pm \sqrt{49}}{-6}$

$x = \frac{-5+7}{-6}$ or $x = \frac{-5-7}{-6}$

$x = \frac{2}{-3}$ or $x = \frac{-12}{-6}$

$x = -\frac{2}{3}$ or $x = 2$

c) By substituting the values of x into the original equation and if the answer is 0 in each case, then the values are solutions.

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C. If $b^2 - 4ac > 0$ and not a perfect square, then the quadratic equation has real, unequal and irrational roots.

Worked example 3.18

a) Discuss the nature of the roots of the quadratic equation $x^2 + 2x - 2 = 0$.

b) Solve for x: $x^2 + 2x - 2 = 0$

Solution

a) $b^2 - 4ac = (2)^2 - 4(1)(-2)$

$= 4 + 8$

$= 12$

The discriminant is greater than zero but is not a perfect square, so the roots are real, unequal and irrational.

b) $x^2 + 2x - 2 = 0$

$x = \frac{-2 + \sqrt{12}}{2}$ or $x = \frac{-2 - \sqrt{12}}{2}$

$x = \frac{-2 + 2\sqrt{3}}{2}$ or $x = \frac{-2 - 2\sqrt{3}}{2}$

$x = -1 + \sqrt{3}$ or $x = -1 - \sqrt{3}$

D. If $b^2 - 4ac < 0$, then the quadratic equation has no real roots.

Worked example 3.19

Discuss the nature of the roots of the equation $2x^2 + 3x + 5 = 0$.

Solution

$b^2 - 4ac = (3)^2 - 4(2)(5)$

$= 9 - 40$

$= -31$

Since $b^2 - 4ac = -31$ then $\sqrt{b^2 - 4ac} = \sqrt{-31}$, is a non-real number or imaginary.

Since the discriminant has a negative value, the roots of the quadratic equation are non-real or imaginary.

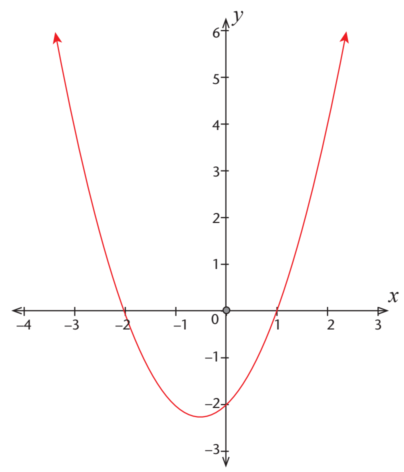
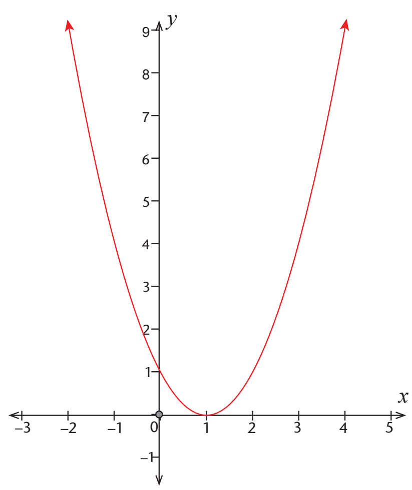
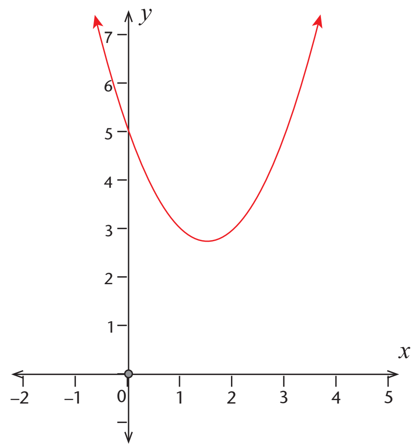
EXERCISE 3.10

Copy and complete the table below in your exercise book.

	Equation	Value of the discriminant: $b^2 - 4ac$	Nature of the roots
a)	$2x^2 - 4x - 3 = 0$		
b)	$9x^2 - 12x + 4 = 0$		
c)	$x^2 + x + 1 = 0$		
d)	$3x^2 + x + 5 = 0$		
e)	$x^2 - 5x + 3 = 0$		
f)	$-2x^2 + 5x - 3 = 0$		
g)	$-x^2 + 1 = 0$		
h)	$2x^2 + 5x + 6 = 0$		
i)	$x^2 - 4x + 4 = 0$		

3.5 The relationship between the discriminant and the graph of the quadratic function

The relationship between the value of the discriminant and the number of x-intercepts of a parabola is summarised in the table below.

$x^2 + x - 2 = 0$	$x^2 - 2x + 1 = 0$	$x^2 - 3x + 5 = 0$
$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-2)}}{2(1)}$ $x = \frac{-1 \pm \sqrt{9}}{2}$ $x = \frac{-1+3}{2}$ or $x = \frac{-1-3}{2}$ $x = 1$ or $x = -2$	$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)}$ $x = \frac{2 \pm \sqrt{0}}{2}$ $x = \frac{2+0}{2}$ or $x = \frac{2-0}{2}$ $x = 1$ or $x = 1$	$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(5)}}{2(1)}$ $x = \frac{3 \pm \sqrt{-11}}{2}$ $x = \frac{3+\sqrt{-11}}{2}$ or $x = \frac{3-\sqrt{-11}}{2}$
Two real solutions or two real roots.	One real solution or one real root.	No real solution or real root.
		

3.6 Solving inequalities graphically

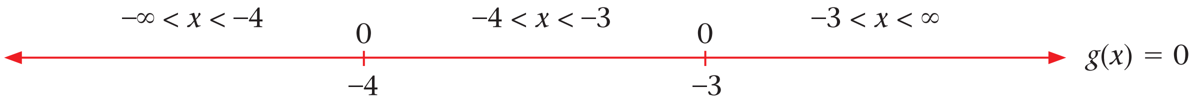
Let's consider the function $f(x) = x^2 + 7x + 12$.

Solving the quadratic inequality $x^2 + 7x + 12 < 0$ is equivalent to finding the values of x for which the graph of $f(x) = x^2 + 7x + 12$ is below the graph of $g(x) = 0$.

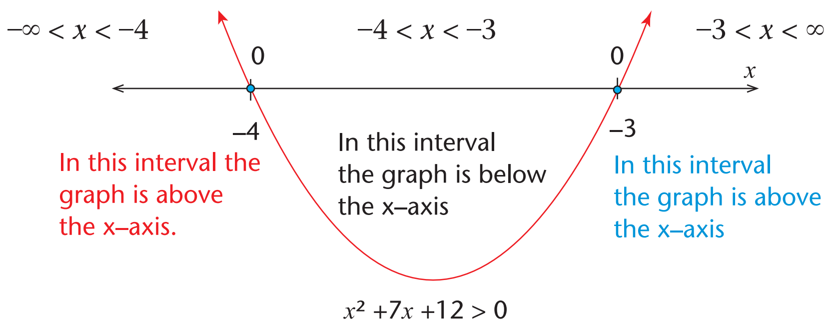
Note:
An inequality is a relationship between two expressions that are not equal, and uses the symbols < or >.

Follow the following steps:

- Calculate the roots of $x^2 + 7x + 12 = 0$ either using the factorisation method or by using quadratic formula. We determine that $x = -3$ and $x = -4$ are the roots.
- Show the roots on the x-axis as shown below.



- Observe that the roots divide the x-axis into three intervals as shown.
  - Sketch graph of $f(x) = x^2 + 7x + 12$.
- The graph opens upwards because $a > 0$ (that is, $a = 1$).



- Zeros are placed above the values $x = -4$ and $x = -3$ because they are the solutions, the roots or the zeros of the equation $x^2 + 7x + 12 = 0$.
- From the sketch we see that $x^2 + 7x + 12 < 0$ lies in the interval $-4 < x < -3$ or $x = (-4; -3)$.

EXERCISE 3.11

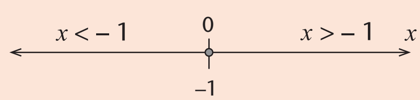
Solve the following inequalities.

a) $x^2 - 6x + 9 > 0$	d) $-3x^2 + 4x + 4 < 0$	g) $2x^2 - 7x + 3 < 0$
b) $x^2 - x > 0$	e) $-x^2 + 2x - 1 < 0$	h) $x^2 + \frac{5}{2}x + 1 > 0$
c) $x^2 - 1 < 0$	f) $x^2 + 2x - 3 > 0$	i) $-5x^2 - 9x + 2 < 0$

Worked example 3.20

Solve the inequality $x^2 + 2x + 1 \geq 0$

Solution

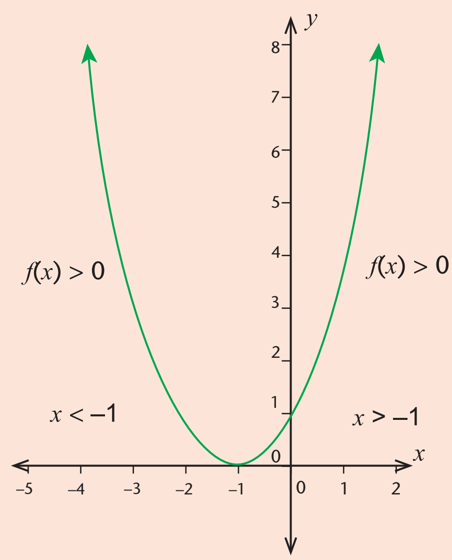


$x^2 + 2x + 1 \geq 0$

$(x + 1)(x + 1) \geq 0$

The inequality has a repeated root, $x = -1$

Solve $x^2 + 2x + 1 \geq 0$ by looking for values of x where: $x^2 + 2x + 1 = 0$ as well as for values of x where $x^2 + 2x + 1 > 0$. When we sketch the graph of $f(x) = x^2 + 2x + 1$ we get the graph shown alongside:



The graph is always above the x-axis except at $x = -1$ where it lies on x-axis. Thus, the solution for $x^2 + 2x + 1 \geq 0$ is $x \leq -1$ or $x \geq 1$.

EXERCISE 3.12

Solve the inequalities below; where $x \in \mathbb{R}$.

a) $4x^2 + 4x + 1 \geq 0$

b) $(x - 1)^2 < 0$

c) $-x^2 + 6x - 9 > 0$

d) $-x^2 + 4x - 4 \geq 0$

e) $2x^2 - 50 \geq 0$

f) $2x^2 + 5x - 12 \leq 0$

g) $-2x^2 + x + 3 < 0$

h) $7x^2 + x \leq 0$

i) $4x^2 - 9 < 0$

3.7 Simultaneous equations in two unknowns

Simultaneous equations are pairs of equations that have the same solution(s).

We will work with simultaneous equations in two unknowns, where one equation is quadratic and the other is linear. For example, $y = x^2 - 4$ and $y = x - 2$.

Solving a system of simultaneous equations simply means calculating a value or values of x that make(s) the two equations have the same value or values of y.

Worked example 3.21

Solve the given system of simultaneous equations:

$y = x^2 - 4$

$y = x - 2$

Solution

We are calculating a value or values of x that will give the same value of y when substituted in both equations.

$x^2 - 4 = x - 2$

$x^2 - x - 4 + 2 = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x - 2 = 0$ or $x + 1 = 0$

$x = 2$ or $x = -1$

When $x = 2$: $y = 2^2 - 4 = 4 - 4 = 0$

$y = 2 - 2 = 0$

If the value of x is 2 then y is zero and in coordinate form (2; 0).

When $x = -1$: $y = (-1)^2 - 4 = 1 - 4 = -3$

$y = -1 - 2 = -3$

What this means is that for the same value of x, the two equations have the same value of y.

We write this solution in coordinate form as $(-1; -3)$.

Worked example 3.22

Solve for x:

$y = x^2 + xy - 4$

$y - x = 2$

Solution

$y = x^2 + xy - 4$ .... (1)

$y - x = 2$ .... (2)

$y = 2 + x$ .... (3)

Substitute (3) into (1)

$x + 2 = x^2 + x(x + 2) - 4$

$x + 2 = x^2 + x^2 + 2x - 4$

$x + 2 = 2x^2 + 2x - 4$

$2x^2 + 2x - x - 4 - 2 = 0$

$2x^2 + x - 6 = 0$

$(2x - 3)(x + 2) = 0$

$x = \frac{3}{2}$ or $x = -2$

$y = \frac{7}{2}$ or $y = 0$

$\left(1\frac{1}{2}; 3\frac{1}{2}\right)$ and $(-2; 0)$

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EXERCISE 3.13

Solve the following systems of equations; where $x, y \in \mathbb{R}$.

a) $y = x^2 + 4x + 3$
$y = x + 3$

b) $y = x^2 + 2x + 5$
$y - x = 5$

c) $y = x^2 - 6x + 8$
$y = \frac{x}{2} + 2$

d) $y = x^2$
$y - 2 = x$

e) $y = x^2 + 2x + 5$
$y - x = 2$

f) $y = x^2 - 1$
$y = x + 1$

g) $y = 2x^2 - 5x - 3$
$y = -6x + 3$

h) $y = 6x^2 + 11x - 10$
$y = 4x$

i) $y = x^2 - 1{,}5x - 2$
$y = 0{,}5x$

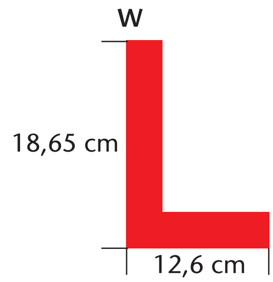
j) $y = x^2 - 1{,}3x + 0{,}3$
$y = 0{,}3$

k) $y = x^2 + 5{,}5x - 3$
$y = 5{,}5x$

l) $y = 3x^2 + xy + 1$
$y + 2x = 5$

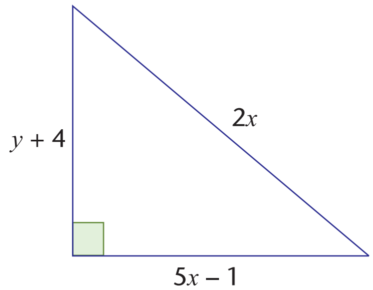
3.8 Word problems

1. The angle iron shown has a cross-sectional area of 70,56 cm^2.
   If the width of the angle iron is w, then the cross-sectional area is $12{,}6w + (18{,}65 - w)w$.
   What is the width of this angle iron?



2. A gutter is made by folding up the edges of a strip of metal.
   If the metal is 30,5 cm wide and the cross-sectional area of the gutter must be 108,2 cm^2, what is:
   a) the width of the gutter?
   b) depth of the gutter?
3. A rectangular piece needs to be cut from a large flat piece of sheet metal.
   The length of the cut piece is to be 18,3 cm longer than the width, and the piece is to contain 5,5 m^2. Calculate:
   a) the length of the cut piece.
   b) the width of the cut piece.
4. The bending moment of a beam is given by the formula $I = 7x^2 + 4x - 3$, where x is the distance along a beam from the other end. Calculate the value of x for which $I = 0$
5. The perimeter of the rectangle is represented by $8y$ metres and the area is represented by $(6y + 3)$ square metres.
   a) Write down the equations for the perimeter and the area of the rectangle in terms of x and y.
   b) Determine the perimeter and the area.

6. The perimeter of the right-angled triangle is 60 m, and its area is $10y$ square meters.



a) Write a simplified expression for the triangle's perimeter in terms of x and y.
b) Write a simplified expression for the triangle's area in terms x and y.
c) Write a system of equations and explain how it relates to this problem.
d) Solve the system of equations for x and y.
e) What are the dimensions of the triangle?
f) Verify your solution.

7. The number of centimetres in the circumference of a circle is three times the number of square centimetres in the area of the circle.
a) Write the system of linear-quadratic equations, in two variables, that model the circle with the given property.
b) What is the radius of the circle with this property?
c) What is the circumference of the circle with this property?
d) What is the area of the circle with this property?

8. The total surface area of a solid cone is 486,2 cm^2 and its slant height is 15,3 cm. Calculate the diameter of the base.

9. A shed is 4 metres long and 2 metres wide. A concrete path of constant width is laid all the way around the shed. If the area of the path is 9,5 cm^2, calculate its width (correct to the nearest centimetre).

3.9 Changing the subject of the formula

A formula shows a relationship between quantities. For example, the formula $A = \pi r^2$, shows the relationship between the area of a circle and the radius. Using this formula, we can calculate the area for circles of various radii.

But what if we, for example, have pre-determined the area a circular surface must have and want to calculate the radii that would result in the pre-determined area?

Such a calculation would require the re-arrangement of the formula $A = \pi r^2$.

Let's look at an example of this.

Worked example 3.23

Given the formula for calculating the area of a circle, $A = \pi r^2$, make r the subject of the formula.

Solution

$A = \pi r^2$

$\pi r^2 = A$

$r^2 = \frac{A}{\pi}$

$r = \sqrt{\frac{A}{\pi}}$ [a radius is a length and can only be a positive value]

EXERCISE 3.14

1. The power, P dissipated in a resistor of resistance, R, is given by the formula $P = \frac{V^2}{R}$. Make:
   a) R the subject of the formula.
   b) V the subject of the formula.
2. The total resistance, R, in a circuit containing two resistors in parallel is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.
   Make R_1 the subject of the formula.
3. Newton's law is expressed as $F = ma$.
   Rearrange the formula such that you can calculate the acceleration of a body given its mass and the force acting on it.
4. Given the formula $PV^n = C$, make each of the indicated quantities the subject of the formula:
   a) P b) n
5. Ohm's Law is given by $V = IR$.
   Rewrite the formula such that you can calculate the resistance of any conductor given the other two quantities.
6. The circumference of a circle is given by $C = 2\pi r$.
   Write a formula that will help you calculate the length of the radius of the circle given the circumference.
7. The formula $A = 2\pi rh + 2\pi r^2$ is used to calculate the surface area of a cylindrical object.
   Write down a formula that will enable you to calculate the radius given the surface area and the height of the object.
8. Make l, the length of a pendulum, the subject of the formula given that $T = 2\pi\sqrt{\frac{l}{g}}$
9. Given that $V = \frac{ER}{R + r}$ make each of the indicated quantities the subject of the formula.
   a) E b) R c) r
10. The velocity of a free falling body through a certain height, h, is given by $v^2 = 2gh$.
    Write down a formula that you would use to calculate the height through which the object is falling.

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CONSOLIDATION EXERCISE

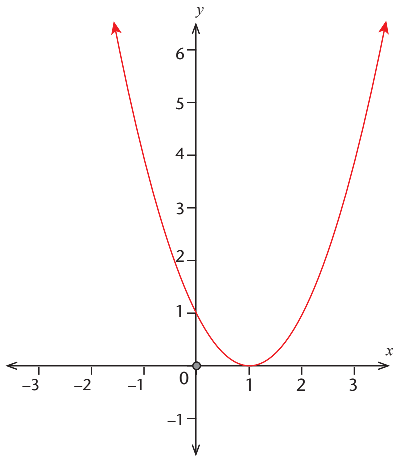
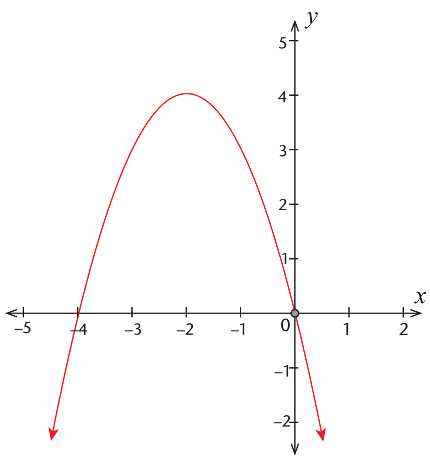
1. Which of the quadratic equations given below can be solved by factorising?
   a) $x^2 + 13x + 22 = 0$
   b) $x^2 - 10x + 25 = 0$
   c) $3x^2 - 6x + 5 = 0$
   d) $x^2 + 12x + 32 = 0$
   e) $-3x^2 + 5x + 1 = 0$
   f) $-3{,}1x^2 + 2{,}5x + 3{,}75 = 0$
2. Explain how you arrived at your decision(s) in answering question 1 above.
3. Now solve each of the equations in the question.
4. Solve for x in each of the cases below:
   a) $(x - 1)(x + 7) < 0$
   b) $3(x - 3)(x - 3) \geq 0$
   c) $(x - 1)(x + 2) > 0$
   d) $3x^2 + 18x + 15 \leq 0$
   e) $-2x^2 - 19x - 9 < 0$
   f) $-3x^2 - x + 2 < 0$
5. Solve the simultaneous equations given below:
   a) $y = (x - 4)(x + 13)$
      $y = x - 4$
   b) $y = x^2 - 36$
      $y - x + 34 = 0$

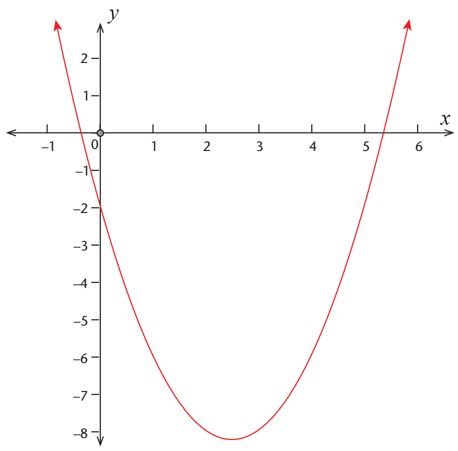
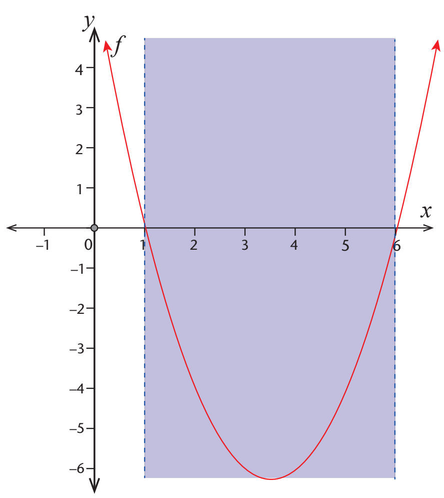
Summary

- a quadratic equation can be solved by:
  - factorisation
  - using the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
  - completing a square (Not examinable in Grade 11)
- to decide on the nature of the roots we use the discriminant: $b^2 - 4ac$
- the four cases of the discriminant are:

Value of $\Delta = b^2 - 4ac$	Is $\Delta = b^2 - 4ac$ a perfect square?	Number of solutions	Type of solution	Equal/unequal
equal to 0	No	2 or one unique solution	Rational	Equal
greater than 0	Yes	2	Rational	Unequal
greater than 0	No	2	Irrational	Unequal
less than 0			Imaginary	

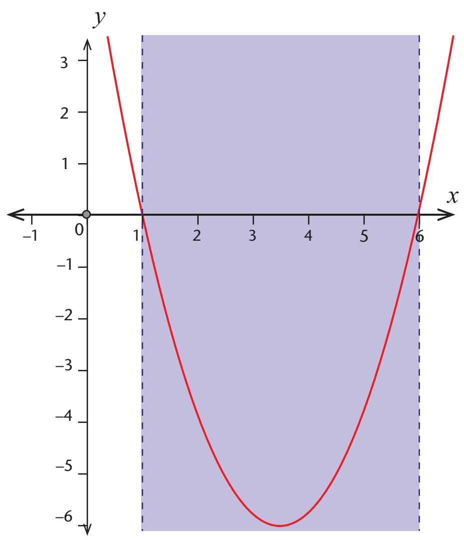
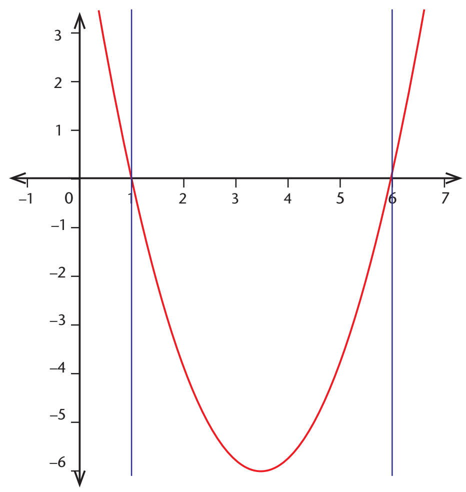
- the four cases of the discriminant shown graphically are as follows:

Value of the discriminant	Possible Graph		Nature of the roots
$b^2 - 4ac = 0$		The parabola intersects the x-axis at one rational value of x. The turning point is on the x-axis.	There are two equal or one unique root(s) that is (are): • Real • Rational
$b^2 - 4ac = 0$ and a perfect square		The parabola intersects the x-axis at two distinct rational values of x.	There are two roots that are: • Real • Rational • Unequal

Value of the discriminant	Possible Graph		Nature of the roots
$b^2 - 4ac > 0$ and not a perfect square		The parabola intersects the x-axis at two distinct irrational values of x.	There are two roots that are: • Real • Irrational • Unequal
$b^2 - 4ac < 0$		The parabola does not intersect the x-axis	The roots are non-real or imaginary

- a mathematical statement such as $x^2 - 7x + 6 \leq 0$ is called an inequality. It is similar to the equation $x^2 - 7x + 6 = 0$. In the case of the equation $x^2 - 7x + 6 = 0$ there are only two values that make the equation true, ($x = 1$ or $x = 6$). However, in the case of the inequality $x^2 - 7x + 6 \leq 0$ there is an infinite number of x-values that make the inequality true. The solution of the inequality is thus not a single number, but a range of numbers between 1 and 6 and includes both 1 and 6. It can be written mathematically as $1 \leq x \leq 6$.
- we solve inequalities in the same way as we solve equations.

- we can represent solutions of both equations and inequalities graphically.

Inequality: $x^2 - 7x + 6 \leq 0$	Equation: $x^2 - 7x + 6 = 0$
	
Solution: $1 \leq x \leq 6$	Solution: $x = 1$ and $x = 6$

- to solve simultaneous equations is to calculate solutions that satisfy all the given equations.
- in the case where one of the simultaneous equations is quadratic and the other linear, the maximum number of possible solutions is two for both x and y.
- we change the subject of the formula to create a new formula that has the variable or unknown whose value we want to calculate – it is similar to solving an equation.
- when changing the subject of the formula we apply the rules we apply when solving equations.

4 Analytical geometry

Objectives

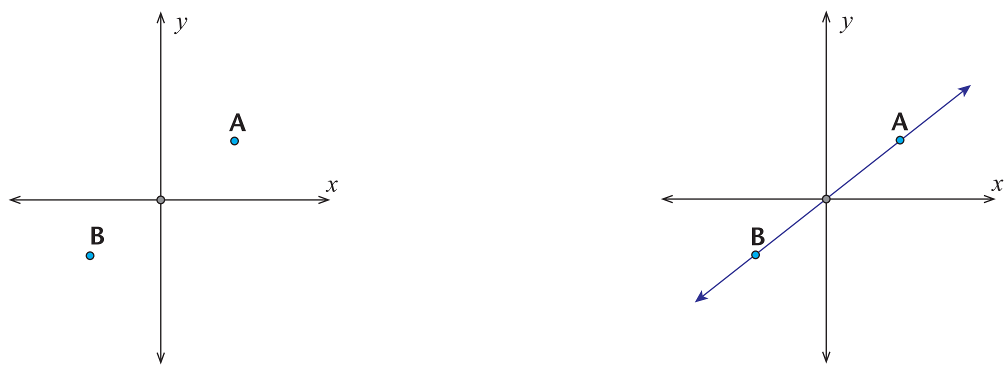
In this chapter you will:

- revise how to find the equation of the line through two given points
- determine:
    - the equation of the line through one point and parallel or perpendicular to the given line
    - the inclination (θ) of a line, where m = tan is the gradient of the line and $0^\circ \leq \theta \leq 180^\circ$.

Scientists use analytical geometry to calculate the angle at which to launch a rocket.

4.1 The equation of a straight line through two points

A geometric fact that you may have come across is that 'only one straight line can be drawn through two different points'. What this means is that if we are given two different points on a line, we can reconstruct the line in the *x*-*y* plane by simply lining a ruler up with the two points and drawing in a line (as illustrated below).



The gradient of a line ensures its straightness.

The gradient of the line has to be constant irrespective of the points on the line that are chosen to calculate it.

Let's consider a line in the *x*-*y* plane, with two points $A\ (x_1; y_1)$ and $B\ (x_2; y_2)$ on the line.

The gradient (m) of the line passing through the points A and B is given by the equation:

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} \text{ where } x_2 \neq x_1$$

Substituting in $y - y_1 = m(x - x_1)$ we get $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ where $x_2 \neq x_1$.

We can also re-arrange the above equation and rewrite is as:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \text{ where } x_2 \neq x_1 \text{ and } x \neq x_1$$

Worked example 4.1

Write the equation of the straight line passing through the given points (2; 1) and (–3; 2).

Solution

Let $(x_1; y_1) = (2; 1)$ and $(x_2; y_2) = (-3; 2)$

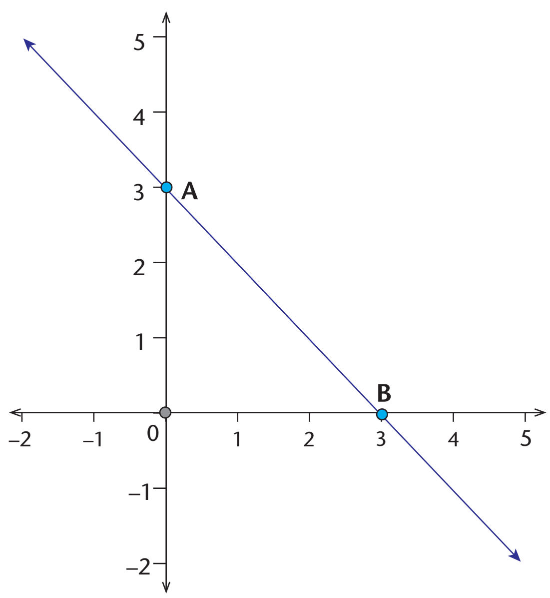
$$\begin{aligned}
\frac{y - y_1}{x - x} &= \frac{y_2 - y_1}{x_2 - x_1}\\
\frac{y - 1}{x - 2} &= \frac{2 - 1}{-3 - 2}\\
\frac{y - 1}{x - 2} &= \frac{1}{-5}\\
y - 1 &= -\frac{1}{5}(x - 2)\\
y &= -\frac{1}{5}(x - 2) + 1\\
y &= -\frac{1}{5}x + \frac{2}{5} + 1\\
y &= -\frac{1}{5}x + \frac{7}{5}
\end{aligned}$$

or

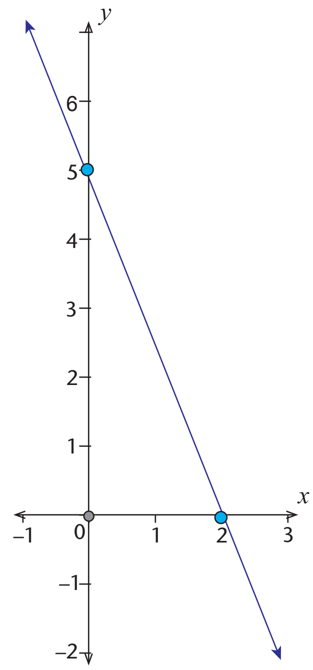
$$\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1}\\
&= \frac{2 - 1}{-3 - 2}\\
&= -\frac{1}{5}\\
y - y_1 &= \mathrm{m}(x - x_1)\\
y - 1 &= -\frac{1}{5}(x - 2)\\
y &= -\frac{1}{5}(x - 2) + 1\\
y &= \frac{1}{5}x + \frac{7}{5}
\end{aligned}$$

EXERCISE 4.1

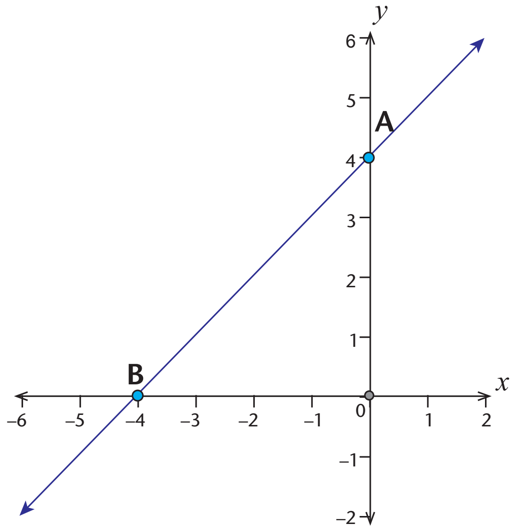
1. Write the equation of the straight line passing through the given points.
   a) (5; 4) and (0; 1)
   b) (1; 3) and (2; 6)
   c) (0; 3) and (2; –3)
   d) (–1; –3) and (–4; 0)
   e) (–2; 3) and (0; –3)
   f) (0; –1) and (9; 4)
2. Write the equation of the straight line through the points A and B in each graph below.
   a)



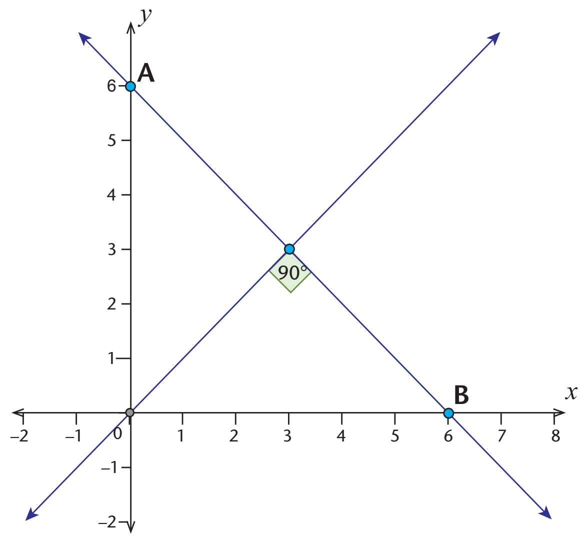
   b)



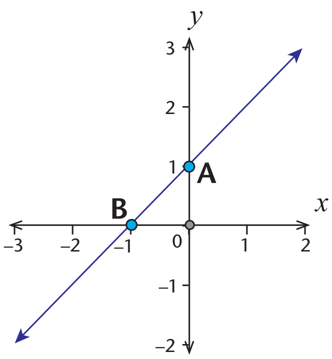
c)



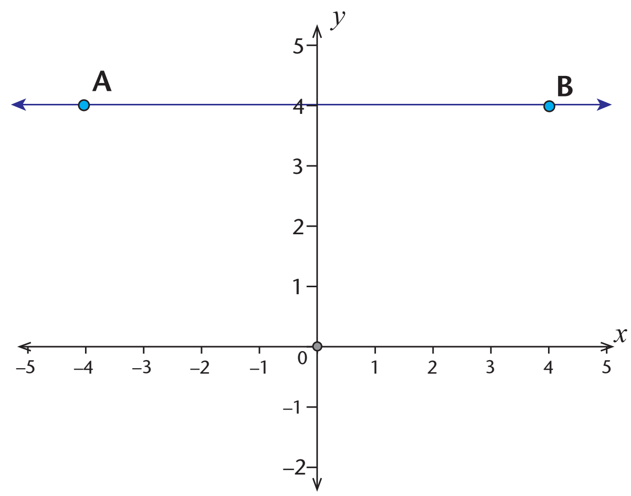
d)



e)



f)



3. Calculate p such that the points:
   a) $(p; -11)$ is on the line passing through the points $(1; 1)$ and $(5; 13)$.
   b) $(6; -14)$ is on the line passing through the points $(4; -6)$ and $(p; 2p)$.
   c) $\left(p\ ; \frac{p}{2}\right)$ is on the line passing through the points $(0; -6)$ and $(-1; -13)$.

4.2 The angle of inclination of a straight line

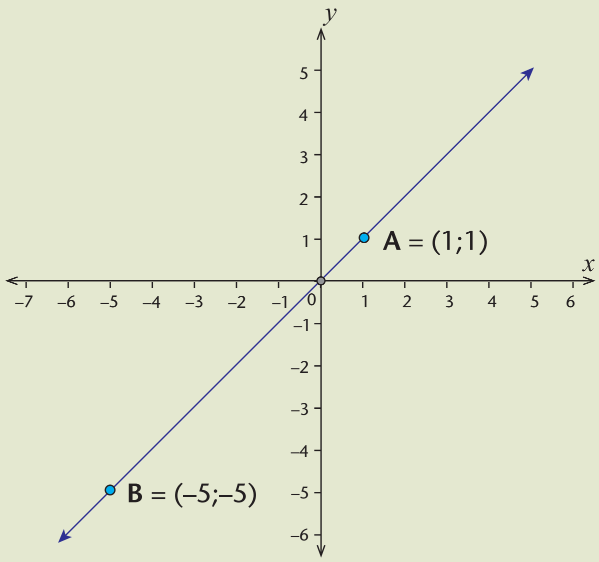
When we draw any non-horizontal straight line in the x-y plane it intersects with the x-axis at a certain angle. This is called the angle of inclination.

We use the x-axis as our point of reference in defining the angle of inclination of a line. We define the angle of inclination of a non-horizontal line to be the positive angle θ that the line or the part of the line makes with the x-axis. It is common practice to use Greek symbols such as θ to represent the angle of inclination.

The angle is measured from the positive x-axis in an anti-clockwise direction. The value of measured angle is (by convention) always given as a positive angle between 0˚ and 180˚.

Worked example 4.2

Calculate the angle of inclination for a straight line through (1; 1) and (–5; –5)



Solution

$m = \frac{-5-1}{-5-1} = \frac{-6}{-6} = 1$

$m = \frac{\Delta y}{\Delta x} = \tan\theta$

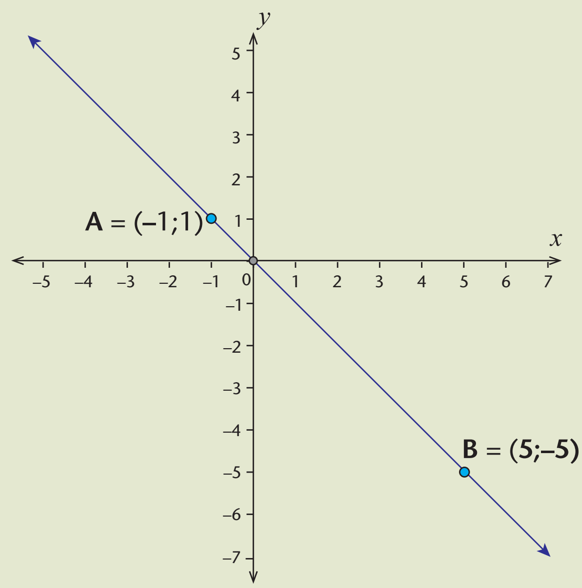
$\tan\theta = 1$

$\theta = \tan^{-1}(1)$ [calculator]

$\theta = 45°$

Worked example 4.3

Calculate the angle of inclination for a straight line through (–1; 1) and (5; –5)



Solution

$m = \frac{-5-1}{-5-(-1)} = \frac{-6}{5+1} = \frac{-6}{6} = -1$

$\tan\theta = -1$

$\theta = \tan^{-1}(-1)$ [calculator]

$\theta = -45°$

$\theta = 180° - 45°$

$\theta = 135°$

Note:
When a gradient is negative, the calculator gives a negative angle. We then use this negative angle to obtain a positive angle of inclination less than 180˚.

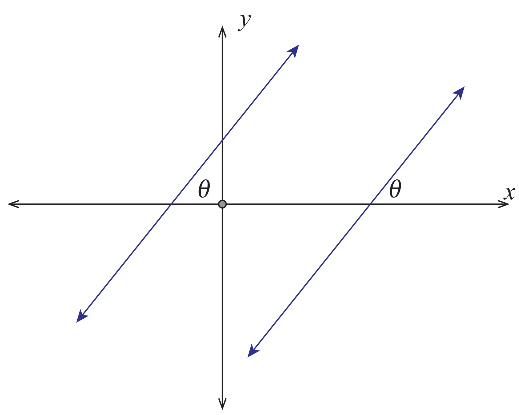
EXERCISE 4.2

1. Calculate the angle of inclination of a line passing through the given points. Give the answer correct to two decimal places (where appropriate).
    a) (4; 1) and (–4; 3)
    b) (7; 4) and (0; 6)
    c) (1; 5) and (6; 8)
    d) (0; 0) and (3; 1)
    e) (0; 3) and (–3; 0)
    f) (0,1; 1,3) and (1; 4)
2. What is the angle of inclination of a line having the given gradient?
    a) $m = 3$
    b) $m = -3$
    c) $m = 1.5$
    d) $m = -0,6$
    e) $m = 1,67$
    f) $m = -2,7$

4.3 The equation of a line through a point and parallel to a given line

Before we learn how to determine the equation of a line through a point and parallel to a given line, let's revise some useful facts regarding parallel lines that we learnt in previous grades:

- if two non-vertical lines are parallel, then their gradients are equal
- conversely, if two non-vertical lines have equal gradients then they can be said to be parallel to each other
- two lines that are parallel to each other are equally inclined to the x-axis.



Worked example 4.4

What is the equation of a straight line which passes through the point (4; 3) and is parallel to the line $y = -\frac{5}{4}x + 2$.

Solution

The straight line through the point (4; 3) will have the same gradient as that of the line $y = -\frac{5}{4}x + 2$ because they are parallel. That is, the gradient of both lines is $-\frac{5}{4}$ $\left(m = -\frac{5}{4}\right)$.

This means that for the line whose equation we are to determine we know:

1. the gradient $\left(m = -\frac{5}{4}\right)$
2. the point through which it passes (4; 3)

So, to determine the equation of the line we can use the slope-point form: $y - y_1 = m(x - x_1)$

$$y - 4 = -\frac{5}{4}(x - 3)$$

$$y - 4 = -\frac{5}{4}x + \frac{15}{4}$$

$$y = -\frac{5}{4}x + \frac{31}{4}$$

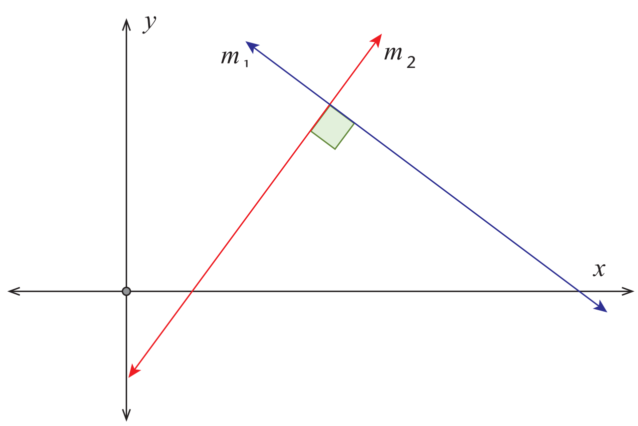
EXERCISE 4.3

1. What is the equation of a straight line that passes through the point:
   a) (–1; 1) and is parallel to the line $y = x - 2$.
   b) (0; –3) and is parallel to the line $y = x - 2$.
   c) (–1; 0) and is parallel to the line $y = x$.
   d) (–1; 1) and is parallel to the line $y = -x + 2$.
   e) (0; –3) and is parallel to the line $y = -x + 2$.
   f) (0,2; 3) and is parallel to the line $y = -x$.
   g) (2,1; 10) and is parallel to the line $y = -0{,}25x + 1$.
   h) (4; 3) and that is parallel to the line $y = 4x + 1$.
2. Are the lines in the x–y plane given by the following equations parallel to each other? Explain your answer in each case.
   a) $y = 3x - 6$ and $y = 3x + 1\ 000$
   b) $y = -2x + 17$ and $y = -\frac{5}{10}x = 1\ 003$
   c) $y = 5x - 1$ and $y = -5x - 1$
   d) $y = 7x + \frac{1}{3}$ and $y = \frac{1}{3}x + 7$
   e) $y = \frac{3}{8}x + 0{,}75$ and $y = -\frac{8}{3}x + 13$
3. Which of the lines in question 2 above have the same angles of inclination? Explain.
4. Given the line $y = \frac{3}{7}x + 1\ 001$, write the equations of two lines parallel to it.

5. Calculate k such that the line passing through the points.
   a) $(-3; k)$ and $(2;-5)$ is parallel to the line passing through the points $(3; 6)$ and $(-2; 4)$.
   b) $(1; 0)$ and $(k; 3)$ is parallel to the line passing through the points $(3; 0)$ and $(0; -3)$
   c) $(2,5; 3,8)$ and $(2,75; k)$ is parallel to the line $y = -2,25x + 1\,000$ (k must be given correct to two decimal places)
   d) $\left(\frac{1}{2}; k\right)$ and $\left(\frac{3}{4}; \frac{7}{4}\right)$ is parallel to the line $y = x - 1$.
   e) $(k; -1,5)$ and $(3,1; -3,2)$ is parallel to the line $y = -2x - 45$.
6. The points $(0; -5)$ and $(0; 2)$ are on the straight line, l_1 and the point $(3; -4)$ is on another straight line l_2 parallel to l_1. Calculate the co-ordinates of any point on l_2.
7. Points A (6; 1), B (3; −1), C (−4; 2) and D (x; y) are the vertices of a parallelogram. Calculate the coordinates of D.
8. What should the coordinates of the point B be so that quadrilateral ABCD is a parallelogram?
   a) $A(1; 1)$
   b) $B(x; y)$
   c) $C(-5; 5)$
   d) $D(-7; 6)$

4.4 The equation of a line through a point and perpendicular to a given line

The figure below shows two non-vertical perpendicular lines drawn in the x–y plane with gradients m_1 and m_2, respectively.



In previous grades you learned that two lines are perpendicular to each other if they intersect at 90˚. To determine the equation of a line through a point and perpendicular to another line we use the following theorem: Two non-vertical lines are perpendicular if and only if the product of their gradients is -1 ($m_1.m_2 = -1$).

Now consider the example below:

Worked example 4.5

Show that the lines in the x-y plane given by the following equations are perpendicular to each other: $y = 3x - 5$ and $y = \frac{x}{3} + 7$.

Solution

The gradients of the two lines are $m_1 = 3$ and $m_2 = -\frac{1}{3}$

$m_1 \times m_2 = 3 \times -\frac{1}{3} = -1$

This means that the lines given by equations $y = 3x - 5$ and $y = -\frac{x}{3} + 7$ are perpendicular to each other.

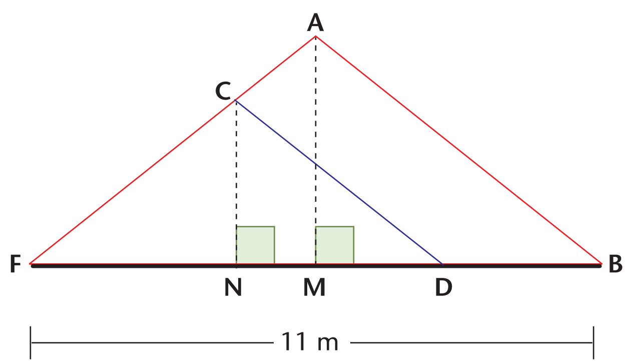
EXERCISE 4.4

1. Which of the pairs of lines given by the indicated equations are perpendicular to each other? Give an explanation in each case.
   a) $y = \frac{2}{5}x + 100$ and $y = 2{,}5x - 300$
   b) $y = x$ and $y = -x$
   c) $y = 2x - 0{,}375$ and $y = -2x + 1{,}5$
   d) $y = \frac{2x}{3} + 17$ and $y = \frac{3x}{2} - 17$
   e) $y = \frac{2x}{3} + 17$ and $y = \frac{-3x}{2} - 17$
2. Calculate the value of h such that the line passing through the points:
   a) $(0; 1)$ and $(8; h)$ is perpendicular to the line passing through the points $(5; -1)$ and $(4; 6)$.
   b) $(3; 4)$ and $(1; -8)$ is perpendicular to the line passing through the points $(5; 0)$ and $(h; h)$.
   c) $(h; 0)$ and $(-2; -5)$ is perpendicular to the line $y = 2x - 7$.
   d) $(h; -2)$ and $(1; -1)$ is perpendicular to the line $y = -\frac{4}{3}x + 1$.
3. ΔABC is a right-angled triangle with vertices A $(-4; 1)$, B $(0; y)$ and C $(2; -1)$.

   Calculate the value of y if angle A equals 90°.
4. Given the points A $(-2; 2)$, B $(-7; -1)$ and C $(x; 0)$ in the x-y plane, and given that AB is perpendicular to AC, calculate the value of x.

EXERCISE 4.5

1. AB and CD are roof lines of a house, with A at the centre of the house. AM and CN are vertical supports of the roof.
   a) If AM = 2,7 m; CN = 2,4 m, and EB = 11 m, how long should ND be if the slopes of the roof lines, AB and CD are equal?
   b) Does EN equal to ND?
   c) How long is DB?



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2. A ladder leans against a straight line wall defined by $y = 2x - 4$.
   a) Determine the length of the ladder.
   b) How far does the ladder reach up the wall?
   c) Calculate the ladder's inclination with the floor.

CONSOLIDATION EXERCISE

1. Given the co-ordinates for the points A, B, C and D, say whether AB and CD are parallel, perpendicular, or neither.

	A	B	C	D
a)	(3; 5)	(7; –1)	(–4; 4)	(0; –2)
b)	(4; 7)	(8; –1)	(0; 1)	(–4; –1)
c)	(–2; –3)	(4; –1)	(5; –2)	(6; 1)
d)	(5; 2)	(10; 4)	(–6; 2)	(–4; –3)

2. In each of the cases in question 1 a) to d), write the equations of both AB and CD.
3. What is the equation of a perpendicular bisector of the line segment joining the points A (0; 3) and B (8; –1)?
4. What is the equation of a straight line that satisfies the given condition(s)?
   a) x-intercept $= 1$, y-intercept $= 2$
   b) y-intercept $= 3$, parallel to the line $3x - 2y + 4 = 0$
   c) x-intercept $= 2$, perpendicular to the line $3x - 2y + 4 = 0$
5. Write the equation of the line that contains the point (–1; 8) and is:
   a) parallel to $x = 2$
   b) perpendicular to $x = -2$
   c) parallel to $y = 3x + 6$
   d) perpendicular to $y = 3x + 6$

Summary

- the equation of a straight line that passes through two points can be determined by means of the following formulae: $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ or $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ or $y - y_1 = m(x - x_1)$
- the gradients of parallel lines are equal.
- the product of the gradients of perpendicular lines = −1.
- the angle of inclination of a line is the positive angle a line makes with the positive x-axis. It is always calculated in an anti-clockwise direction.
- the angle of inclination of a line is calculated by setting the equation:

  $m = \tan\theta$

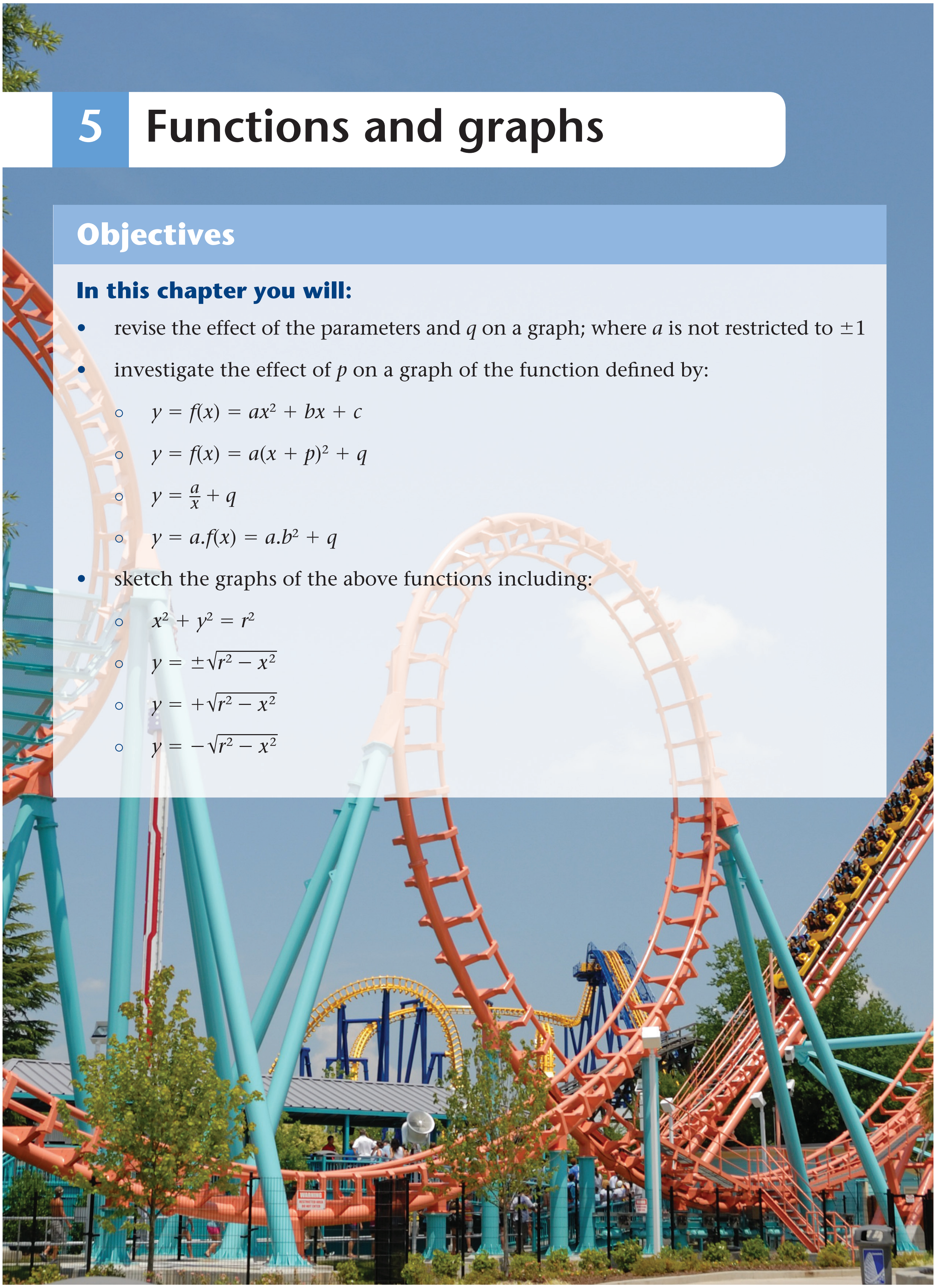
  $\theta = \tan^{-1}(m)$ for $0° \leq \theta \leq 180°$

5 Functions and graphs

Objectives

In this chapter you will:

- revise the effect of the parameters and q on a graph; where a is not restricted to ± 1
- investigate the effect of p on a graph of the function defined by:
  - $y = f(x) = ax^2 + bx + c$
  - $y = f(x) = a(x + p)^2 + q$
  - $y = \frac{a}{x} + q$
  - $y = a.f(x) = a.b^2 + q$
- sketch the graphs of the above functions including:
  - $x^2 + y^2 = r^2$
  - $y = \pm\sqrt{r^2 - x^2}$
  - $y = +\sqrt{r^2 - x^2}$
  - $y = -\sqrt{r^2 - x^2}$



Engineers use functions to determine a rollercoaster's height above the ground.

5.1 Introduction

In mathematics we often want to talk about the value of a quantity without ever calculating it. To make it easier to work with, we use function notation such as $f(x)$ or $g(x)$ to refer to the rule for calculating a quantity's value without having to write the rule over and over again. In this chapter we will use the function notation to express relationships between input and output values.

A function consists of:

- a set of input values – called the domain of the function
- a set of output values – called the range of the function, and
- a rule that tells us how to calculate the output values when we know the input values.

In previous grades we drew the graphs of $f(x) = x^2$ and $f(x) = -x^2$ by:

- completing a table of input and output values
- plotting the points on the Cartesian plane, and
- joining the points to form a continuous smooth curve.

In this chapter we will investigate the effects of a, b and c on the graph of $f(x) = ax^2 + bx + c$.

5.2 Revision: The graph of $f(x) = x^2$; $a = 1$

Remember that the simplest form of the function is $f(x) = x^2$, which results in a parabola. The graph is symmetrical about the y-axis, which is called the axis of symmetry of this function. The coefficient of x^2 is called a.

Let's work through an example to refresh our memories.

Worked example 5.1

Consider the function defined by $f(x) = x^2$.

a) Calculate the output values of $f(x) = x^2$ for $x \in [-4; 4]$.
b) Represent the function $f(x) = x^2$ using a table of values.
c) Plot the points on a Cartesian plane and then join them to form a graph.
d) Discuss the features of the graph.

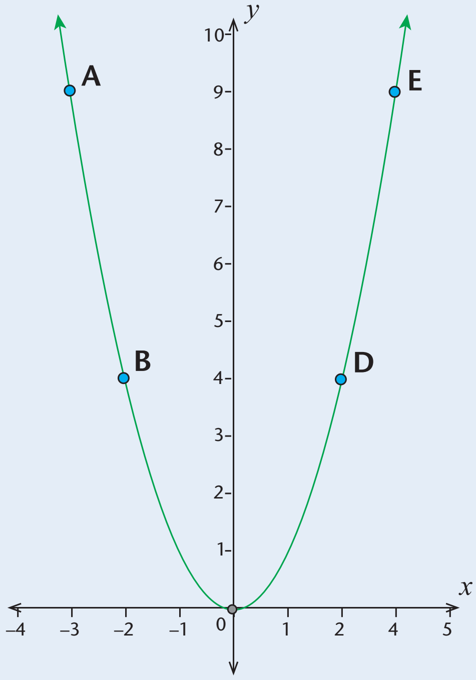
Solution

a) $f(-4) = (-4)^2 = 16$ $f(-1) = (-1)^2 = 1$ $f(2) = (2)^2 = 4$
$f(-3) = (-3)^2 = 9$ $f(0) = (0)^2 = 0$ $f(3) = (3)^2 = 9$
$f(-2) = (-2)^2 = 4$ $f(1) = (1)^2 = 1$ $f(4) = (4)^2 = 16$

b) Table of values

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	16	9	4	1	0	1	4	9	16

c) The graph of $f(x) = x^2$



d) Features (characteristics) of the graph

(i) The shape of the graph of $f(x) = x^2$:
The graph opens upwards (a > 0).

(ii) Intercepts of the graph:
The graph cuts the x-axis at (0; 0) and the y-axis at (0; 0).

(iii) Axis of symmetry:
The graph is symmetric with respect to the y-axis.
For example, the points (2; 4) and (–2; 4) are on either side of the y-axis.
The equation of the axis of symmetry is $x = 0$.

(iv) The turning point (TP):
The graph has a turning point at (0; 0).

(v) Maximum or minimum value:
The graph has a minimum value of $y = 0$ ($a < 0$).
The minimum value is the y-coordinate of the turning point.

(vi) The domain:
The domain is $x \in \mathbb{R}$ or $(-\infty; \infty)$.

(vii) The range:
The smallest output value for the function $f(x) = x^2$ is 0.
The range for this function is $y \geq 0$ or $[0; \infty)$.

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5.3 The effect of a on the graph of $f(x) = x^2$

The value of a affects how the graph will be drawn. If the value of a is positive, the graph bends upwards (like a smile), but if a is negative, the graph bends downwards (like a frown).

The effect of $a > 1$ on the graph of $f(x) = ax^2$

Let's investigate the effect of a on the graph of $f(x) = x^2$ when $a > 1$.

EXERCISE 5.1

1. a) Copy and complete the table of values for the function defined by $f(x) = ax^2$.

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$									

b) Plot the points in the Cartesian plane and then join them.

You now have a graph of the function defined by $f(x) = x^2$.

2. Copy and complete the tables below for the functions defined by:

a) $h(x) = 2x^2$.

x	–4	–3	–2	–1	0	1	2	3	4
$h(x)$									

b) $g(x) = 3x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$g(x)$									

c) On the same system of axes draw the graphs of the functions f, g and h.

3. a) Use the functions and their graphs from in question 2 to complete the table below.

		$f(x) = x^2$	$h(x) = 2x^2$	$g(x) = 3x^2$
(i)	Value of	$a = 1$ $b = 0$ $c = 0$	$a =$ $b =$ $c =$	$a =$ $b =$ $c =$
(ii)	x-intercepts			
(iii)	y-intercept			
(iv)	Coordinates of TP	(0; 0)		
(v)	Axis of symmetry			
(vi)	Domain			
(vii)	Range			

b) In what way are the graphs of the functions $f(x) = x^2$, $h(x) = 2x^2$ and $g(x) = 3x^2$ the:

(i) same? (ii) different

4. What is the effect of a on the graph of $f(x) = x^2$ for $a > 1$?

The effect of a on the graph of $f(x) = ax^2$ for $0 \leq a \leq 1$

Let's investigate the effect of a on the graph of $f(x) = x^2$ when $0 \leq a \leq 1$.

EXERCISE 5.2

1. Copy and complete the tables of values for the given functions:

a) $f(x) = x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$									

b) $f(x) = \frac{1}{2}x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$									

c) $f(x) = \frac{1}{3}x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$									

2. a) On the same system of axes, plot and join the set of points, then draw the graphs of each function.

b) Complete the table below.

		$f(x) = x^2$	$f(x) = \frac{1}{2}x^2$	$f(x) = \frac{1}{3}x^2$
(i)	Value of	$a = 1$ $b = 0$ $c = 0$	$a =$ $b =$ $c =$	$a =$ $b =$ $c =$
(ii)	x-intercepts			
(iii)	y-intercept			
(iv)	Coordinates of TP	(0; 0)		
(v)	Axis of symmetry			
(vi)	Domain			
(vii)	Range			

c) In what way are the graphs of the functions $f(x) = x^2$, $f(x) = \frac{1}{2}x^2$ and $f(x) = \frac{1}{3}x^2$ the:

(i) same?

(ii) different?

3. How does the value of a for $0 \leq a \leq 1$ affect the graph of $f(x) = x^2$?

The effect of a, $a < 0$, on the graph of $f(x) = ax^2$

Let's investigate the effect of a on the graph of $f(x) = x^2$ when $a < 0$.

EXERCISE 5.3

1. Copy and complete the tables of values for the given functions:

a) $f(x) = -x^2$

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$									

b) $f(x) = -2x^2$

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$									

c) $f(x) = -3x^2$

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$									

2. a) On the same system of axes, plot and join the set of points, then draw the graphs of each function.

b) Complete the table below.

		$f(x) = x^2$	$f(x) = -x^2$	$f(x) = -2x^2$	$f(x) = -3x^2$
(i)	Value of	$a = 1$ $b = 0$ $c = 0$	$a =$ $b =$ $c =$	$a =$ $b =$ $c =$	$a =$ $b =$ $c =$
(ii)	x-intercepts				
(iii)	y-intercept				
(iv)	Coordinates of TP	(0; 0)			
(v)	Axis of symmetry				
(vi)	Domain				
(vii)	Range				

3. What effect does the value of a have on the graph of:

a) $f(x) = ax^2$ for $a < 0$?

b) $f(x) = -x^2$ for $a < -1$?

The effect of a, $-1 < a < 0$, on the graph of $f(x) = ax^2$

Let's investigate the effect of a on the graph of $f(x) = x^2$ when $-1 < a < 0$.

EXERCISE 5.4

1. Copy and complete the tables of values for the given functions:

a) $f(x) = -x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$									

b) $f(x) = -\frac{1}{2}x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$									

c) $f(x) = -\frac{1}{3}x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$									

2. On the same system of axes, plot and join the set of points, then draw the graphs of each function.

3. a) Complete the table below.

	Features	$f(x) = -x^2$	$f(x) = -\frac{1}{2}x^2$	$f(x) = -\frac{1}{3}x^2$
(i)	Value of	$a = 1$ $b = 0$ $c = 0$	$a =$ $b =$ $c =$	$a =$ $b =$ $c =$
(ii)	x-intercepts			
(iii)	y-intercept			
(iv)	Coordinates of TP	(0; 0)		
(v)	Axis of symmetry			
(vi)	Domain			
(vii)	Range			

b) Using your own words, describe the effect of a on the graph $f(x) = -x^2$ for $-1 < a < 0$.

5.4 The effect of b on the graph of $f(x) = ax^2$

The value of b affects how the graph will be drawn. The value of b moves the line of reflection, but how it moves, still depends on the value of a.

An increasing value of b; $a > 0$ and $c = 0$

Let's investigate the effect of b on the graph of $f(x) = x^2$ when $a > 0$ and $c = 0$.

EXERCISE 5.5

1. Copy and complete the tables of values for the given functions:

a) $k(x) = 2x^2 - 3x$

x	–4	–3	–2	–1	0	1	2	3	4
$k(x)$									

b) $h(x) = 2x^2 - 2x$

x	–4	–3	–2	–1	0	1	2	3	4
$h(x)$									

c) $g(x) = 2x^2 - x$

x	–4	–3	–2	–1	0	1	2	3	4
$g(x)$									

d) $f(x) = 2x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$									

e) $s(x) = 2x^2 + x$

x	–4	–3	–2	–1	0	1	2	3	4
$s(x)$									

f) $t(x) = 2x^2 + 2x$

x	–4	–3	–2	–1	0	1	2	3	4
$t(x)$									

g) $p(x) = 2x^2 + 3x$

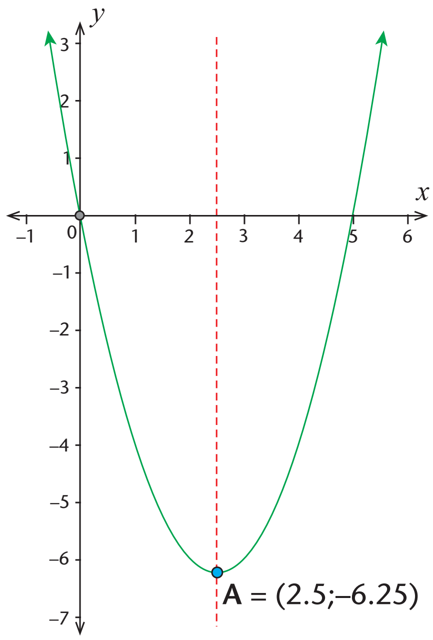
x	–4	–3	–2	–1	0	1	2	3	4
$p(x)$									

2. On the same system of axes, plot and join the set of points, then draw the graphs of each function.
3. Copy and complete the table below

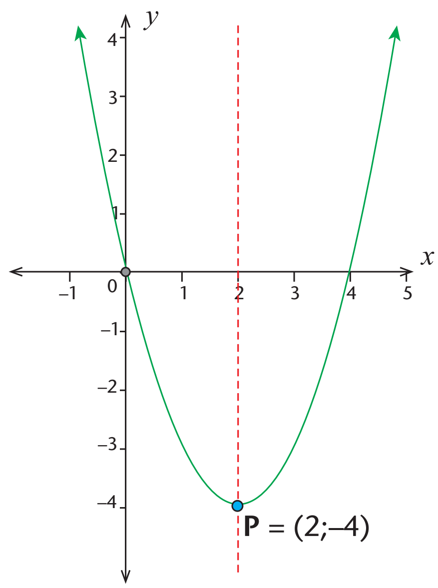
	Features	$k(x) = 2x^2 - 3x$	$h(x) = 2x^2 - 2x$	$g(x) = 2x^2 - x$	$f(x) = 2x^2$	$s(x) = 2x^2 + x$	$t(x) = 2x^2 + 2x$
(i)	Value of	$a = 2$ $b = -3$ $c = 0$		$a =$ $b =$ $c =$		$a =$ $b =$ $c =$	$a =$ $b =$ $c =$
(ii)	x-intercepts						
(iii)	y-intercept						
(iv)	Coordinates of TP	(0; 0)					
(v)	Axis of symmetry						
(vi)	Domain						
(vii)	Range						

4. How does increasing the value of b affect the graph of $f(x) = x^2$?
   a) To answer the given question, consider the graphs given on the next three pages.
      The value of b increases from $b = -5$ to $b = 2$.
   b) Plot the coordinates of the turning points for the graphs above on a system of x-y axes.
      What do you observe?

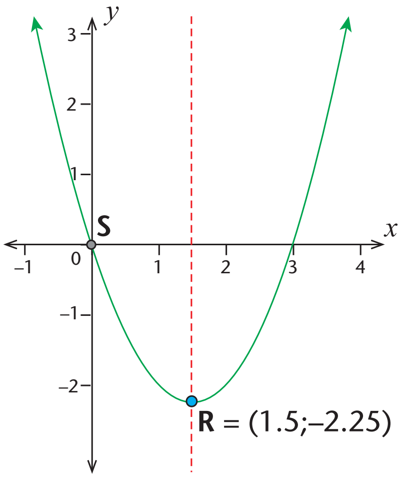
(i) $f(x) = 1x^2 + (-5)x$



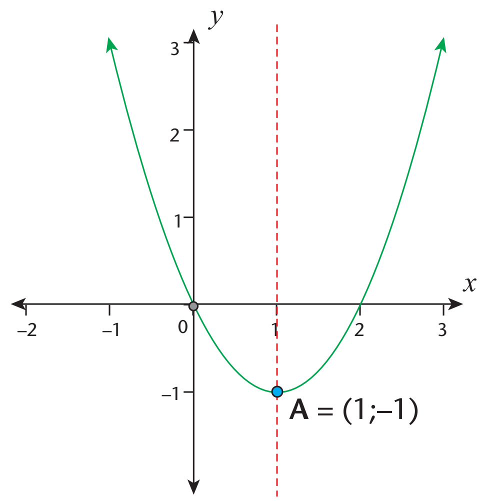
(ii) $f(x) = 1x^2 + (-4)x$



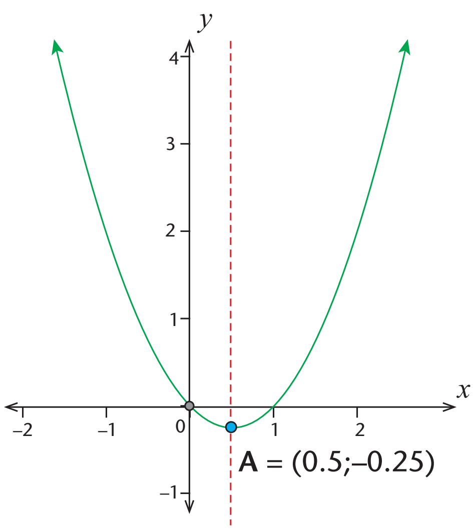
(iii) $f(x) = x^2 + (-3)x$



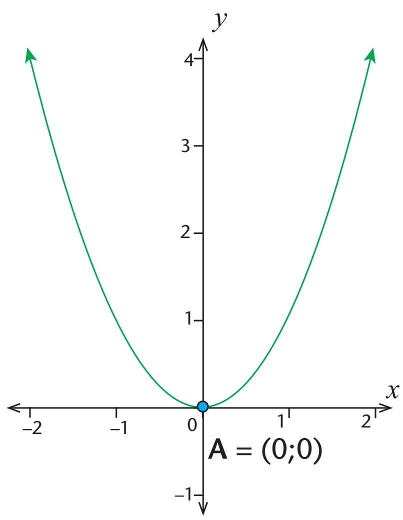
(iv) $f(x) = x^2 + (-2)x$



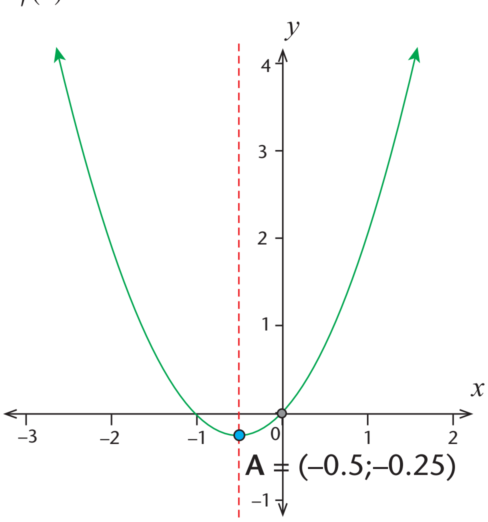
(v) $f(x) = x^2 + (-1)x$



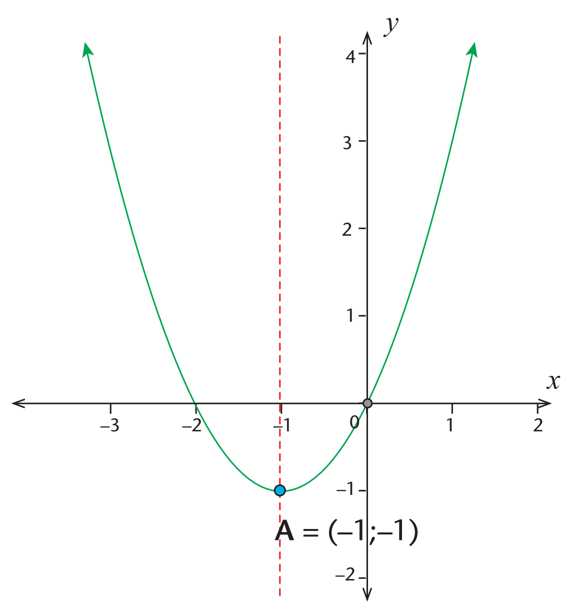
(vi) $f(x) = x^2 + (0)x$



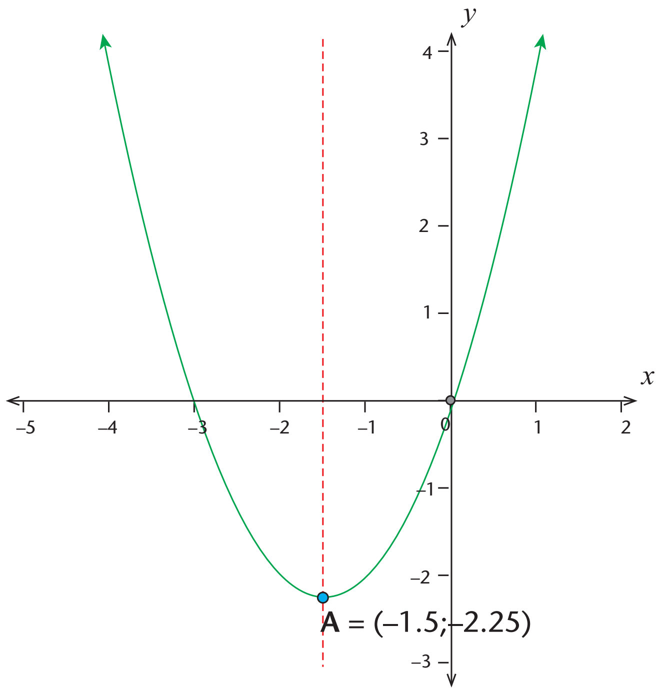
(vii) $f(x) = x^2 + x$



(viii) $f(x) = x^2 + (2)x$



(ix) $f(x) = x^2 + (3)x$



The effect of b on the graph of $f(x) = ax^2$, where $a < 0$, $c = 0$

Let's investigate the effect of b on the graph of $f(x) = x^2$ when $a < 0$ and $c = 0$.

EXERCISE 5.6

1. Copy and complete the tables of values for the given functions:

a) $s(x) = -2x^2 - 3x$

x	–4	–3	–2	–1	0	1	2	3	4
$s(x)$									

b) $r(x) = -2x^2 - 2x$

x	–4	–3	–2	–1	0	1	2	3	4
$r(x)$									

c) $p(x) = -2x^2 - x$

x	–4	–3	–2	–1	0	1	2	3	4
$p(x)$									

d) $s(x) = -2x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$									

e) $p(x) = -2x^2 + x$

x	−4	−3	−2	−1	0	1	2	3	4
$p(x)$									

f) $r(x) = -2x^2 + 2x$

x	−4	−3	−2	−1	0	1	2	3	4
$r(x)$									

g) $w(x) = -2x^2 + 3x$

x	−4	−3	−2	−1	0	1	2	3	4
$w(x)$									

2. On the same system of axes, plot and join the set of points, then draw the graphs of each function.
3. How is the graph of $f(x) = ax^2$ affected by an increasing value of b when $a < 0$?

5.5 The effect of c on the graph $f(x) = ax^2$

If there is no b term, changing c moves the parabola up or down so that the y-intercept is $(0, c)$.

The effect of c on the graph of $f(x) = ax^2$, where $a > 0$, $b = 0$ and $c > 0$

Let's investigate the effect of c on the graph of $f(x) = x^2$ when $a > 0$; $b = 0$ and $c > 0$.

EXERCISE 5.7

1. Copy and complete the tables of values for the given functions:

a) $f(x) = 2x^2$

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$									

b) $g(x) = 2x^2 + 1$

x	−4	−3	−2	−1	0	1	2	3	4
$g(x)$									

c) $h(x) = 2x^2 + 2$

x	−4	−3	−2	−1	0	1	2	3	4
$h(x)$									

d) $k(x) = 2x^2 + 3$

x	−4	−3	−2	−1	0	1	2	3	4
$k(x)$									

2. a) On the same system of axes draw the graphs of $f(x) = 2x^2$, $g(x) = 2x^2 + 1$, $h(x) = 2x^2 + 2$ and $k(x) = 2x^2 + 3$.

   b) How does a positive value of c affect the graph of $f(x) = 2x^2$?

The effect of c on the graph of $f(x) = ax^2$, where $a > 0$ and $c < 0$

Let's investigate the effect of c on the graph of $f(x) = x^2$ when $a > 0$; $b = 0$ and $c < 0$.

EXERCISE 5.8

1. Copy and complete the tables of values for the given functions:

   a) $f(x) = 2x^2$

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$									

   b) $g(x) = 2x^2 - 1$

x	−4	−3	−2	−1	0	1	2	3	4
$g(x)$									

   c) $h(x) = 2x^2 - 2$

x	−4	−3	−2	−1	0	1	2	3	4
$h(x)$									

   d) $k(x) = 2x^2 - 3$

x	−4	−3	−2	−1	0	1	2	3	4
$k(x)$									

2. a) On the same system of axes draw the graphs of $f(x) = 2x^2$, $g(x) = 2x^2 - 1$, $h(x) = 2x^2 - 2$ and $k(x) = 2x^2 - 3$.

   b) How does a negative value of c affect the graph of $f(x) = 2x^2$?

The effect of c: $f(x) = ax^2 + c$, $a < 0$ and $c > 0$

Let's investigate the effect of c on the graph of $f(x) = x^2$ when $a < 0$; $b = 0$ and $c > 0$.

EXERCISE 5.9

1. Copy and complete the tables of values for the given functions:

a) $g(x) = -2x^2$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$									

b) $g(x) = -2x^2 + 1$

x	-4	-3	-2	-1	0	1	2	3	4
$g(x)$									

c) $h(x) = -2x^2 + 2$

x	-4	-3	-2	-1	0	1	2	3	4
$h(x)$									

d) $k(x) = -2x^2 + 3$

x	-4	-3	-2	-1	0	1	2	3	4
$k(x)$									

2. a) On the same system of axes draw the graphs of $f(x) = -2x^2$, $g(x) = -2x^2 + 1$, $h(x) = -2x^2 + 2$ and $k(x) = -2x^2 + 3$.

   b) How does a positive value of c affect the graph of $f(x) = -2x^2$?

The effect of c: $f(x) = ax^2$, where $a < 0$ and $c < 0$

Let's investigate the effect of c on the graph of $f(x) = x^2$ when $a < 0$; $b = 0$ and $c < 0$.

EXERCISE 5.10

1. Copy and complete the tables of values for the given functions:

a) $f(x) = -2x^2$

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$									

b) $g(x) = -2x^2 - 1$

x	-4	-3	-2	-1	0	1	2	3	4
$g(x)$									

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c) $h(x) = -2x^2 - 2$

x	−4	−3	−2	−1	0	1	2	3	4
$h(x)$									

d) $k(x) = -2x^2 - 3$

x	−4	−3	−2	−1	0	1	2	3	4
$k(x)$									

2. a) On the same system of axes draw the graphs of $f(x) = -2x^2$, $g(x) = -2x^2 - 1$, $h(x) = -2x^2 - 2$ and $k(x) = -2x^2 - 3$.
   b) How does a negative value of c affect the graph of $k(x) = -2x^2 - 3$?
   c) How does $a = -2$ affect the graph of:
      (i) $f(x) = x^2$
      (ii) $f(x) = -x^2$

The effect of p: $f(x) = a(x - p)^2$, where $a > 0$, $p < 0$

Let's investigate the effect of p on the graph of $a(x - p)^2$, where $a > 0$, and $p < 0$.

EXERCISE 5.11

1. Copy and complete the tables of values for the given functions:

a) $f(x) = 2x^2$

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$									

b) $s(x) = 2(x + 1)^2$

x	−4	−3	−2	−1	0	1	2	3	4
$s(x)$									

c) $p(x) = 2(x + 2)^2$

x	−4	−3	−2	−1	0	1	2	3	4
$p(x)$									

d) $k(x) = 2(x + 3)^2$

x	−4	−3	−2	−1	0	1	2	3	4
$k(x)$									

2. On the same system of axes draw the graphs of the functions.

3. Use the graphs you have drawn (and any other information) to complete the table below.

		$f(x) = x^2$	$f(x) = 2x^2$	$s(x) = 2(x + 1)^2$	$s(x) = 2(x + 2)^2$	$s(x) = 2(x + 3)^2$
	Value of	$a = 1$ $p = 0$ $q = 0$	$a = 2$ $p = 0$ $q = 0$	$a = 2$ $p = -1$ $q = 0$	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$
a)	x-intercept					
b)	y-intercept					
c)	Axis of symmetry			$p = -1$		
d)	Coordinates of TP	(0; 0)				
e)	Domain					
f)	Range					

4. How does the value of p affect the graph of $f(x) = 2x^2$.

Note:
The graph of $f(x) = (x + 1)^2$ is the same as the graph of $f(x) = x^2$ we move the graph one unit to the left. We say that the basic graph ($f(x) = x^2$) has been translated by -1 unit horizontally. The axis of symmetry is now $x = -1$.

The effect of p: $f(x) = a(x + p)^2$, where $a < 0$

Let's investigate the effect of p on the graph of $a(x + p)^2$, where $a < 0$.

EXERCISE 5.12

1. Copy and complete the tables of values for the given functions:

a) $f(x) = -2x^2$

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$									

b) $q(x) = -2(x + 1)^2$

x	−4	−3	−2	−1	0	1	2	3	4
$q(x)$									

c) $n(x) = -2(x - 1)^2$

x	−4	−3	−2	−1	0	1	2	3	4
$n(x)$									

2. On the same system of axes draw the graphs of $f(x) = -2x^2$, $q(x) = -2(x + 1)^2$ and $n(x) = -2(x - 1)^2$.

3. Use the graphs you have drawn (and any other information) to complete the table below.

		$f(x) = -x^2$	$g(x) = -2x^2$	$q(x) = -2(x + 1)^2$	$n(x) = -2(x - 1)^2$
a)	Value of	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$
b)	x-intercepts				
c)	y-intercept				
d)	Axis of symmetry				
e)	Coordinates of TP	(0; 0)			
f)	Domain				
g)	Range				

4. How does the value of p affect the graph of $f(x) = -2x^2$?

The effect of q: $f(x) = a(x + p)^2 + q$, where $a > 0$, $q > 0$

Let's investigate the effect of p on the graph of $a(x + p)^2 + q$ where $a > 0$ and $q > 0$.

EXERCISE 5.13

1. Copy and complete the tables of values for the given functions:

a) $f(x) = 2x^2$

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$									

b) $t(x) = 2(x + 1)^2 + 1$

x	−4	−3	−2	−1	0	1	2	3	4
$t(x)$									

c) $v(x) = 2(x + 1)^2 + 2$

x	−4	−3	−2	−1	0	1	2	3	4
$v(x)$									

2. On the same system of axes draw the graphs of $f(x) = 2x^2$, $t(x) = 2(x + 1)^2 + 1$ and $v(x) = 2(x + 1)^2 + 2$.

3. Use the graphs you have drawn (and any other information) to complete the table below.

		$f(x) = -2x^2$	$t(x) = 2(x + 1)^2 + 1$	$v(x) = 2(x + 1)^2 + 2$
a)	Value of	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$
b)	x-intercepts			
c)	y-intercept			
d)	Axis of symmetry			
e)	Coordinates of TP	(0; 0)		
f)	Domain			
g)	Range			

5.6 The effect of q on $f(x) = a(x + p)^2 + q$

Writing a function as $f(x) = a(x + p)^2 + q$ tells us how $f(x) = 2x^2$ is translated vertically and horizontally, where the axis of symmetry is and where the points of intersection with the x- and y-axes are.

The effect of q: $f(x) = a(x + p)^2 + q$, where $a > 0$, $p = 1$, $q < 0$

Let's investigate the effect of p on the graph of $a(x + p)^2 + q$ where $a > 0$, $p = 1$, $q < 0$.

EXERCISE 5.14

1. Copy and complete the tables of values for the given functions:

a) $f(x) = 2x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$									

b) $g(x) = 2(x - 1)^2 - 1$

x	–4	–3	–2	–1	0	1	2	3	4
$g(x)$									

c) $h(x) = 2(x - 1)^2 - 2$

x	–4	–3	–2	–1	0	1	2	3	4
$h(x)$									

2. On the same system of axes draw the graphs of $f(x) = 2x^2$, $g(x) = 2(x - 1)^2 - 1$ and $h(x) = 2(x - 1)^2 - 2$.

3. Use the graphs you drew in question 2 and any other information to complete the table below.

		$f(x) = 2x^2$	$g(x) = 2(x - 1)^2 - 1$	$h(x) = 2(x + 1)^2 - 2$
	Value of	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$
a)	x-intercepts			
b)	y-intercept			
c)	Axis of symmetry			
d)	Coordinates of TP	(0; 0)		
e)	Domain			
f)	Range			

The effect of q: $f(x) = a(x + p)^2 + q$, where $a < 0$, $p = -1$, $q > 0$

Let's investigate the effect of p on the graph of $a(x + p)^2 + q$ where $a < 0$, $p = -1$, $q > 0$.

EXERCISE 5.15

1. Copy and complete the tables of values for the given functions:

a) $f(x) = -2x^2$

x	−4	−3	−2	−1	0	1	2	3	4
$f(x)$									

b) $p(x) = -2(x + 1)^2 + 1$

x	−4	−3	−2	−1	0	1	2	3	4
$p(x)$									

c) $w(x) = -2(x + 1)^2 + 2$

x	−4	−3	−2	−1	0	1	2	3	4
$w(x)$									

2. On the same system of axes draw the graphs of $f(x) = -2x^2$, $p(x) = -2(x + 1)^2 + 1$ and $w(x) = -2(x + 1)^2 + 2$.

3. Use the information from the graphs (and any other information) to complete the table below.

		$f(x) = -2x^2$	$p(x) = -2(x + 1)^2 + 1$	$w(x) = -2(x + 1)^2 + 2$
a)	Value of	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$
b)	x-intercepts			
c)	y-intercept			
d)	Axis of symmetry			
e)	Coordinates of TP	(0; 0)		
f)	Domain			
g)	Range			

4. How does a positive value of q affect the graph of $f(x) = -2x^2$?

The effect of q: $f(x) = a(x + p)^2 + q$, where $a < 0$, $p = -1$, $q < 0$

Let's investigate the effect of p on the graph of $a(x + p)^2 + q$ where $a < 0$, $p = -1$, $q < 0$.

EXERCISE 5.16

1. Copy and complete the tables of values for the given functions:

a) $f(x) = -2x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$									

b) $g(x) = -2(x + 1)^2 - 1$

x	–4	–3	–2	–1	0	1	2	3	4
$g(x)$									

c) $h(x) = -2(x + 1)^2 - 2$

x	–4	–3	–2	–1	0	1	2	3	4
$h(x)$									

2. On the same system of axes draw the graphs of $f(x) = -2x^2$, $g(x) = -2(x + 1)^2 - 1$ and $h(x) = -2(x + 1)^2 - 2$.

3. Use the graphs you have drawn (and any other information) to complete the table below.

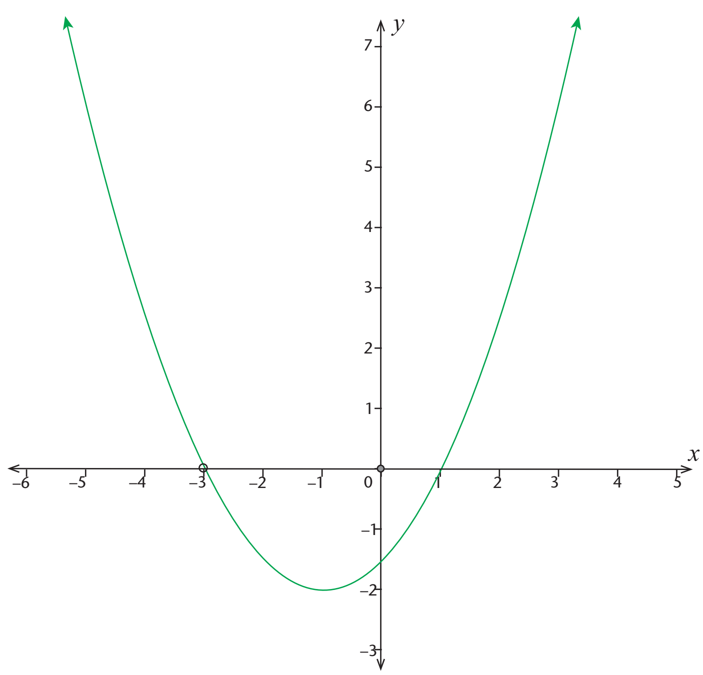
		$f(x) = -2x^2$	$g(x) = 2(x + 1)^2 - 1$	$h(x) = -2(x + 1)^2 - 2$
	Value of	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$	$a =$ $p =$ $q =$
a)	x-intercepts			
b)	y-intercept			
c)	Axis of symmetry			
d)	Coordinates of TP	(0; 0)		
e)	Domain			
f)	Range			

4. How does a negative value of q affect the graph of $f(x) = -2x^2$?

5. Study the following graphs then complete the table below.

Graph	x-intercepts	y-intercept	Axis of symmetry	Coordinates of the TP	Domain	Range	Function
a)							
b)							
c)							
d)							
e)							
f)							

a)



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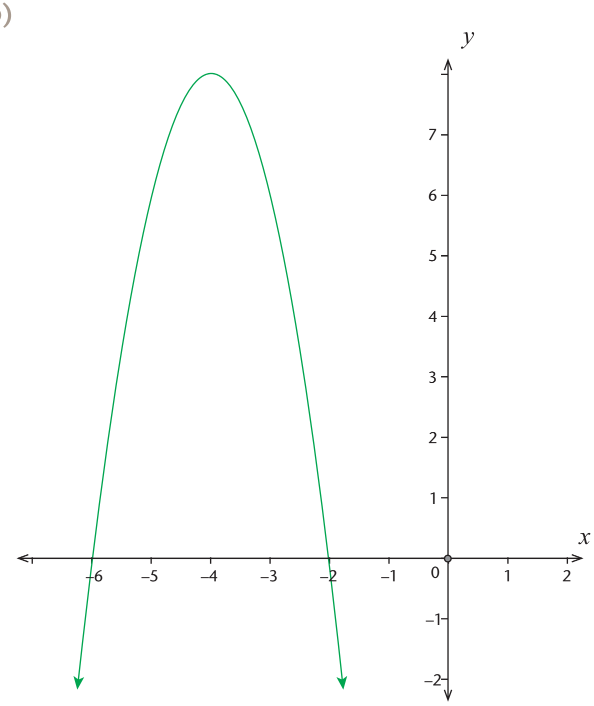

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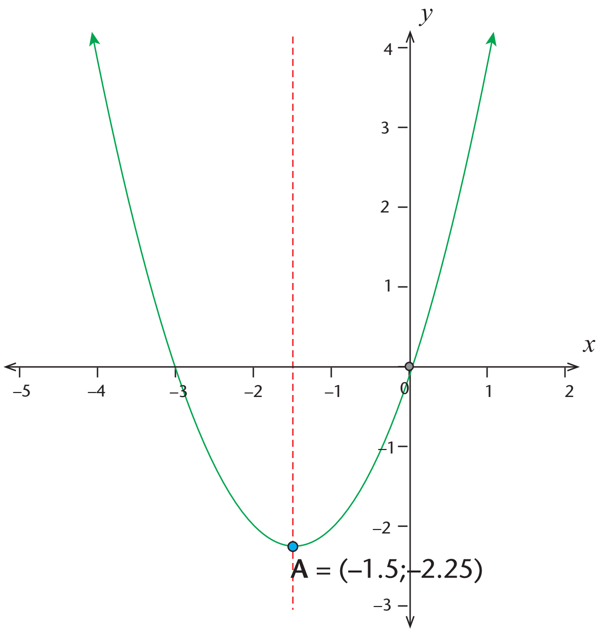

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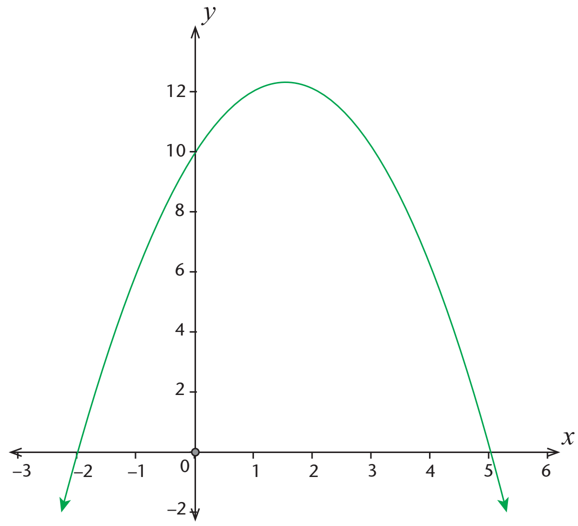

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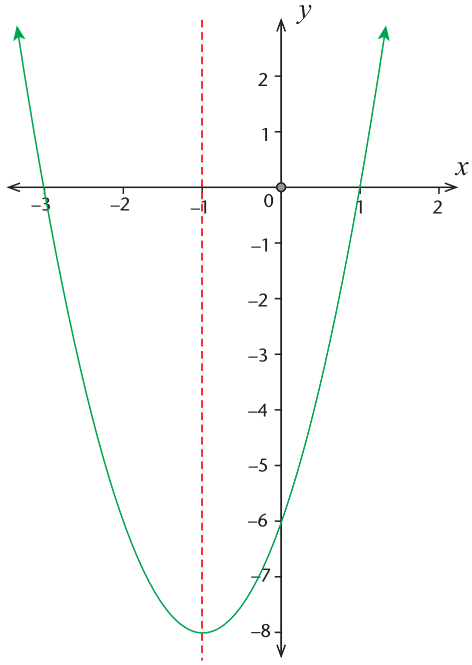

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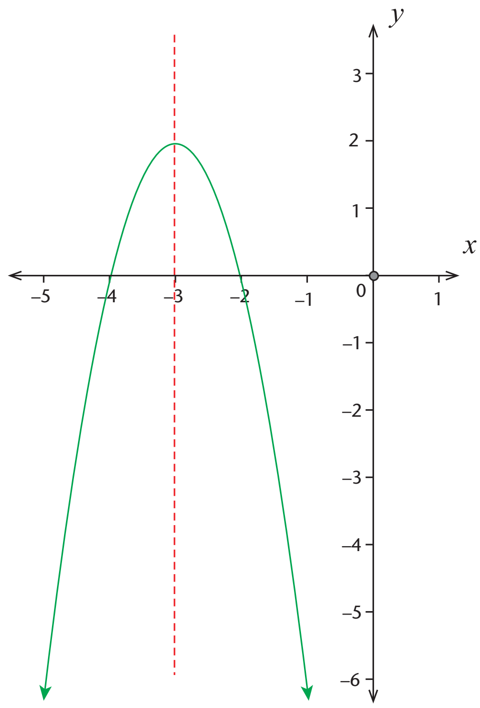


Note:
When we, for example, draw the graph of $f(x) = (x - 2)^2 - 1$ and compare it to the basic graph ($f(x) = x^2$), we can guess that the graph has moved two units to the right and one unit down. The axis of symmetry is now $x = 2$. We can also find where the graph goes over the y-axis without drawing the graph. We do this by calculating $f(0) = 3$ or by multiplying out the brackets and seeing that the constant term (the term without x) is 3. To find where the graph of $f(x) = (x - 2)^2 - 1$ crosses the x-axis, we put $y = f(x) = 0$ and then solve the equation for x.

The function also tells us how the basic graph is translated vertically and horizontally, it also tells us where the axis of symmetry is, and what the points of intersection with the x- and y-axes are.

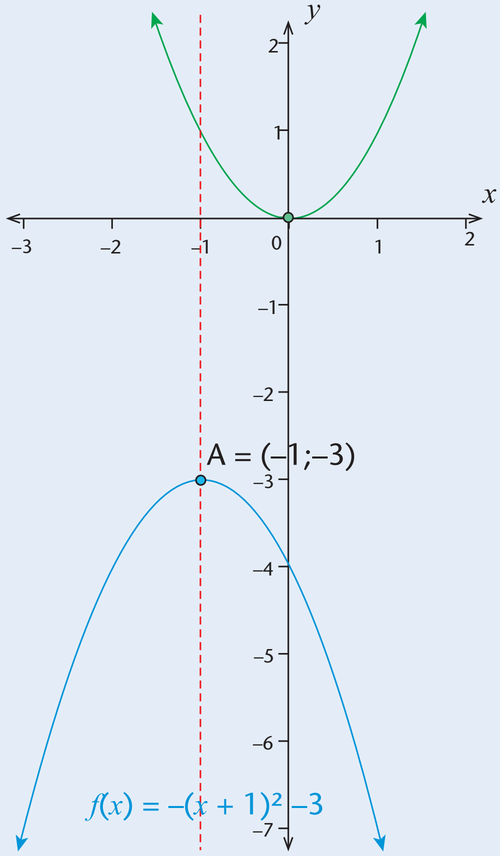
Worked example 5.2

The function $f(x) = -(x + 1)^2 - 3$ is given.

1. List the sequence of steps required to draw the graph of the function.
2. Draw the graph of $f(x) = -(x + 1)^2 - 3$.

Solution

1. Graph of $f(x) = -(x + 1)^2 - 3$ is a transformation of the graph of $f(x) = x^2$.
   a) The graph of $f(x) = x^2$ is translated 1 unit to the left: $(x + 1)^2 = (x - (-1))^2$.
   b) It is then reflected along the x-axis: $-(x + 1)^2$.
   c) It is then translated 3 units downwards: $-(x + 1)^2 - 3$.
2.



The questions below are based on the transformation of the graph $f(x) = x^2$.

EXERCISE 5.17

1. The graph of $f(x) = x^2$ has been transformed so that its line of symmetry is $x = -5$.
   a) What is the new equation?
   b) What are the coordinates of the turning point?
   c) Sketch the graph of the new equation.

2. The graph of $f(x) = x^2$ is transformed so that its new line of symmetry is $x = -5$. It is then translated 2 units up.
   a) What is the new equation?
   b) What are the coordinates of the turning point?
   c) Sketch the graph of the new equation.
3. The graph of $f(x) = x^2$ is transformed so that its new line of symmetry is $x = -5$. It is then translated 0,25 units down.
   a) What is the new equation?
   b) What are the coordinates of the turning point?
   c) Sketch the graph of the new equation.
4. The graph of $f(x) = x^2$ is transformed so that its turning point remains (0; 0) but now it passes through the point (−3; 2).
   a) What is the new equation?
   b) Sketch the graph of the new equation.
5. The graph of $f(x) = x^2$ is transformed so that its turning point (4; −9). It is exactly 3 times as steep as the graph of $f(x) = x^2$.
   a) What is the new equation?
   b) Sketch the graph of the new equation.

5.7 Using critical points to draw the graph of $f(x) = ax^2 + bx + c$

When drawing the graph of $f(x) = ax^2 + bx + c$, we need to determine the following critical points:

- the x-intercept
- the y-intercept
- the axis of symmetry, and
- the coordinates of the turning point.

To determine the critical points for:

	$f(x) = ax^2 + bx + c$	$f(x) = a(x - p)^2 + q$
Critical point	How it is calculated or information to look for:	
The x-intercepts	Let $y = 0$ and then solve the quadratic equation using: a) factorisation, or b) the quadratic formula.	
The y-intercept	Let $x = 0$ and calculate c	Let $x = 0$ or use $y = ap^2 + q$
The axis of symmetry	$x = -\frac{b}{2a}$	$x = p$
The coordinates of the turning point	By substituting $x = -\frac{b}{2a}$ in $f(x) = ax^2 + bx + c$ or by using $\left(-\frac{b}{2a};\ c - \frac{b^2}{4a}\right)$ to calculate the coordinates.	$(p; q)$

Worked example 5.3

Determine the critical points for the function $y = x^2 - 5x + 6$, then sketch the graph.

Solution

a) x-intercepts: Let $y = 0$

$x^2 - 5x + 6 = 0$

$(x - 3)(x - 2) = 0$

$x - 3 = 0$ or $x - 2 = 0$

$x = 3$ or $x = 2$

b) y-intercept: Let $x = 0$

$y = 0^2 - 5(0) + 6$

$y = 6$

c) Axis of symmetry: $x = -\frac{b}{2a} = -\frac{-(-5)}{2(1)} = \frac{5}{2}$

d) Turning point: y-coordinate

$f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right)$ or $y = c - \frac{b^2}{4a}$ or $y = \frac{4ac - b^2}{4a}$

$= \frac{25}{4} - \frac{25}{2} + 6$ or $y = 6 - \frac{5^2}{4(1)}$

$= \frac{25 - 50 + 24}{4}$ or $y = 6 - \frac{25}{4}$

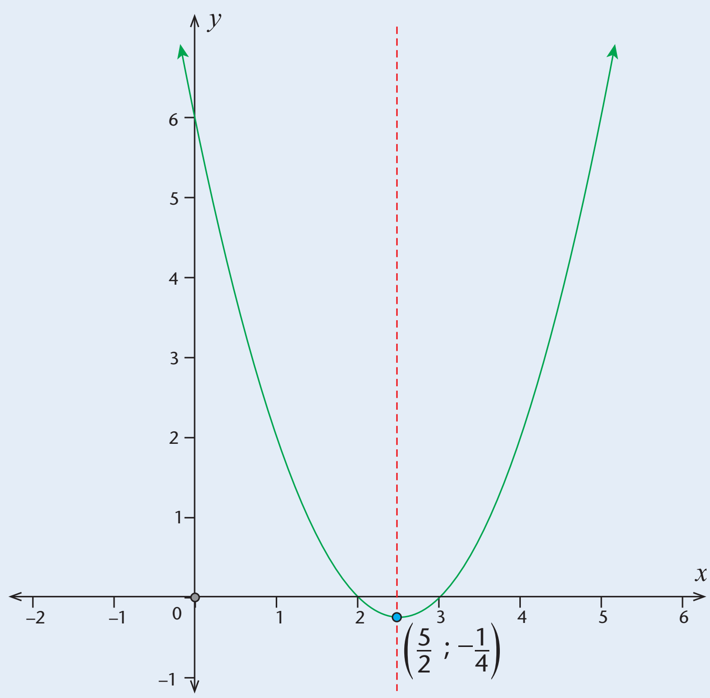
$= -\frac{1}{4}$ or $y = \frac{24 - 25}{4}$

or $y = -\frac{1}{4}$

Turning point: $\left(\frac{5}{2}; -\frac{1}{4}\right)$

$\left(2\frac{1}{2}; -\frac{1}{4}\right)$

e) Graph



$y = x^2 - 5x + 6$

EXERCISE 5.18

Determine the critical points for each function, then sketch each graph on a separate system of x-y axes:

a) $y = x^2 - x - 2$

b) $y = x^2 + 6x + 8$

c) $y = x^2 + 2x - 8$

d) $y = -x^2 + x + 2$

e) $y = -x^2 - 2x + 8$

f) $y = -x^2 - 7x - 6$

g) $y = 2x^2 + 5x - 3$

h) $y = 3x^2 + 5x - 2$

i) $y = -3x^2 + 7x - 2$

j) $y = -x^2 - \frac{3}{2}x + 1$

k) $y = (x - 2)(x + 3)$

l) $y = -3(x + 1)(x - 2)$

Worked example 5.4

Determine the critical points for the function $y = 2(x + 1)^2 - 3$, then sketch the graph.

Solution

a) Axis of symmetry:

$y = 2(x + 1)^2 - 3$

$y = 2(x - (-1))^2 - 3$

$p = -1$

b) Coordinates of the turning point:

$(p; q) = (-1; -3)$

c) y-intercept: Let $x = 0$

$y = 2(0 + 1)^2 - 3$

$y = -1$

d) x-intercepts: Let $y = 0$

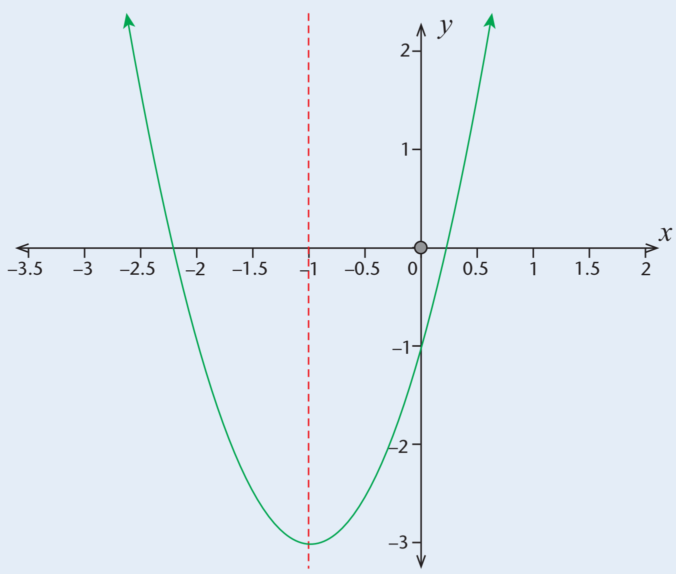
$2(x + 1)^2 - 3 = 0$

$2(x + 1)^2 = 3$

$(x + 1)^2 = \frac{3}{2}$

$x + 1 = \pm\sqrt{\frac{3}{2}}$

$x = 1 \pm \sqrt{\frac{3}{2}}$



EXERCISE 5.19

Determine the critical points for each function, then sketch each graph on a separate system of x-y axes:

a) $y = -(x + 2)^2 + 1$

b) $y = 3(x - 3)^2 - 3$

c) $y = -(x + 2)^2 - 1$

d) $y = \frac{1}{4}(x + 4)^2$

e) $y = 4\left(x - \frac{1}{4}\right)^2$

f) $y = -3(x + 3)^2 - 3$

5.8 An exponential graph: $y = ab^x$

An exponential function with base, b is defined by the function $f(x) = ab^x$, where $a \neq 0$, $b > 0$ and $b \neq 1$, and x is any real number. The base, b, is constant and the exponent, x, is a variable.

The features of an exponential graph are:

- the domain is all Real numbers
- the range is all positive Real numbers (not zero)
- the graph has a y-intercept at (0; 1).
- when $b > 1$, the graph increases. The greater the base, b, the faster the graph rises from left to right
- when $0 < b < 1$, the graph decreases
- has an asymptote (a line that the graph gets very, very close to, but never crosses or touches). For this graph the asymptote is the x-axis ($y = 0$).

Note:
Any number to the power of zero equals 1.

Worked example 5.5

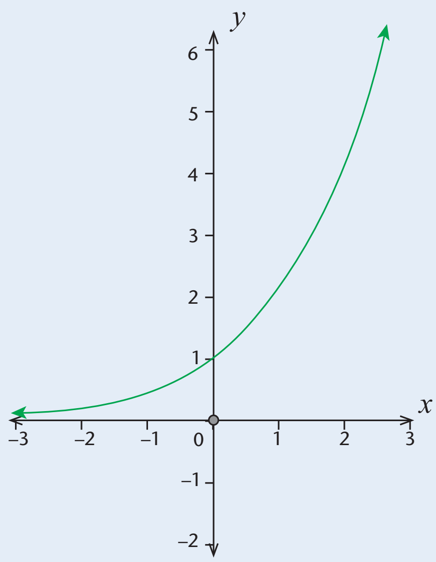
1. Draw the graph of $y = 2^x$ by:
   a) completing a table of values and plotting the points on a system of axis.
   b) joining the points you have plotted to form a smooth curve.
2. List the features of the graph.

Solution

1. a)

x	–4	–3	–2	–1	0	1	2	3	4
$y = 2^x$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

b)



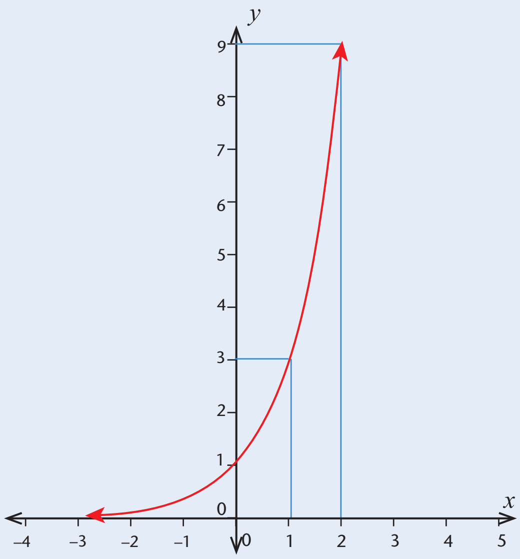
2. Features of the graph
   a) Domain: $x \in \mathbb{R}$
   b) Range: $y > 0$
   c) Intercept(s): $y = 1$ or (0; 1)
   d) As the values of x get smaller and smaller (in the example they decrease from 4 to –4), the values of y get closer and closer to 0.
   e) As the values of x increase (from –4 to 4 in our example), the values of y also increase. We say the function $y = 2^x$ is an increasing function.
   f) The graph is asymptote to the x-axis.

Worked example 5.6

The equation for the graph alongside is in the form $y = a^x$. Calculate the value of a and write the equation for the graph.

Solution

Reading from the graph, the following is established:
when $x = 0$ the corresponding y value is 1, that is, $1 = a^0$
when $x = 1$ the corresponding y value is 3, that is $3 = a^1$
when $x = 2$ the corresponding y value is 9, that is $9 = a^2$.
$\therefore a = 3$ and the equation of the graph is $y = 3^x$



EXERCISE 5.20

1. Use the table method to draw each graph below on a separate set of axes:
   a) $g(x) = 3^x$
   b) $q(x) = 4^x$
   c) $t(x) = 5^x$
   d) $f(x) = 1{,}25^x$
   e) $h(x) = 2{,}5^x$
   f) $y = 3{,}2^x$
   g) $y = 2{,}2^x$
   h) $y = 2{,}3^x$
   i) $y = 2{,}4^x$
2. Copy and complete the table below in your exercise book.

	Function	Intercept	Domain	Range	Asymptote
a)	$g(x) = 3^x$				
b)	$q(x) = 4^x$				
c)	$t(x) = 5^x$				
d)	$f(x) = 1{,}25^x$				
e)	$h(x) = 2{,}5^x$				
f)	$y = 3{,}2^x$				
g)	$y = 2{,}2^x$				
h)	$y = 2{,}3^x$				
i)	$y = 2{,}4^x$				

The effect of a and q on the exponential graph: $y = ab^x + q$, where $a > 0$

Let's investigate the effect of a and q on the exponential graph of $y = ab^x + q$, where $a > 0$.

EXERCISE 5.21

1. Copy and complete the tables below.
   a) $g(x) = 3^x + 2$

x	−6	−5	−4	−3	−2	1	2	3	4	5	6
$g(x)$											

   b) $q(x) = 4^x - 1$

x	−6	−5	−4	−3	−2	1	2	3	4	5	6
$q(x)$											

   c) $y = 3{,}2^x + 1{,}5$

x	−6	−5	−4	−3	−2	1	2	3	4	5	6
y											

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d) $y = 2{,}3^x - 3$

x	−6	−5	−4	−3	−2	1	2	3	4	5	6
y											

2. On the same system of axes draw the graphs of the defined functions:
   a) $y = 3^x$ and $g(x) = 3^x + 2$
   b) $y = 4^x$ and $q(x) = 4^x - 1$
   c) $y = 2^x$; $p(x) = 3{,}2^x$ and $r(x) = 3{,}2^x + 1{,}5$
   d) $y = 3^x$; $t(x) = 2{,}3^x$ and $k(x) = 2{,}3^x - 3$
3. Complete the table below.

	Function	Intercepts	Asymptotes	Domain	Range
a)	$y = 3^x$				
	$g(x) = 3^x + 2$				
b)	$y = 4^x$				
	$q(x) = 4^x - 1$				
c)	$y = 2^x$				
	$p(x) = 3{,}2^x$				
	$r(x) = 3{,}2^x + 1{,}5$				
d)	$y = 3^x$				
	$t(x) = 2{,}3^x$				
	$k(x) = 2{,}3^x - 3$				

4. Write a sentence to describe the effect of:
   a) a on the graph of an exponential function; and
   b) q on the graph of an exponential function.

The effect of a and q on the exponential graph: $y = ab^x + q$, where $a < 0$

Let's investigate the effect of a and q on the exponential graph of $y = ab^x + q$, where $a < 0$.

EXERCISE 5.22

1. Copy and complete the table below.
   a) $h(x) = -2^x$

x	−6	−5	−4	−3	−2	1	2	3	4	5	6
$h(x)$											

b) $g(x) = -3^x + 2$

x	−6	−5	−4	−3	−2	1	2	3	4	5	6
$g(x)$											

c) $q(x) = -4^x - 1$

x	−6	−5	−4	−3	−2	1	2	3	4	5	6
$q(x)$											

d) $r(x) = -3{,}2^x + 1{,}5$

x	−6	−5	−4	−3	−2	1	2	3	4	5	6
y											

e) $k(x) = -2{,}3^x - 3$

x	−6	−5	−4	−3	−2	1	2	3	4	5	6
y											

2. On the same system of axes draw the graphs of the defined functions:
   a) $y = 2^x$ and $h(x) = -2^x$
   b) $y = 3^x$ and $g(x) = -3^x + 2$
   c) $y = 4^x$ and $q(x) = -4^x - 1$
   d) $y = 2^x$, $p(x) = -3{,}2^x$ and $r(x) = -3{,}2^x + 1{,}5$
   e) $y = 3^x$, $t(x) = -2{,}3^x$ and $k(x) = -2{,}3^x - 3$
3. Complete the table below.

	Function	Intercepts	Asymptotes	Domain	Range
a)	$y = 2^x$ $h(x) = -2^x$				
b)	$y = 3^x$				
	$g(x) = -3^x + 2$				
c)	$y = 4^x$				
	$q(x) = -4^x - 1$				
d)	$y = 2^x$				
	$p(x) = -3{,}2^x$				
	$r(x) = -3{,}2^x + 1{,}5$				
e)	$y = 3^x$				
	$t(x) = -2{,}3^x$				
	$k(x) = -2{,}3^x - 3$				

4. Consider the function given by $f(x) = 7^x$. What should the new equation be that would result in the graph of the given function being translated:
   a) 3 units downwards?
   b) 1,5 units downwards?
5. Write down the equation of the new function
   a) if $f(x) = 6^x$ is reflected along the x-axis.
   b) if f is translated 1,25 units up and reflected along the y-axis.

Sketching the graph of $f(x) = ab^x + q$

So far we have drawn graphs by plotting the points from the tables on a system of axes and joining them. By doing this, we should be able to visualise what a particular graph should look like.

We can also use the guidelines below to draw a graph of an exponential function $f(x) = ab^x + q$:

- if $a > 0$, then the graph of $f(x) = abx + q$ is about growth.
- basic exponential functions, $f(x) = bx$, pass through the point (0; 1).
  - the graph of $f(x) = abx + q$, shifts the graph of $f(x) = bx$ along the y-axis and the point (0; 1) will move up by q units to the point $(0; 1 + q)$.
  - the graph of $f(x) = abx - q$, shifts the graph of $f(x) = bx$ along the y-axis and the point (0; 1) will move down by q units to the point $(0; 1 - q)$.
  - basic exponential functions, $f(x) = bx$, have a horizontal asymptote at $y = 0$.
  - the graph of $f(x) = abx + q$ shifts the horizontal asymptote q units up.
  - the graph of $f(x) = abx - q$ shifts the horizontal asymptote q units down.
  - plot a few points that are on the graph of $f(x) = abx + q$.

EXERCISE 5.23

1. On a separate system of axes sketch the graphs of the functions defined by:
   a) $y = 3^x + 3$
   b) $y = 2{,}3^x + 3$
   c) $y = 3{,}2^x + 3$
   d) $y = 3^x - 3$
   e) $y = 2{,}3^x - 3$
   f) $y = 3{,}2^x - 3$
2. Copy and complete the table.

	Function	Domain	Range	Horizontal asymptote
a)	$y = 3^x + 3$			
b)	$y = 2{,}3^x + 3$			
c)	$y = 3{,}2^x + 3$			
d)	$y = 3^x - 3$			
e)	$y = 2{,}3^x - 3$			
f)	$y = 3{,}2^x - 3$			

5.9 The hyperbola: $y = \frac{a}{x}$; where $a \neq 0$

Suppose we want to draw the graph of a function given by $y = \frac{12}{x}$.
Let's look at an example of the steps to follow.

Worked example 5.7

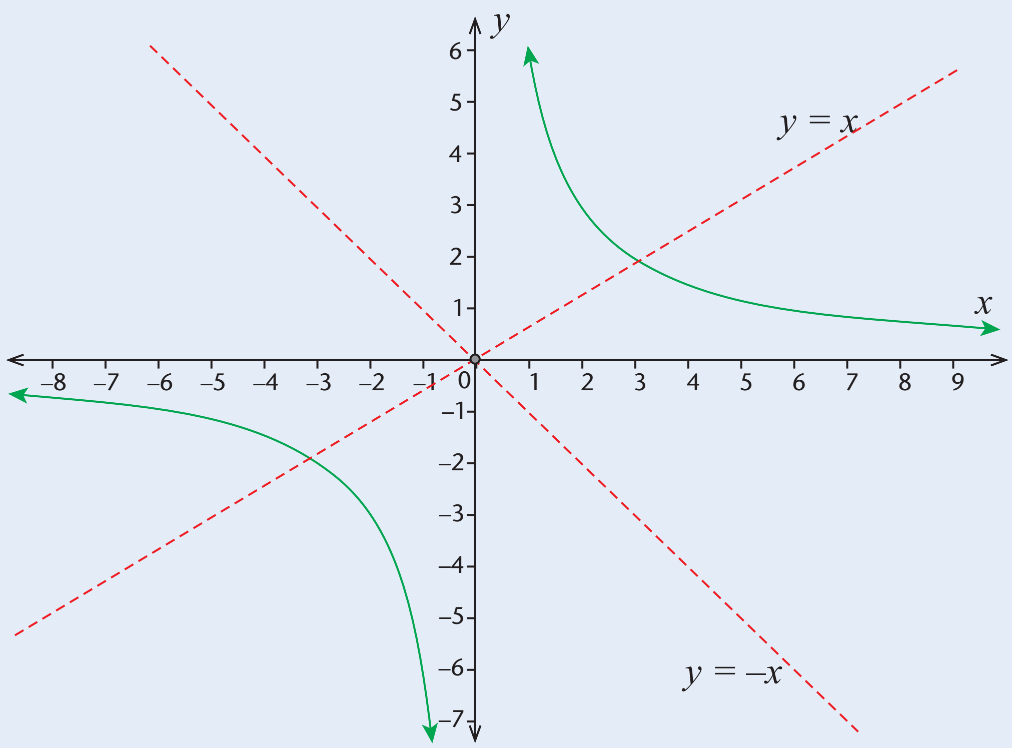
Draw the graph of a function given by $y = \frac{12}{x}$.

Solution

Step 1: Rewrite this function as $xy = 12$ and then complete a table showing the relationship between the x- and y-values:

x	–6	–3	–2	–1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	6
y	–1	–2	–3	–6	–12	12	6	3	2	1

Step 2: Use the values in the table of values to plot the graph:



Step 3: List the features of the hyperbola you have drawn:

a) Domain: $\{x : x \in \mathbb{R}, x \neq 0\}$
b) Range: $\{y : y \in \mathbb{R}, y \neq 0\}$
c) The function has a discontinuity at $x = 0$.
d) There are two asymptotes:
   (i) a horizontal asymptote at $y = 0$. (ii) a vertical asymptote at $x = 0$.
e) The graph has axes of symmetry $y = -x$ and $y = x$. About this line, one half of the hyperbola is a mirror image of the other half.

EXERCISE 5.24

1. Use the table method to draw the graphs of the functions defined below. Use separate axes for each graph.

a) $y = \frac{-12}{x}$
b) $y = \frac{6}{x}$
c) $y = \frac{-6}{x}$
d) $y = \frac{8}{x}$
e) $y = \frac{-8}{x}$
f) $y = \frac{-10}{x}$
g) $y = \frac{16}{x}$
h) $y = \frac{20}{x}$
i) $xy = 1$

2. Complete the table

	Function	Domain	Range	Line of symmetry	Asymptote(s)
a)	$y = \frac{-12}{x}$				
b)	$y = \frac{6}{x}$				
c)	$y = \frac{-6}{x}$				
d)	$y = \frac{8}{x}$				
e)	$y = \frac{-8}{x}$				
f)	$y = \frac{-10}{x}$				
g)	$y = \frac{16}{x}$				
h)	$y = \frac{20}{x}$				
i)	$xy = 1$				

The effect of q: $y = \frac{a}{x} + q$; where $a \neq 0$

Let's investigate the effect of q on $y = \frac{a}{x} + q$; where $a \neq 0$.

EXERCISE 5.25

1. Copy and complete the tables for each function given below.

a) $y = -\frac{12}{x}$

x									
y									

b) $y = -\frac{12}{x} + 1$

x									
y									

c) $y = -\frac{12}{x} - 1$

x									
y									

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d) $y = \frac{8}{x}$

x									
y									

e) $y = \frac{8}{x} + 1$

x									
y									

f) $y = \frac{8}{x} - 1$

x									
y									

Sketching the hyperbola: $y = \frac{a}{x} + q$; where $a \neq 0$

To sketch a hyperbola, we need to know (i) its shape; (ii) the intercepts and (iii) the asymptotes.

Worked example 5.8

Sketch the graph of $y = \frac{4}{x} + 2$; $x \neq 0$

Solution

(i) x-intercept:

Let $y = 0$

$y = \frac{4}{x} + 2 = 0$

$\frac{4}{x} = -2$

$-2x = 4$

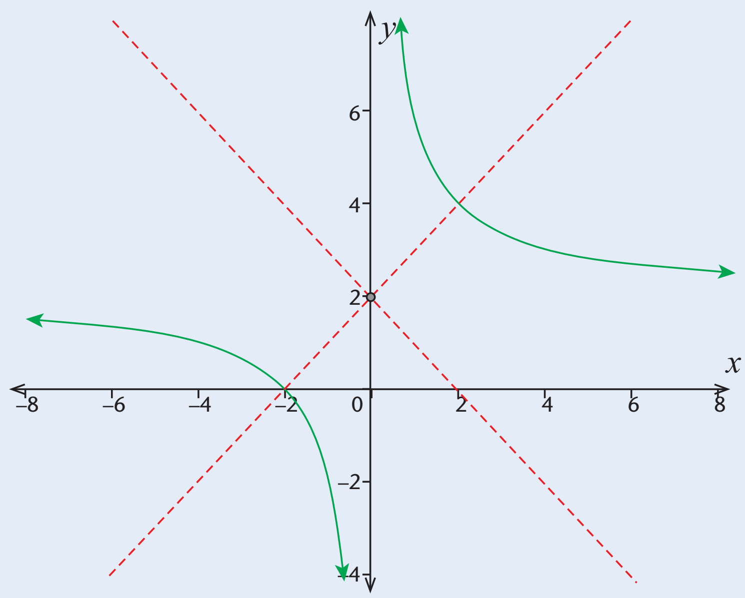
$x = -2$

(ii) The asymptotes:

$y = 2$ and $x = 0$

(iii) Lines of symmetry:

$y = -x + 2$ and $y = x + 2$



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EXERCISE 5.26

1. On the same system of x-y axes draw the graphs of:
   a) $y = \frac{-12}{x}$ and $y = -\frac{12}{x} + 1$
   b) $y = \frac{-12}{x}$ and $y = -\frac{12}{x} - 1$
   c) $y = \frac{8}{x}$ and $y = \frac{8}{x} + 1$
   d) $y = \frac{8}{x}$ and $y = \frac{8}{x} - 1$
2. Copy and complete the table below.

	Function	Domain	Range	Lines of symmetry	Asymptote(s)	
					Horizontal	Vertical
a)	$y = \frac{-12}{x}$					
b)	$y = -\frac{12}{x} + 1$					
c)	$y = -\frac{12}{x} - 1$					
d)	$y = \frac{8}{x}$					
e)	$y = \frac{8}{x} + 1$					
f)	$y = \frac{8}{x} - 1$					

3. What is the effect of q on the graph of $y = \frac{a}{x}$; where $a \neq 0$? Give examples.
4. Sketch the hyperbolas below, showing the:
   (i) asymptotes (ii) intercepts, and (iii) lines of symmetry
   a) $y = \frac{6}{x} + 3$; where $x \neq 0$
   b) $y = \frac{4}{x} - 1$; where $x \neq 0$
   c) $y = -\frac{6}{x} + 3$; where $x \neq 0$
   d) $y = -\frac{4}{x} - 1$; where $x \neq 0$
   e) $y = \frac{4}{x} + 1$; where $x \neq 0$

5.10 The equation of the circle with the center at the origin: $x^2 + y^2 = r^2$

We define the circle as a set of points that are at a fixed distance from the centre.

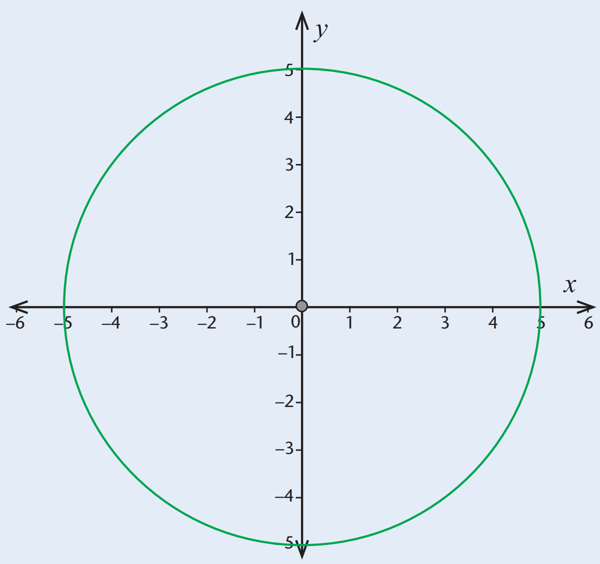
Worked example 5.9

1. Draw the graph of $x^2 + y^2 = 25$ by first constructing a table of values and then plotting the points on the x-y plane.
2. State the domain and range of the graph.

Solution

1.

x	–5	–4	–3	0	0	3	4	5
y	0	–3	–4	–5	5	4	3	0
$x^2 + y^2 = r^2$	25	25	25	25	25	25	25	25
r^2	5^2	5^2	5^2	5^2	5^2	5^2	5^2	5^2



2. The domain: $-5 \leq x \leq 5$
   The range: $-5 \leq y \leq 5$

EXERCISE 5.27

1. Copy and complete the tables for the given function:

a) $x^2 + y^2 = 16$

x	–4	–3	–2	–1	0	0	1	2	3	4
y										
$x^2 + y^2 = 16$										

b) $x^2 + y^2 = 16$

x	–3	–2	–1	0	0	1	2	3
y								
$x^2 + y^2 = 9$								

c) $x^2 + y^2 = 36$

x	–6				0	0				6
y										
$x^2 + y^2 = 36$										

d) $x^2 + y^2 = 49$

x	–7				0	0				7
y										
$x^2 + y^2 = 49$										

e) $x^2 + y^2 = 64$

x	–8				0	0				8
y										
$x^2 + y^2 = 64$										

f) $x^2 + y^2 = 81$

x	–9				0	0				9
y										
$x^2 + y^2 = 81$										

g) $x^2 + y^2 = 100$

x	–10				0	0				10
y										
$x^2 + y^2 = 100$										

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2. Draw each graph on a separate systems of axes.
3. Complete the table below:

	Equation	Domain	Range
a)	$x^2 + y^2 = 16$		
b)	$x^2 + y^2 = 9$		
c)	$x^2 + y^2 = 36$		
d)	$x^2 + y^2 = 49$		
e)	$x^2 + y^2 = 64$		
f)	$x^2 + y^2 = 81$		
g)	$x^2 + y^2 = 100$		

4. What is the equation of a circle whose centre is at the origin and whose diameter is:
   a) 10 cm b) 12 cm c) 7 cm
5. Write down the equation of the circle with the centre at the origin and the given radii.
   a) $r = 4$ units b) $r = 13$ units c) $r = 17$ units

Writing the equation of a circle through a point and centred at the origin

If we take a circle and place it on the Cartesian plane so that its centre is at the origin, then we can calculate its equation. The equation of a circle placed at the centre of the Cartesian plane is $x^2 + y^2 = r^2$; where r is the radius and $(x; y)$ is any point through which the circle passes.

Worked example 5.10

Let's consider a circle centred at the origin, which passes through the point (3; 1).

a) What is the equation of the circle?
b) Calculate the radius of the circle.
c) Give the coordinates of another point through which the circle passes.

Solution

a) Using the equation of a circle whose origin is at the centre we have:

$x^2 + y^2 = r^2$

$(3)^2 + (1)^2 = r^2$

$9 + 1 = r^2$

$10 = r^2$

The equation of the circle is $x^2 + y^2 = 10$

b) The radius of the circle is $r = \sqrt{10}$.

c) Another point which is the same distance from the origin as point (1; 3).

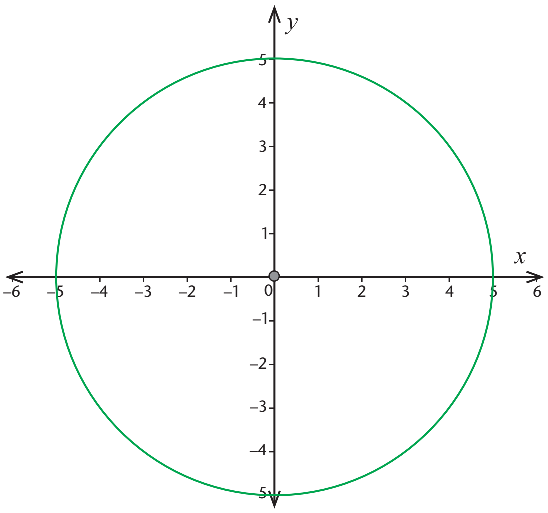
EXERCISE 5.28

1. Write the equation of the circle that passes through the given point and is centred at the origin.

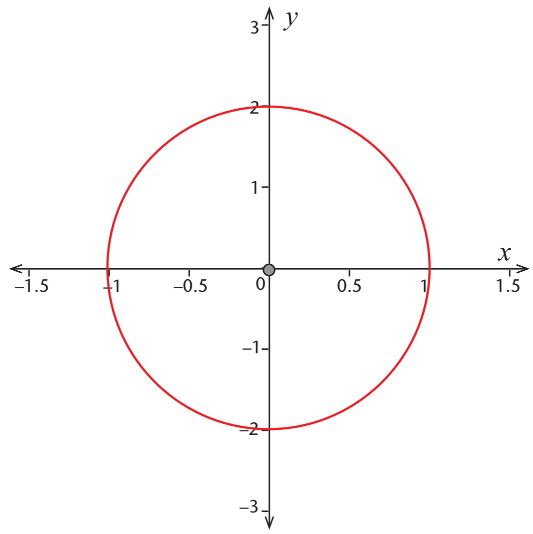
a) (1; 4)	e) (2; −1)	i) (0; 5)
b) (−4; −1)	f) (1; 2)	j) (5; 0)
c) (−1; 4)	g) (−2; −1)	k) (−5; 0)
d) (4; 1)	h) (−2; 1)	l) (0; −5)

2. Write down the equations of each of the circles drawn below:

a)



b)



Sketching circles given their equations

Follow these guidelines for drawing the graph of a circle given its equation. To draw a graph of a circle with its centre at the origin:

1. first identify and label the origin as the center of the circle
2. identify and label the two x-intercepts as $(r; 0)$ and $(-r; 0)$
3. identify and label the two y-intercepts as $(0; r)$ and $(0; -r)$
4. sketch the circle containing the intercepts.

EXERCISE 5.29

1. Draw the graphs of the following circles on the same system of x-y axes.

a) $x^2 + y^2 = 1$	e) $x^2 + y^2 = 25$	i) $x^2 + y^2 = 81$
b) $x^2 + y^2 = 4$	f) $x^2 + y^2 = 36$	j) $x^2 + y^2 = 100$
c) $x^2 + y^2 = 9$	g) $x^2 + y^2 = 49$	k) $x^2 + y^2 = 17$
d) $x^2 + y^2 = 16$	h) $x^2 + y^2 = 64$	l) $x^2 + y^2 = 13$

The equation of the semi- circle with the center at the origin: $y = \pm\sqrt{r^2 - x^2}$

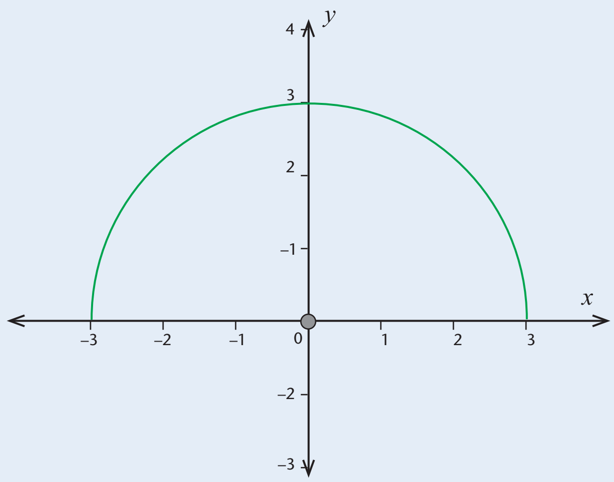
Consider the equation of the circle centred at the origin $x^2 + y^2 = r^2$. If we make y the subject of the formula, we get $y = \pm\sqrt{r^2 - x^2}$. We can rewrite this equation as two separate equations: $y = \sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$. These are equations of the two halves of the circle, and each half is called a semi-circle.

Worked example 5.11

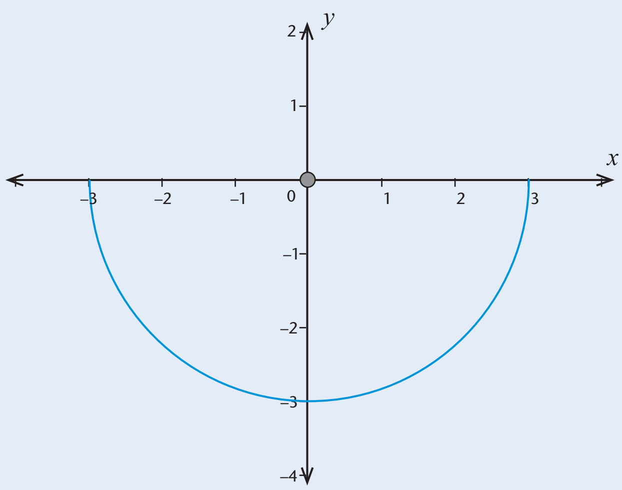
1. Draw the graphs of the functions defined by:
   a) $y = \sqrt{9 - x^2}$
   b) $y = -\sqrt{9 - x^2}$
2. State (i) the domain and (ii) the range of each function.

Solution

1. a) The graph of $y = \sqrt{9 - x^2}$



b) The graph of $y = -\sqrt{9 - x^2}$



2.		$y = \sqrt{9 - x^2}$	$y = -\sqrt{9 - x^2}$
a)	Domain	$-3 \le x \le 3$	$-3 \le x \le 3$
b)	Range	$0 \le x \le 3$	$-3 \le x \le 0$

EXERCISE 5.30

1. Draw the graphs for:
   a) $y = -\sqrt{121 - x^2}$
   b) $y = \sqrt{225 - x^2}$
   c) $y = \sqrt{625 - x^2}$
   d) $y = -\sqrt{196 - x^2}$
   e) $y = -\sqrt{961 - x^2}$
   f) $y = \sqrt{169 - x^2}$
2. Copy and complete the table below.

	Function	Domain	Range
a)	$y = -\sqrt{121 - x^2}$		
b)	$y = \sqrt{225 - x^2}$		
c)	$y = \sqrt{625 - x^2}$		
d)	$y = -\sqrt{196 - x^2}$		
e)	$y = -\sqrt{961 - x^2}$		
f)	$y = \sqrt{169 - x^2}$		

CONSOLIDATION EXERCISE

1. Sketch the graphs of the following functions:
   a) $y = -2(x - 1)^2$
   b) $y = -2(x - 1)^2 + 3$
   c) $y = (x + 3)(x - 6)$
   d) $y = x^2 - 4x + 3$
   e) $y = x^2 + 2x + 3$
   f) $y = 2x^2 - 4x + 1$
2. Copy and complete the table below. You may refer to the graphs you have already drawn when completing the table.

	Function	x-inter-cept(s)	y-inter-cept	Axis of symmetry	Turning point	Max/ Min Value	Domain	Range
a)	$y = -2(x - 1)^2$							
b)	$y = -2(x - 1)^2 + 3$							
c)	$y = (x + 3)(x - 6)$							
d)	$y = x^2 - 4x + 3$							
e)	$y = x^2 + 2x + 3$							
f)	$y = 2x^2 - 4x + 1$							

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3. Describe the transformation of $f(x) = x^2$ represented by g. Then make a neat rough sketch of the graph of each function.

a) $g(x) = (x-4)^2$
b) $g(x) = (x-4)^2 + 1$
c) $g(x) = 2(x-4)^2 + 1$
d) $g(x) = -2(x-1)^2 + 1$
e) $g(x) = \frac{1}{2}(x-4)^2 + 1$

4. Copy and complete the table below.

	Function	x-inter-cept(s)	y-inter-cept	Axis of symmetry	Turning point	Max/Min Value	Domain	Range
a)	$g(x) = (x-4)^2$							
b)	$g(x) = (x-4)^2 + 1$							
c)	$g(x) = 2(x-4)^2 + 1$							
d)	$g(x) = -2(x-1)^2 + 1$							
e)	$g(x) = \frac{1}{2}(x-4)^2 + 1$							

5. Sketch the following graphs.

a) $y = 2^x - \frac{1}{2}$
b) $y = 3^x + 1$
c) $y = 5^x - \frac{5}{4}$
d) $y = -3^x + 1$

6. Copy and complete the table below.

	Function	x-intercept	y-intercept	Domain	Range	Horizontal asymptote
a)	$y = 2^x - \frac{1}{2}$					
b)	$y = 3^x + 1$					
c)	$y = 5^x - \frac{5}{4}$					
d)	$y = -3^x + 1$					

7. Sketch the graphs of the functions.

a) $y = \frac{8}{x} - 3$
b) $y = \frac{4}{x} + 1$
c) $y = -\frac{2}{x} + 1$

8. Copy and complete the table below.

	Function	x-intercept	y-intercept	Domain	Range	Equation of the asymptote
a)	$y = \frac{8}{x} - 3$					
b)	$y = \frac{4}{x} + 1$					
c)	$y = -\frac{2}{x} + 1$					

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9. a) Write the equation of the circle centered at the origin with radius $\frac{7}{2}$.
   b) State the domain of the circle.
   c) State the range of the circle.
   d) Draw the circle.
10. a) Write the equation of the circle whose centre is (0; 0) and passes through the point (–3; 3).
   b) State the domain of the circle.
   c) State the range of the circle.
   d) Draw the circle.
11. a) Sketch the function $y = \sqrt{25 - x^2}$.
   b) State the domain of the function $y = \sqrt{25 - x^2}$.
   c) State the range of the function $y = \sqrt{25 - x^2}$.
12. a) Sketch the function $y = -\sqrt{169 - x^2}$.
   b) State the domain of the function $y = -\sqrt{169 - x^2}$.
   c) State the range of the function $y = -\sqrt{169 - x^2}$.

Summary

- the equation of a straight line that passes through two points can be determined by using the following formula: $y = mx + c$
- when you sketch the graph of a quadratic function (parabola) you need to identify the following:
  - the x-intercept(s), if they exist, by solving the equation $ax^2 + bx + c = 0$
  - the y-intercept, $(0; c)$
  - the coordinates of the turning point, are given by $\left(-\frac{b}{2a}; f\left(-\frac{b}{2a}\right)\right)$
  - the vertex form of a quadratic function is given by $f(x) = a(x - p)^2 + q$, where $a \neq 0$ and the coordinates of the turning point are $(p; q)$.

$$f(x) = a(x - p)^2 + q$$

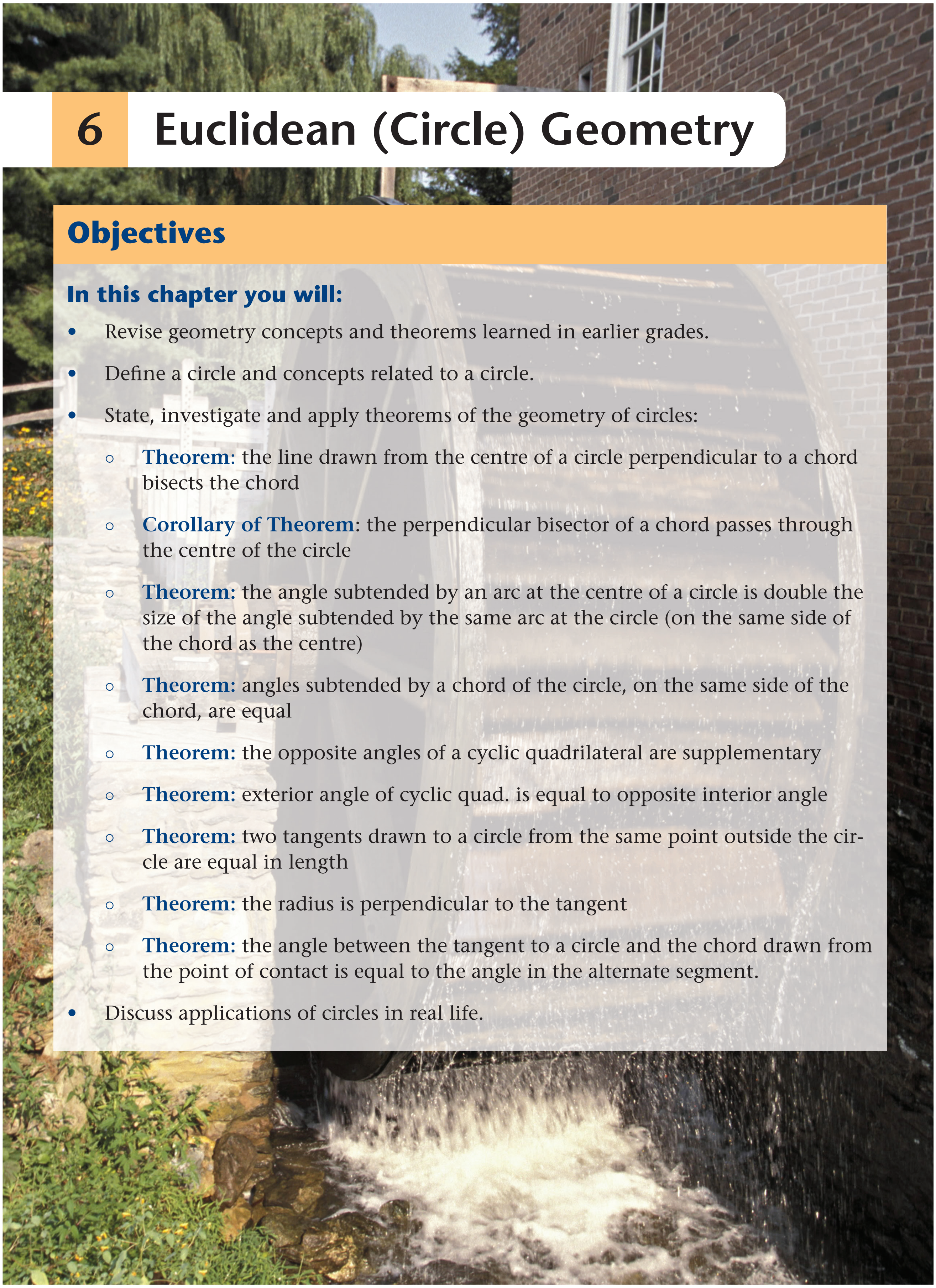
a indicates a reflection in the *x*-axis and/or a vertical stretch or shrink

p indicates a horizontal translation

q indicates a vertical translation

- in an exponential function $f(x) = ab^x + q$, q shifts the graph of $f(x) = ab^x$, q units along the y-axis:
  - the graph of $f(x) = ab^x + q$ passes through the point $(0; a \pm q)$
  - for $a < 0$, the graph of $f(x) = ab^x + q$ is reflected about the x-axis
  - the domain is the set of all real numbers
  - the range is $y > 0$, $y \in \mathbb{R}$
  - the graph has a horizontal asymptote at $y = q$.
- the equation $y = \frac{a}{x} + q$ is that of a hyperbola with the following features:
  - the horizontal asymptote has the equation $y = q$. This is the line passing through the point (0; q) and parallel to the x-axis
  - the vertical asymptote is $x = 0$ This is the line passing through the point (0; 0) and parallel to the y-axis
  - the domain is the set of all real numbers except $x = 0$
  - the range is the set of all real numbers except $y = q$
  - if a is a positive number, then the branches of the hyperbola lie in the first and third quadrants
  - if a is a negative number, then the branches of the hyperbola lie in the second and fourth quadrants
  - if q is positive the graph shifts up
  - if q is negative the graph shifts down
  - q also shifts the horizontal asymptote
- the equation of the circle with the centre at the origin is given by $x^2 + y^2 = r^2$ and it has the following features:
  - the domain of $x^2 + y^2 = r^2$ is given by $-r \leq y \leq r$
  - the range of $x^2 + y^2 = r^2$ is given by $-r \leq y \leq r$
  - the semi-circle centred at the origin
  - the general form of a semi-circle whose centre is at the origin is given by $y = \pm\sqrt{r^2 - x^2}$
  - the top half of the circle is represented by $y = \sqrt{r^2 - x^2}$
  - the domain of $y = \sqrt{r^2 - x^2}$ is given by $-0 \leq x \leq r$
  - the range of $y = \sqrt{r^2 - x^2}$ is given by $-r \leq y \leq r$
  - the bottom half of the semi-circle is represented by $y = -\sqrt{r^2 - x^2}$
  - the domain of $y = -\sqrt{r^2 - x^2}$ is given by $-r \leq x \leq r$
  - the range of $y = -\sqrt{r^2 - x^2}$ is given by $-r \leq y \leq 0$

6 Euclidean (Circle) Geometry



Objectives

In this chapter you will:

- Revise geometry concepts and theorems learned in earlier grades.
- Define a circle and concepts related to a circle.
- State, investigate and apply theorems of the geometry of circles:
    - **Theorem:** the line drawn from the centre of a circle perpendicular to a chord bisects the chord
    - **Corollary of Theorem:** the perpendicular bisector of a chord passes through the centre of the circle
    - **Theorem:** the angle subtended by an arc at the centre of a circle is double the size of the angle subtended by the same arc at the circle (on the same side of the chord as the centre)
    - **Theorem:** angles subtended by a chord of the circle, on the same side of the chord, are equal
    - **Theorem:** the opposite angles of a cyclic quadrilateral are supplementary
    - **Theorem:** exterior angle of cyclic quad. is equal to opposite interior angle
    - **Theorem:** two tangents drawn to a circle from the same point outside the circle are equal in length
    - **Theorem:** the radius is perpendicular to the tangent
    - **Theorem:** the angle between the tangent to a circle and the chord drawn from the point of contact is equal to the angle in the alternate segment.
- Discuss applications of circles in real life.

A water turbine is a rotary machine that converts kinetic energy and potential energy of water.

6.1 Revision of concepts and theorems learned in earlier grades

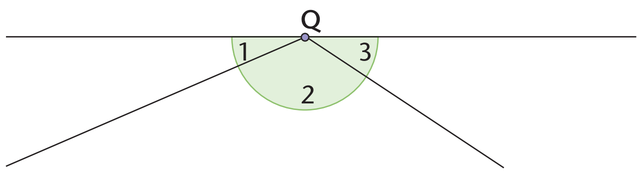
For you to make sense and understand circle geometry theorems and uses, you need to master theorems learned in earlier grades.

Adjacent angles are two angles that have the same vertex and a common side.

Complementary angles are angles that have a sum of 90°.

Supplementary angles are angles that have a sum of 180°.

Theorems relating to a straight line



Theorem

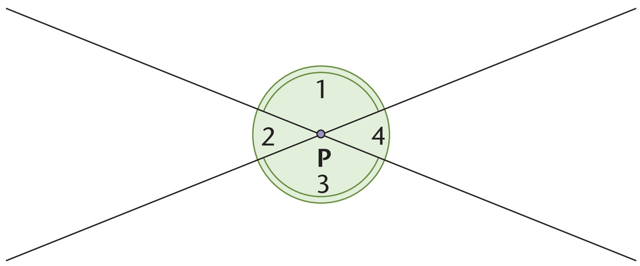
The sum of adjacent angles on a straight line is 180:

$\hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3 = 180°$

$\hat{Q}_1$, $\hat{Q}_2$ and $\hat{Q}_3$ are adjacent supplementary angles.

If line segments form adjacent angles that have a sum of 180°, then the segments form a straight line:

Theorem



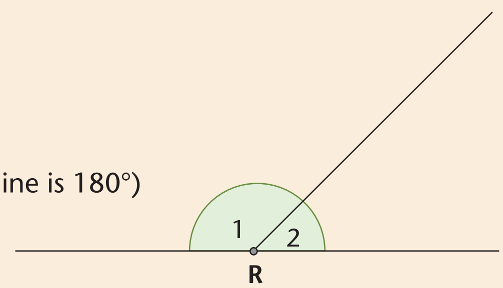
Vertically opposite angles formed by intersecting straight lines are equal:

$\hat{P}_1 = \hat{P}_3$ and $\hat{P}_2 = \hat{P}_4$

If vertically opposite angles formed by intersecting lines are equal, then the intersecting lines are straight:

Worked example 6.1

$\hat{R}_1$ and $\hat{R}_2$ form a straight line. $\hat{R}_1$ is three times $\hat{R}_2$.
Find $\hat{R}_1$ and $\hat{R}_2$.



Solution

$\hat{R}_1 + \hat{R}_2 = 180°$ (The sum of adjacent angles on a straight line is 180°)
$3\hat{R}_2 + \hat{R}_2 = 180°$

$\hat{R}_2 + 3\hat{R}_2 = 180°$
$4\hat{R}_2 = 180°$
$\hat{R}_2 = 45°$
$\therefore \hat{R}_1 = 3 \times \hat{R}_2 = 135°$

EXERCISE 6.1

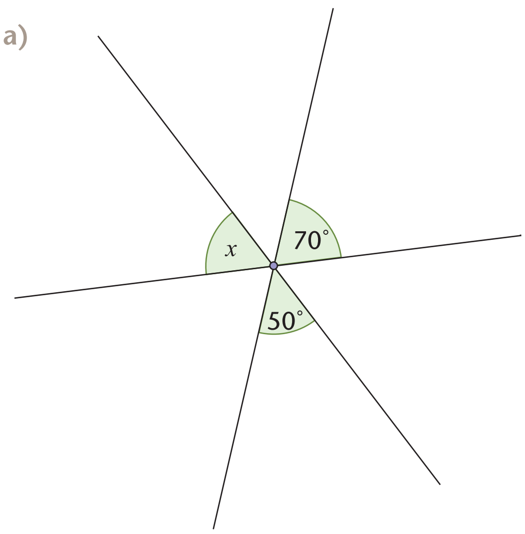
1 Write down the complement and the supplement of each of the following angles:

		Complement	Supplement	Difference between Supplement and Complement
a)	17°			
b)	80°			
c)	29°			
d)	171°			
e)	5°			
f)	23°			
g)	37°			
h)	113°			
i)	90°			
j)	0°			

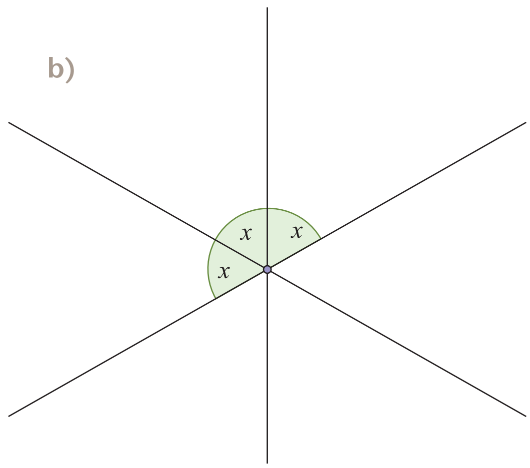
2 What is the difference between the supplement and the complement of any acute angle? Is it always true? Why?

3 Calculate the value of x:

a)



b)



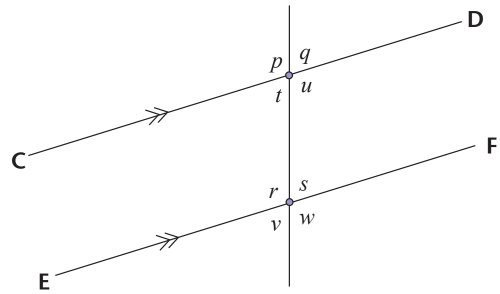
Theorems relating to parallel lines

In plane or Euclidian Geometry parallel lines are a constant distance apart, and do not meet or intersect.

A line that cuts or intersects parallel lines at two different positions is called a transversal.

When parallel lines are cut by a transversal, then the following pairs of angles are formed:

- alternate interior angles t and s; u and r; and alternate exterior angles p and w; q and v.
- corresponding angles q and s; u and w; p and r; t and v.
- co-interior angles u and s; t and r

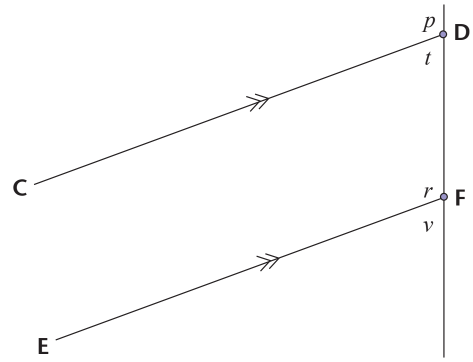


Theorem

If two parallel lines are cut by a transversal, then corresponding angles are equal:

If $CD \parallel EF$ then $p = r$ and $t = v$

If two straight lines are cut by a transversal and a pair of corresponding angles are equal, then the lines are parallel:

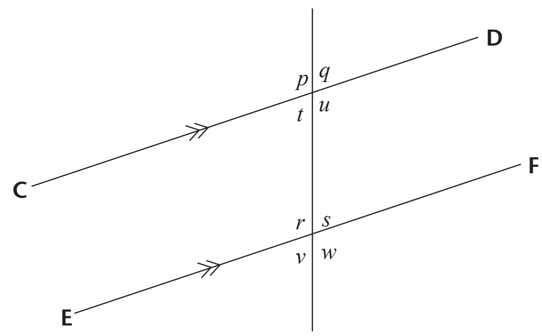


Theorem

If two parallel lines are cut by a transversal, then alternate angles are equal:

If $CD \parallel EF$ then $u = r$ and $t = s$

If two straight lines are cut by a transversal and a pair of alternate angles are equal, then the lines are parallel:

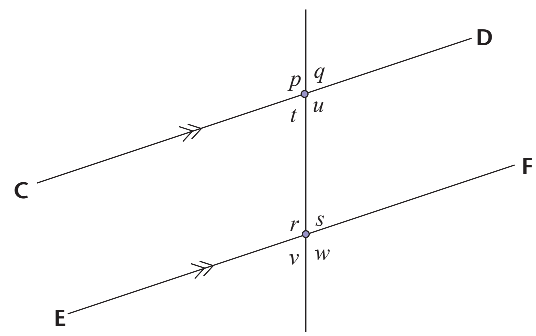


Theorem

If two parallel lines are cut by a transversal, then the pair of co-interior angles are supplementary:

If $CD \parallel EF$ then $u + s = 180°$ and $t + r = 180°$

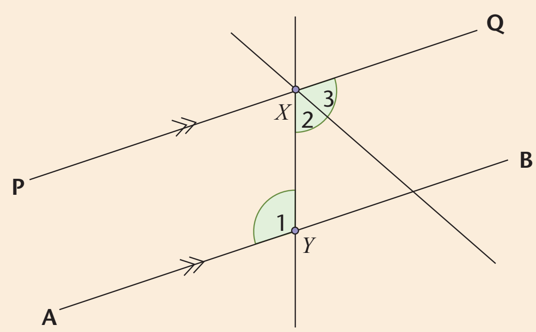
If two straight lines are cut by a transversal and a pair of co-interior angles is supplementary, then the lines are parallel.



Worked example 6.2

In the figure along side $PQ \parallel AB$, $\hat{Y}_1 = 100°$ and $\hat{X}_2 = 55°$.

Calculate the value of $\hat{X}_3$.



Solution

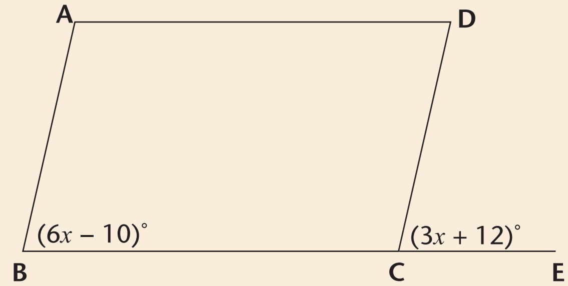
$\hat{Y}_1 = \hat{X}_2 + \hat{X}_3$ *(Alt ∠s are =; PQ ∥ AB)*

$100° = 55° + \hat{X}_3$ *(substitution)*

$\therefore \hat{X}_3 = 45°$

Worked example 6.3

In the figure along side $AB \parallel CD$.
Determine the value of $\hat{B}$.



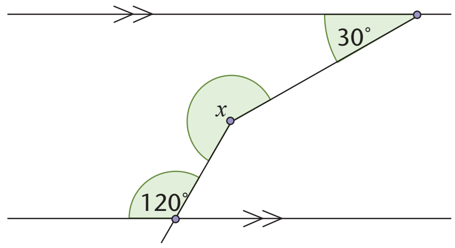
Solution

$\hat{B} = D\hat{C}E$ *(corresponding ∠s are =; AB ∥ CD)*
$6x - 10° + 3x + 20°$ *(substitution)*
$\therefore x = 10°$
$\therefore \hat{B} = 50°$

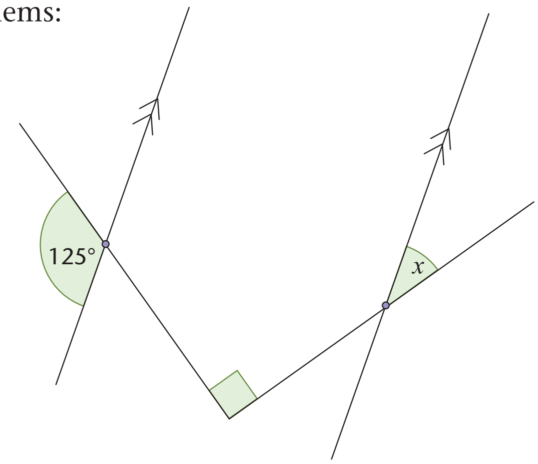
EXERCISE 6.2

1 Determine the value of x in each of the following problems:

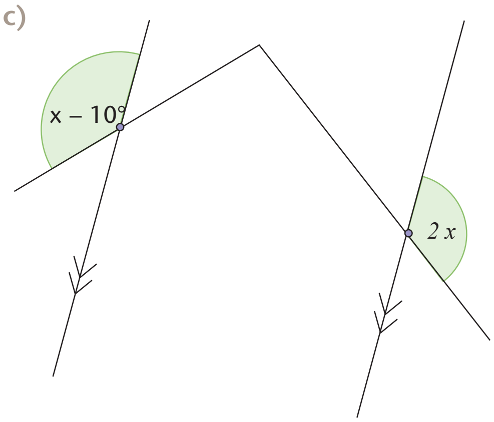
a)



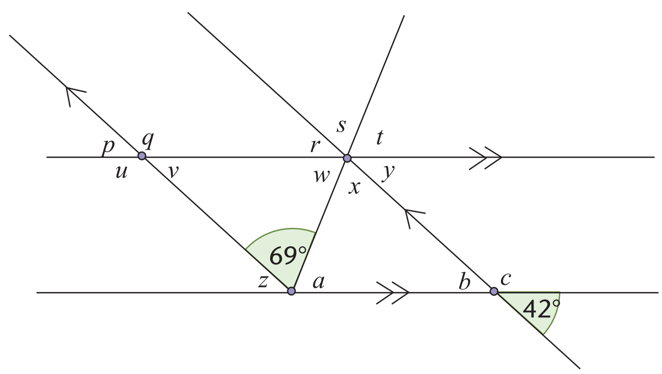
b)



c)



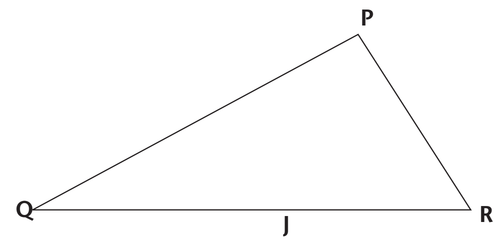
2 Determine the sizes of the angles marked with letters in the following diagram. Give reasons.



Theorems relating to triangles

Angle sum theorem

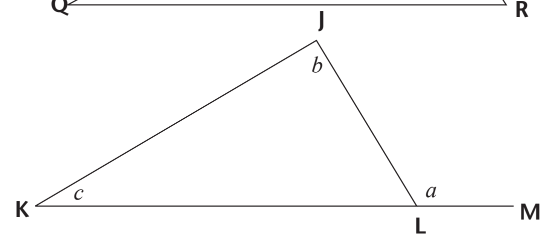
The sum of interior angles of a triangle is 180°:



$\hat{P} + \hat{Q} + \hat{R} = 180°$

$\hat{P}$, $\hat{Q}$ and $\hat{R}$ are supplementary angles.

The exterior angle of a triangle is equal to the sum of the two interior opposite angles: $a = b + c$

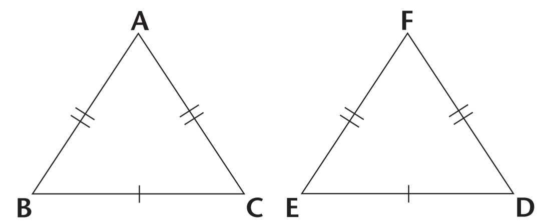


Congruency means equal in every respect

Congruency theorem: Case I

If three sides in one triangle are respectively equal to the three sides in another triangle, the two triangles are congruent.

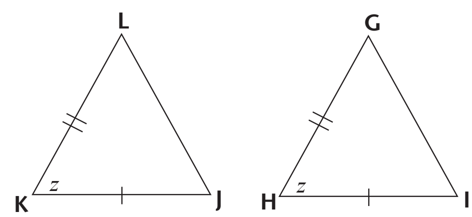
$\Delta ABC = \Delta FED$ (SSS)



Congruency theorem: Case II

If two sides and the included angle in one triangle are respectively equal to two sides and the included angle in another triangle, the two triangles are congruent.

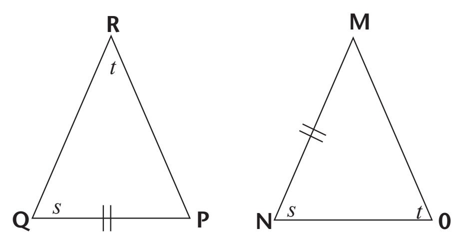
$\Delta LKJ \equiv \Delta GHI$ (S∠S)



Congruency theorem: Case III

If two angles and a side in one triangle are respectively equal to two angles and the corresponding side in another triangle, the two triangles are congruent.

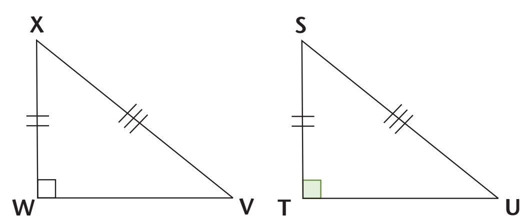
$\Delta RQP \equiv \Delta ONM$ (∠∠S)



Congruency theorem: Case IV

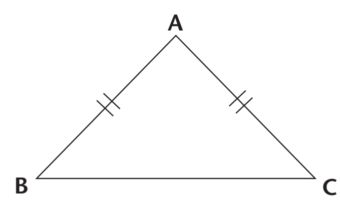
If the hypotenuse and a side of one right-angled triangle is respectively equal to the hypotenuse and the corresponding side of another right angled triangle, the two triangles are congruent:

$\Delta XWV \equiv \Delta STU$ (RHS)



Isosceles triangles theorem

If a triangle has two sides equal, the angles opposite the equal sides are also equal.



> ***Base angles** and **opposite sides** of an **isosceles triangle** are equal. **Angles** and **sides** of an **equilateral triangle** are equal. The measure of each angle of an equilateral triangle is **60°**.*

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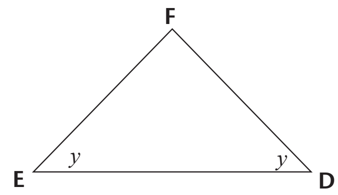
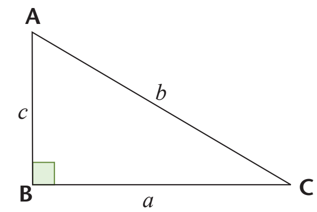
If $AB = AC$, then $\hat{B} = \hat{C}$



If a triangle has two angles equal, the sides opposite the equal angles are also equal.

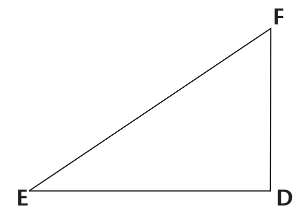
If $\hat{E} = \hat{D}$, the $FE = FD$

Pythagoras' theorem



In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

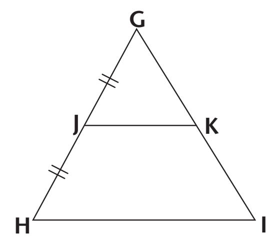
If $\hat{B} = 90°$, then $b^2 = a^2 + c^2$



If the square of the longest side of a triangle is equal to the sum of the squares on the other two sides, then the angle opposite the longest side is a right angle.

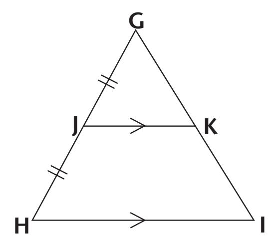
If $EF^2 = FD^2 + ED^2$, then $\hat{D} = 90°$

Midpoint theorem



The line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to half its length.

If $GJ =$ and $GK = KI$, then $JK \parallel HI$ and $JK = \frac{1}{2}HI$

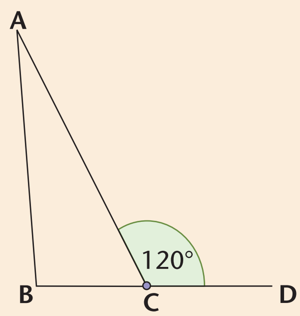


The line passing through the mid-point of one side of a triangle, parallel to a second side, bisects the third side.

If $ST = TR$ and $TU \parallel RQ$, then $SU = UQ$

Worked example 6.4

$\hat{B}$ is five times $\hat{A}$. Determine the value of $\hat{B}$.



Solution

$x + 5x = 120°$ ($\hat{A} + \hat{B} = A\hat{C}D$; exterior angle of a triangle)

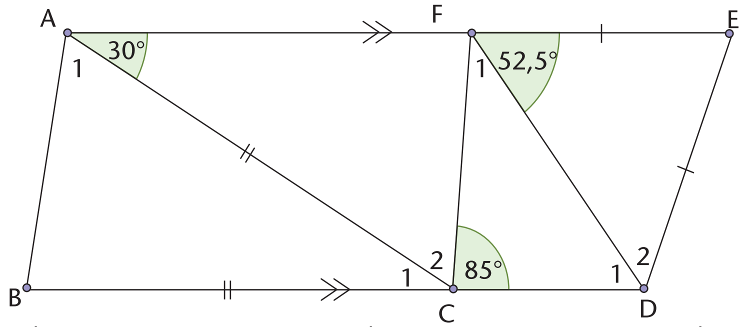
$6x = 120°$

$x = 20°$

$\therefore \hat{B} = 100°$

EXERCISE 6.3

1. $AE \parallel BD$, $FE = ED$ and $AC = CB$



   a) Calculate:

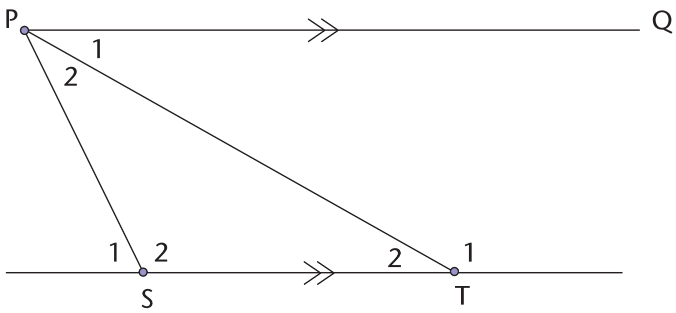
      (i) $\hat{C}_1$ (iii) $\hat{B}$ (v) $\hat{D}_2$ (vii) $\hat{F}_1$

      (ii) $\hat{A}_1$ (iv) $\hat{D}_1$ (vi) $\hat{E}$

   b) Show that AB is parallel to ED.

   c) What is the name of quadrilateral $ABDE$. Justify your answer.

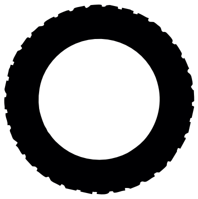
2. Find with reasons the value of x and y given that $\hat{P}_1 = x$, $\hat{P}_2 = y$, $\hat{T}_2 = 24°$ and $\hat{S}_2 = 112°$



6.2 An introduction to circles

Circular objects around us

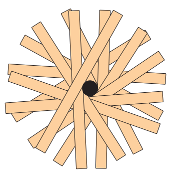
You can see circles and circular objects around us, in our homes, in school etc. Properties of circles are quite important and have significant applications that have made it possible for civilization to roll forward.

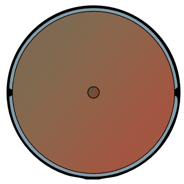


tyre

EXERCISE 6.4

1. What do you know about circles and circular objects?
2. What is the most famous and important circular invention?
3. Name two types of vehicles that use wheels.
4. Why manholes are circular?



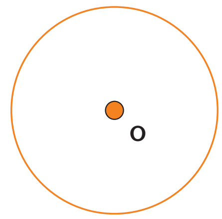
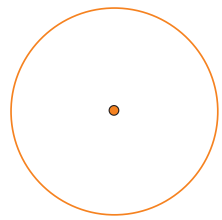
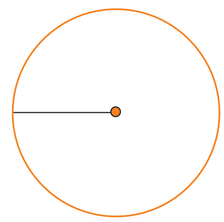
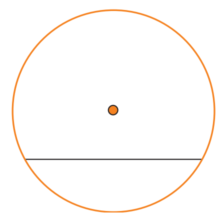
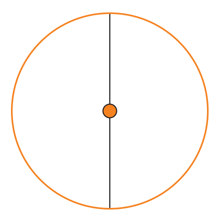
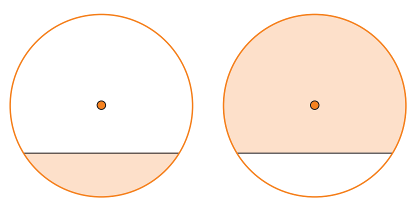


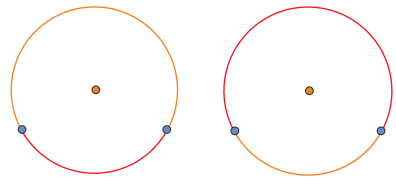
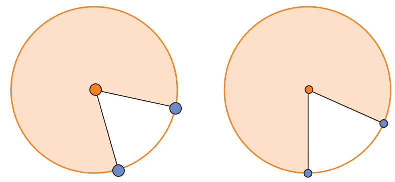
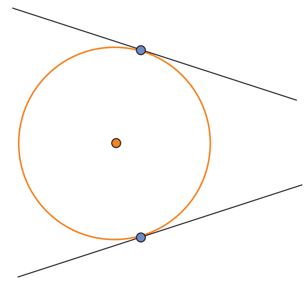
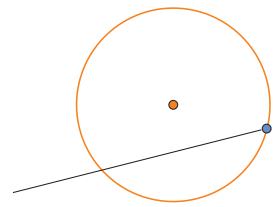
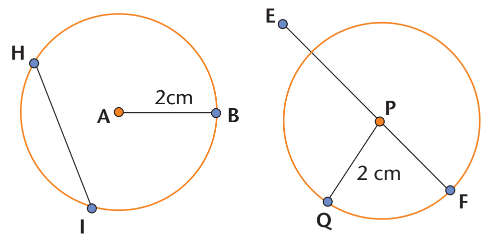
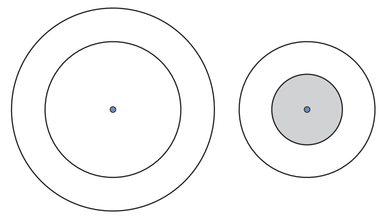
manhole

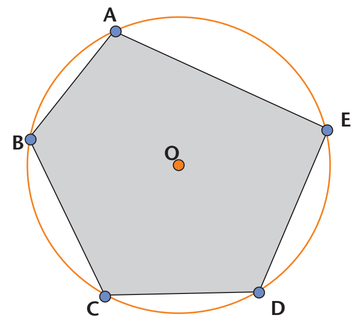
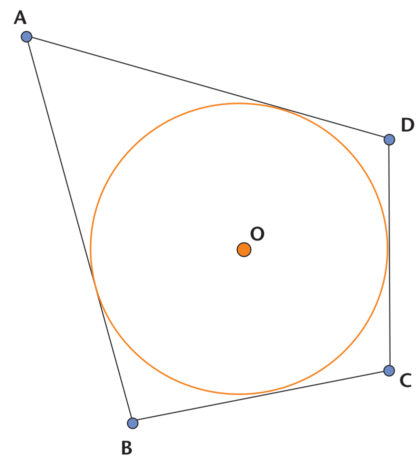
6.3 Parts of a circle and basic terms

Geometry of the circle has many theorems that are used to solve problems. In the study of Euclidean Geometry in general, it is important to know the theorems and be able to prove these theorems.

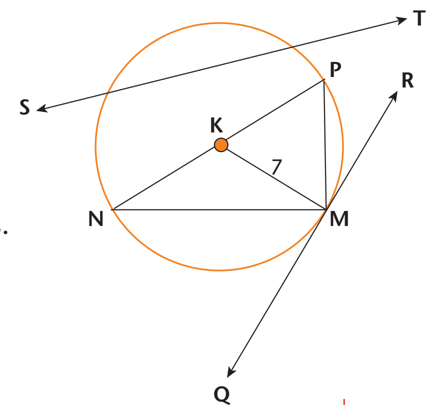
A circle is the set of all points that lie a fixed distance (the radius) from a given point (the centre).

Name (part)	Description	Illustration
Centre	The middle point that is equidistant from all points on the circumference; usually it is indicated by a dot and a label (e.g. O). A circle is named by its centre (e.g. $\odot O$)	
Circumference	The length of the boundary. It is the circle's perimeter	
Radius	A segment with one endpoint the centre of the circle and the other endpoint on the circle.	
Chord	A segment that joins two points on the circle, that is, from one point on the circumference to another.	
Diameter	A chord passing through the centre of the circle	
Segment in a circle	The area of the circle between a chord and the circumference, where the smaller segment is called the minor segment and the bigger segment is called the major segment.	

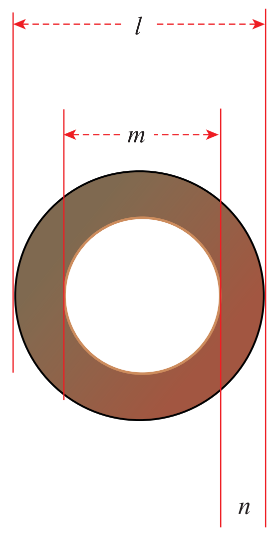
Name (part)	Description	Illustration
Arc	A portion of the circumference.	
Sector	A region formed by two radii and an arc of a circle.	
Tangent	A straight line that intersect the circle in exactly one point.	
Secant	A chord extended beyond the circumference on one side	
Congruent circles	Circles that have equal radii.	
Concentric circles	Circles with the same centre	

Name (part)	Description	Illustration
Inscribed polygon	All vertices of the polygon are on the circle	
Circumscribed polygon	All the sides of the polygon are tangent to the circle	

EXERCISE 6.5



1. Refer to the adjacent figure on the right.
   a) Find the length of the radius of ⊙*K*.
   b) Name two chords of ⊙*K* that are not diameters.
   c) Name a diameter of ⊙*K*.
   d) Find the length of a diameter of ⊙*K*.
   e) Name a tangent of ⊙*K*.
   f) Name a secant of ⊙*K*.



2. Draw a circle and then complete the figure as as outlined below.

   Draw line segment *EF* inside the circle that is not a chord.

   Draw a line segment *GH* that intersect the circle in two points, but is not a diameter or a chord.

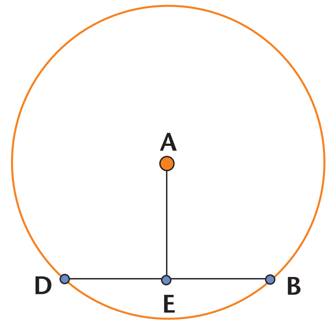
   Draw a line segment *JK* that has one end point on the circle but is not a chord.

3. A flat washer has an inside diameter of 3,2 mm and an outside diameter of 7 mm.
   a) Find the width *n* of the washer.
   b) What do we call the two circles that form a washer?

6.4 Chords and radii/radiuses

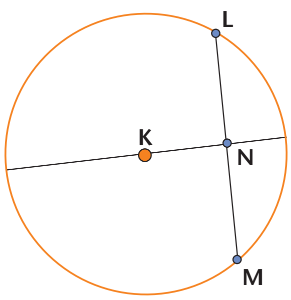
We need to explore the relationship that exists between a line segment drawn from the centre of a circle to a chord of a circle. For that we will do the following exercise.

EXERCISE 6.6



1. Refer to the diagram on the right and answer the following questions. You are given circle ⊙A and chord DB.
   a) Measure the length of line segments DB, DE and EB. What can you say about line segments DE and EB?

      What do we call point E?
   b) Now measure $A\hat{E}D$ and angle $A\hat{E}B$ and write down the magnitudes (i.e. the sizes of $A\hat{E}D$ and $A\hat{E}B$). What can you say about line segments AE and DB?
   c) Write a statement that describes your discovery.
2. Do you think the same discovery about ⊙K can be made as was the case with circle ⊙A above without necessarily measuring line segments LN, NM and $K\hat{N}L$?

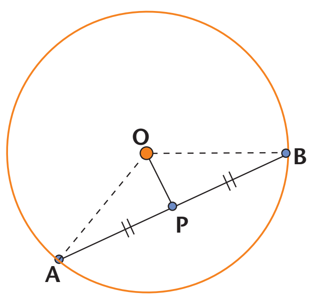


From exercise above, we learn an important statement or Theorem about chords in a circle. We can state the theorem as follows:

Given a circle with centre O with points A and B on the circumference and point P on the chord AB. Then:

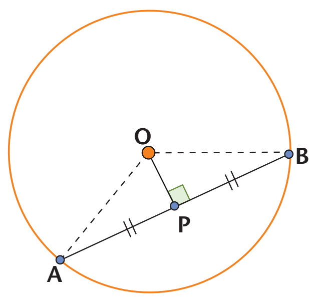
Theorem

If $\overline{OP}$ *bisects* $\overline{AB}$ then $\overline{OP} \perp \overline{AB}$



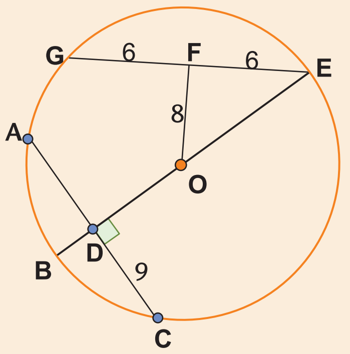
Theorem

If $\overline{OP} \perp \overline{AB}$ then $\overline{OP}$ *bisects* $\overline{AB}$



Worked example 6.5

Refer to the figure below and answer the questions that follow:



Calculate the length of AD and OE:

a) $AD =$ ________________

b) $OE =$ ________________

Solution

a) $AD = 9$ units (OB AC means that OB bisects AC)

b) $OE^2 = EF^2 + FO^2$ (OF bisects EG hence $OF \perp EG$)

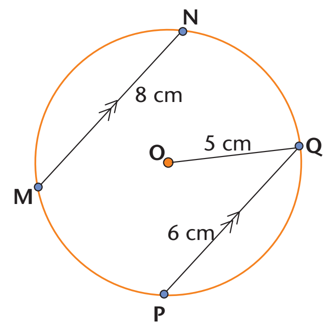
$= 6^2 + 8^2$

$= 100$

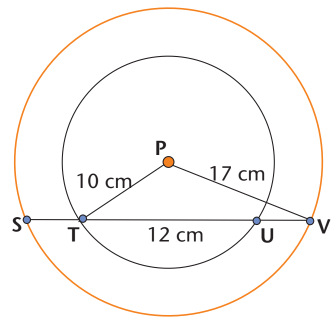
$OE = \sqrt{100}$

$= 10$ units

EXERCISE 6.7



1. In $\odot O$ find the shortest distance between MN and PQ.

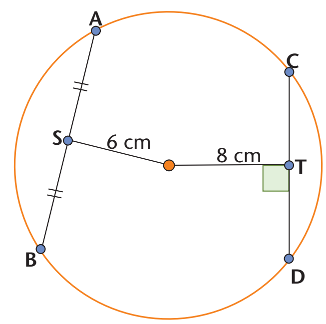


2. The two concentric circles P have radii 10 cm and 17 cm respectively.

   $TU = 12$ cm

   Find SV.

   Show that $ST = 9$ cm.



3. In $\odot O$, $AB = 16$ cm, $OS = 6$ cm, $AS = BS$, $OT \perp CD$ and $OT = 8$ cm. Calculate:

   a) The radius of the circle.

   b) The length of CD.

   (Hint: Join BO and CO)

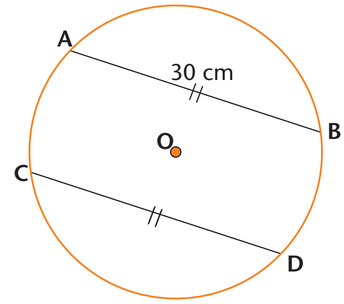
Now the following important statement comes from theorems above:

Corollary: For any circle, the perpendicular bisector of a chord passes through the centre of the circle.

NB: Corollary is a theorem that follows directly from another theorem.

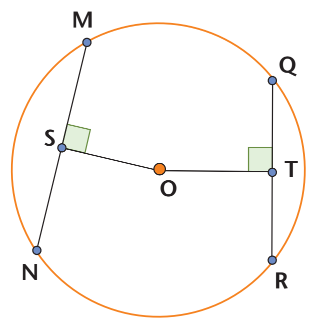
When a point P is the same distance from A as it is from B, we say it is equidistant from A and B.

EXERCISE 6.8



1. Refer to the figure alongside. AB and CD are equal chords of ⊙O. Show that AB and CD are equidistant from the centre given that the radius of ⊙O = 17 cm.

   Do you think this is true for any circle?



2. Refer to the figure alongside. Chords MN and QR are equidistant from the centre O. Show that the chords MN and QR are equal given that the radius of ⊙O = 13 cm and OT = 5 cm.

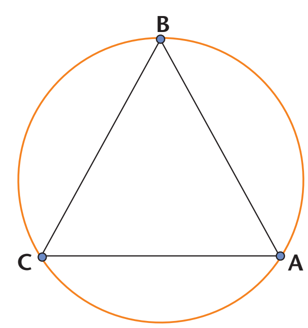
   Do you think this is true for any circle?

3. After doing problems 1 and 2 above, Tshepiso makes the following conclusion:

   "If at least two chords of the same circle (or equal circles) are equal, then the chords are equidistant from the centre of the circle; and if at least two chords of the same circle (or equal circles) are equidistant from the centre of the circle, then the chords are equal."

   Do you agree with Tshepiso?

The circumcentre of a triangle



Another direct consequence of theorem 1 is that a unique circle can be drawn through any three points provided the points are not on a straight line. In other words, any three points on a plane are concyclic. In the diagram a circle passes through the vertices of ΔABC.

ΔABC is circumscribed, and the circle is called the circumcircle of ΔABC. To locate the centre we perpendicularly bisect any two sides of the triangle. The centre of the circle is called the circumcentre. There are three cases to consider in determining the circumcentre:

Inside the triangle if the triangle is acute-angled
Outside the triangle if the triangle is obtuse-angled
On the hypotenuse of the triangle if the triangle is a right-angled triangle.

The exercise below will provide illustrations.

EXERCISE 6.9

Using compass, straight edge and a protractor, construct ΔABC in each of the following cases and construct the circumscribed circle of ΔABC.

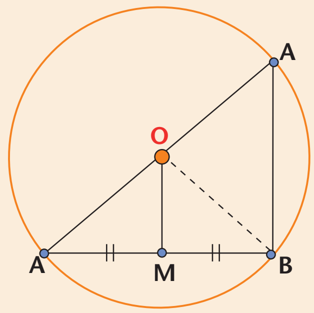
a) $BC = 9$ cm, $AC = 7$ cm, $\angle C = 40°$

b) $BC = 9$ cm, $AC = 9$ cm, $\angle C = 110°$

c) $BC = 9$ cm, $AC = 7$ cm, $\angle C = 90°$

Worked example 6.6

O is the centre of the circle; AC is a diameter and $BM = MC$.
Given that the radius is 5 cm, $BC = 8$ cm and $\hat{C}$ 37°, show that:



a) $\hat{B} = 90°$

b) $AB = 2 \times MO$

c) $C\hat{O}M = C\hat{A}B$

Solution

a) $OM \perp BC$ (OM bisects BC and O is the centre of the circle)

$\therefore O$ is the circumcentre of the circumscribed ΔABC

Hence $\hat{B} = 90°$ (circumcentre is on the hypotenuse)

b) $MO^2 = CO^2 - CM^2$ (Pythagoras Theorem)

$MO^2 = 5^2 - 4^2$ ($CO = 5$ cm is a radius and $CM = 4$ cm; given)

$\therefore MO = 3$ cm

$AB^2 = AC^2 - BC^2$ (Pythagoras Theorem)

$AB^2 = 10^2 - 8^2$ ($AC = 2CO$ and $BC = 10$ cm; given)

$\therefore AB = 6$ cm

Thus $AB = 2 \times MO$

c) $OM \parallel AB$ ($O\hat{M}C = A\hat{B}C = 90°$, corresponding angles =)

$\therefore C\hat{O}M = C\hat{A}B$ (corresponding $\angle$s=)

Alternatively,

$C\hat{O}M = 90° - 37°$ (Sum of interior angles of a triangle)

$\therefore C\hat{O}M = 53°$

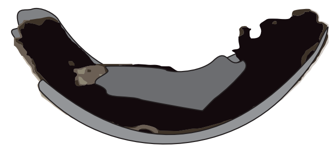
$C\hat{A}B = 90° - 37°$ (Sum of interior angles of a triangle)

$\therefore C\hat{A}B = 53°$

Thus $C\hat{O}M = C\hat{A}B$ (Both = 53°)

EXERCISE 6.10

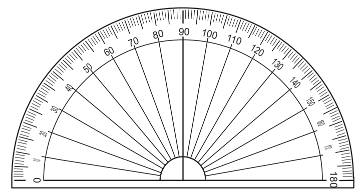
1. Explain how you will determine the radius of the arc that makes the car wash shelter shown on the right.



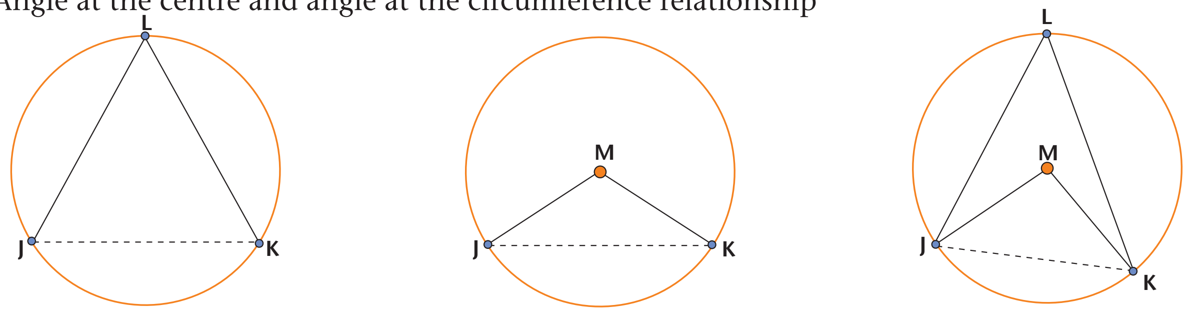
2. A motor mechanic picked up a piece of a car rear brake shoe (pictured). The motor mechanic wishes to calculate the diameter of the piece she found to see which car it might fit. Trace the outer edge of the rear brake shoe. Determine the diameter for the motor mechanic.

6.5 Angles in a circle

In a circle, angles are related. In this section we will investigate the relationship that exists between angles subtended at the circumference of a circle by an arc or chord and angles subtended at the centre of a circle by the same arc or chord.



Angle at the centre and angle at the circumference relationship

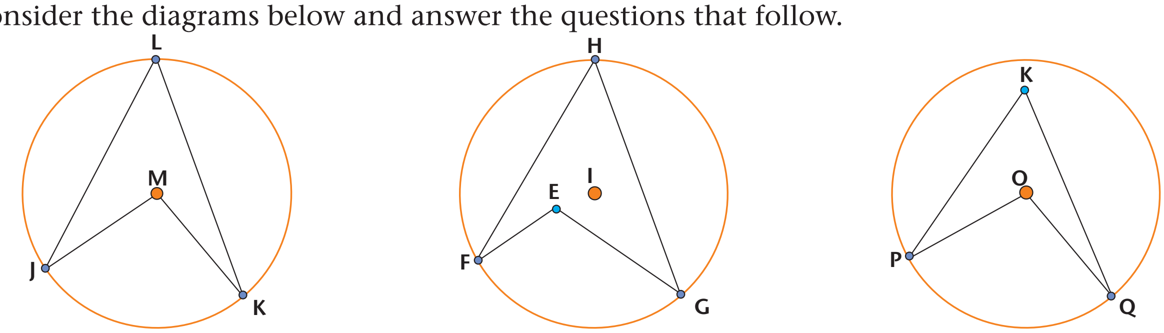


To illustrate, consider circle *JKL* and circle ⊙M above. In circle *JKL* the angle at the circumference is $J\hat{L}K$ and is subtended by arc (chord) *JK*. In circle ⊙*M* the angle at the centre is $J\hat{M}K$ and is subtended by the arc (chord) *JK*. The two angles, $\hat{L}$ and $\hat{M}$, have a relationship and in the following investigations we will explore and establish the relationship.

Before we start with the investigations, let us be sure that we understand the concepts under consideration by doing the following exercise.

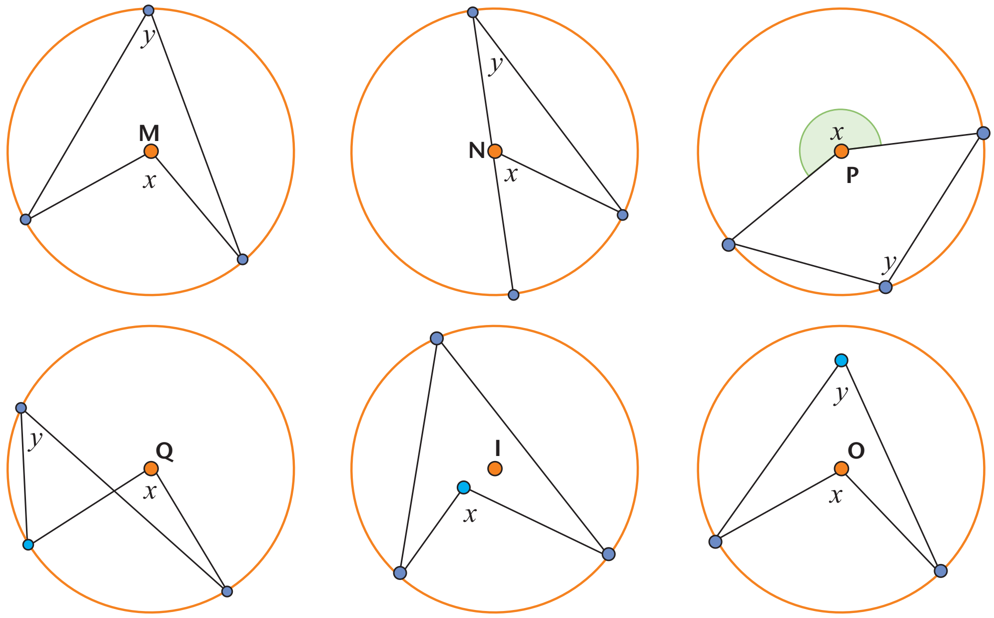
EXERCISE 6.11

1. Consider the diagrams below and answer the questions that follow.



   a) Write down the differences between ⊙*M*, ⊙*I* and ⊙*O* that you observe.
   b) Compare your responses with your classmates.

2. Construct the following circles on a blank sheet of paper.



Using a protractor measure the angles labelled x and y in each circle and record your results in the table on a next page.

	Name of circle	Angle at the centre	Angle at the circumference
		x	y
1.	⊙M		
2.	⊙N		
3.	⊙P		
4.	⊙Q		
5.	⊙I		
6.	⊙O		

a) Is there a relationship between the values of x and y for circles 1 to 4? How is x related to y in this case?

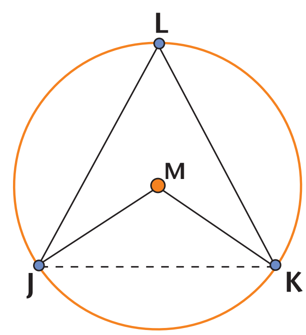
b) Write down a general statement describing the relationship in words and in symbolic form.

c) Does the relationship between the values of x and y for circles 5 and 6 still hold?

Formally we can state the general statement as follows:

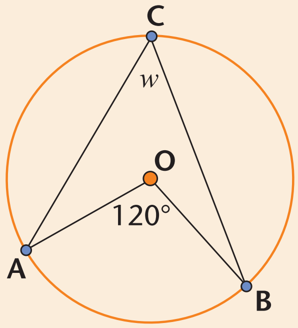
Theorem

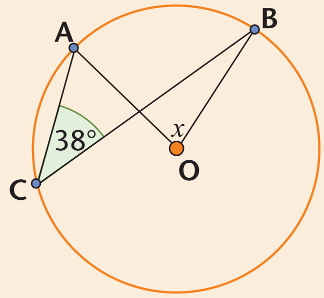
The angle subtended at the centre of a circle by an arc is two times the angle subtended by the same arc at the circumference of a circle.
$J\hat{M}K = 2 \times J\hat{L}K$.

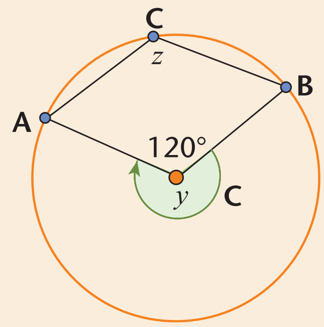


Worked example 6.7

Calculate the value of w, x, y and z.







Solution

$w = 60°$ ($\angle$ at centre $= 2 \times \angle$ at circumference)

$x = 76°$ ($\angle$ at centre $= 2 \times \angle$ at circumference)

$y = 234°$ ($y + 126° = 360°$)

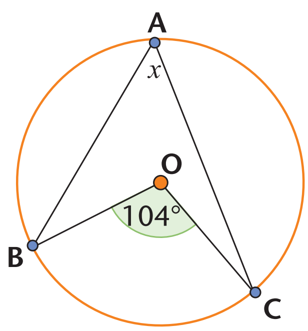
$z = 117°$ ($\angle$ at centre $= 2 \times \angle$ at circumference)

EXERCISE 6.12

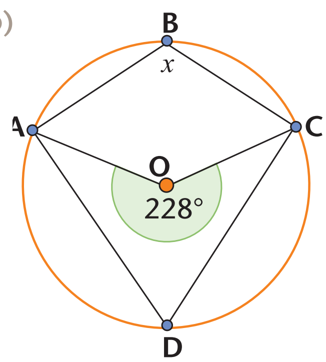
O is the centre of the circle in each case.

Find the sizes of the angles marked x and/or y, giving reasons:

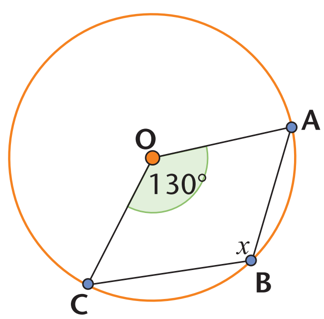
a)



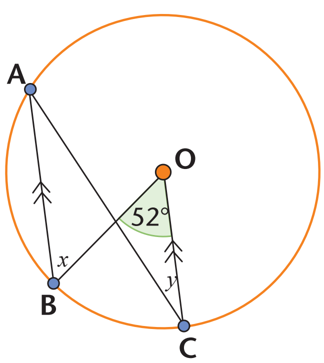
b)



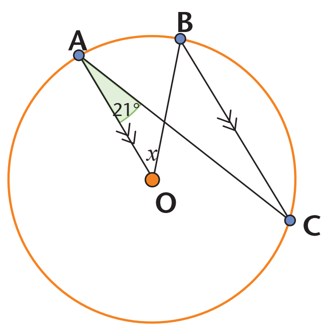
c)



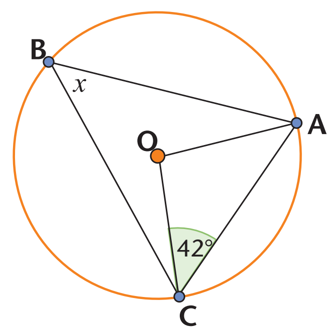
d)



e)



f)

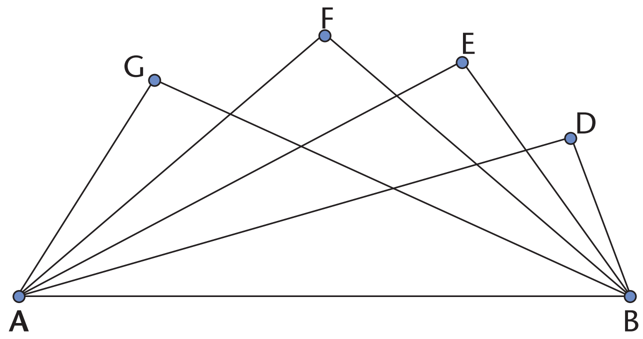


Angles in a semi-circle

EXERCISE 6.13

1. Consider the sketch below and answer the questions that follows:

   Investigate if the circle that passes through point G also passes through points, A, B, D, E and F.

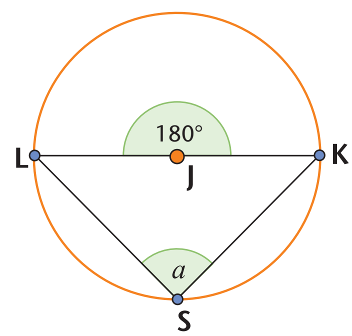


(Hint: Construct the circumcircle of ΔABG.

a) What is the measure of $\hat{D}$, $\hat{E}$, $\hat{F}$ and $\hat{G}$?

b) Justify your answer (Hint: Recall the circumcentre of a triangle).

In the diagram on the right, J is the centre of circle KLS. Use theorem 2 to determine the measure of angle a.



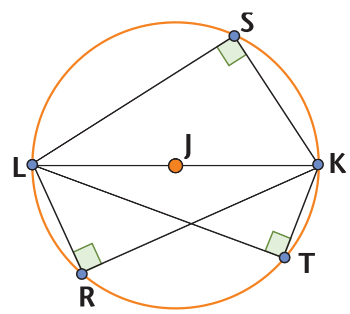
The exercise above establishes another theorem that relates the angle subtended at the circumference of a circle by the circle's diameter. In other words, you have established information about an angle inscribed in a semicircle. Recall that a diameter divides a circle into two equal segments.

Theorem

The angle subtended at the circumference of a circle by the diameter is a right angle.

Or

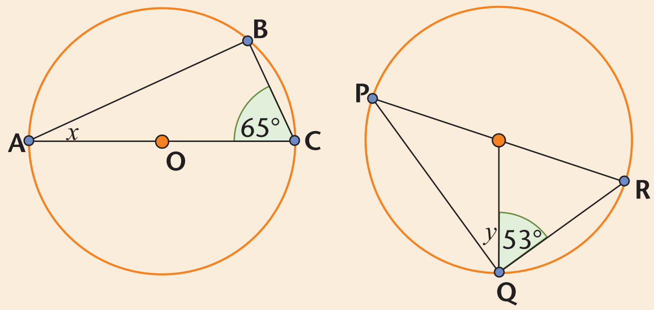
An angle inscribed in a semicircle is a right angle.



Worked example 6.8

O is the centre of the circle in each case.

Find the sizes of the angles marked x and/or y, giving reasons:



Solution

$\hat{B} = 90°$ ($\angle$ in a semicircle)

$x = 25°$ (Sum of angles of Δ)

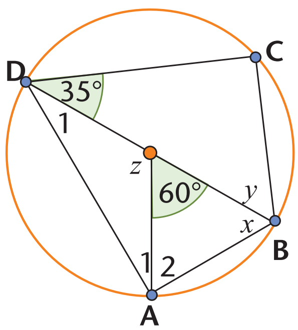
$\hat{Q} = 90°$ ($\angle$ in a semicircle)

$y = 37°$ ($O\hat{Q}R = 90°$)

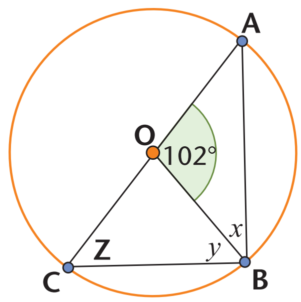
EXERCISE 6.14

1. O is the centre of the circle. Calculate the values of x, y and z in each case.

a)



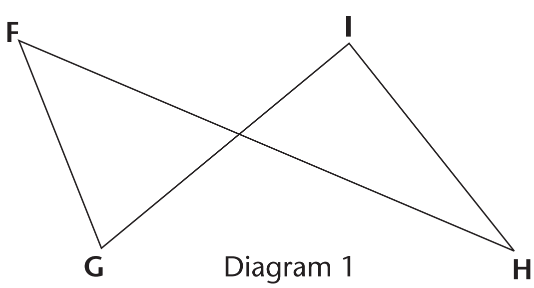
b)

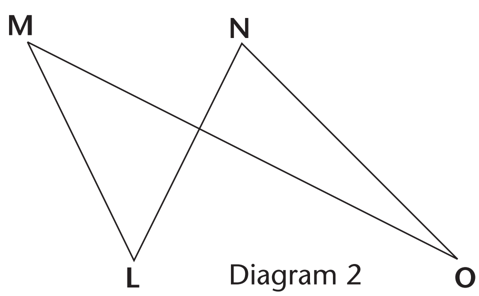


Angles at the circumference of a circle

EXERCISE 6.15

Refer to the diagrams below:





a) Use a protractor to find the measures of $G\hat{F}H$, $G\hat{I}H$, $F\hat{G}I$ and $F\hat{H}I$ in diagram 1.

b) What do you notice about the measures of $G\hat{F}H$ and $G\hat{I}H$?

c) Construct the circumcircle of ΔFGH (Hint: join point G and H)

d) Does the circle that passes through F, G and H also passes through I?

e) Check if you will get the same result for diagram 2 as in diagram 1.

When you have completed exercise c) and d) above the result should look like:

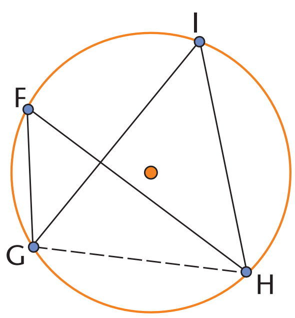


Diagram 3

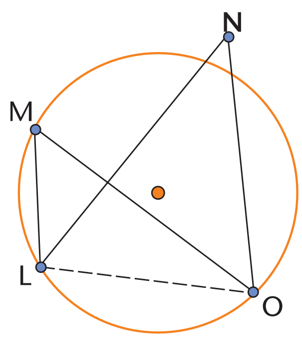


Diagram 4

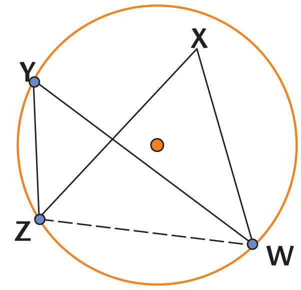


Diagram 5

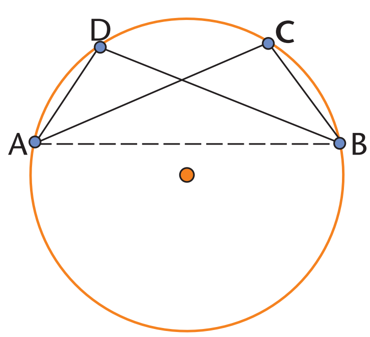
Notice the differences between diagram 3 and 4. We observe in diagram 3 that points *F*, *G*, *H* and *I* are concyclic (lie on the circumference of a circle), whereas in diagram s 4 and 5 not all the points are on the circumference of a circle. Furthermore, as established in exercise 34 (diagram 1).

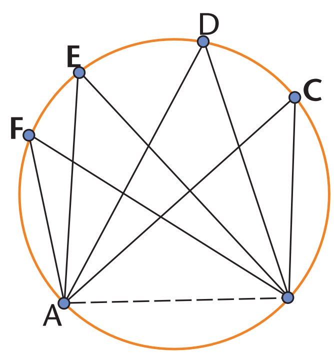
Theorem

Angles in the same segment of a circle are equal.

or

The angles subtended by the chord or arc at the circumference of a circle are equal.





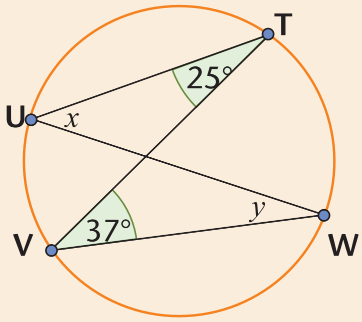
$A\hat{D}B = A\hat{C}B = A\hat{E}B = A\hat{F}B$

Looking at diagram 3, we see that the two angles, that is, $G\hat{F}H$ and $G\hat{I}H$, are on the same side of chord or arc *GH*, stated differently, the two angles are in the same segment; similarly, $F\hat{G}I$ and $F\hat{H}I$ are on the same side of arc (chord) *FI*, that is, the angles are in the same segment. We can generalise these observations and express this generalisation as follows:

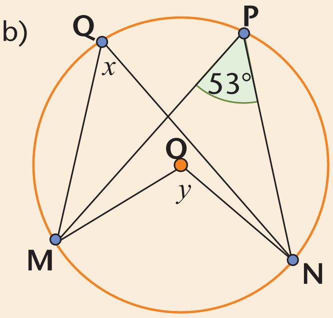
Worked example 6.9

Determine the values of x and y.

a)



b)



Solution

a) $x = 37°$ (arc TR subtends $= \angle$s)

$y = 25°$ (arc TR subtends $= \angle$s)

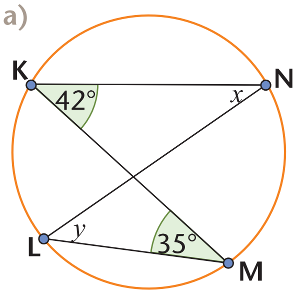
b) $x = 53°$ (arc MN subtends $= \angle$s)

$y = 106°$ ($\angle$ at centre $= 2 \times \angle$ at circumference)

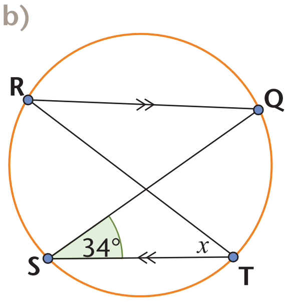
EXERCISE 6.16

Calculate the values of the unknown angles. O is the centre of the circle.

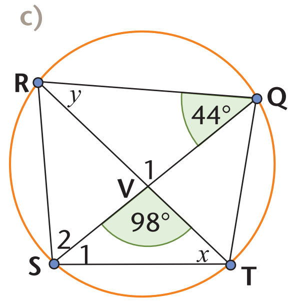
a)



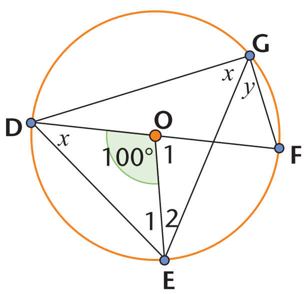
b)



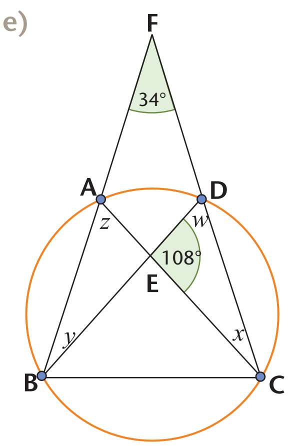
c)



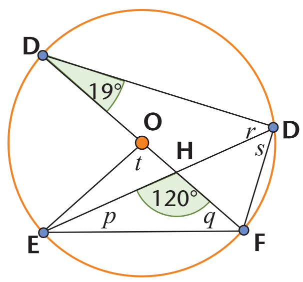
d)



e)



f)



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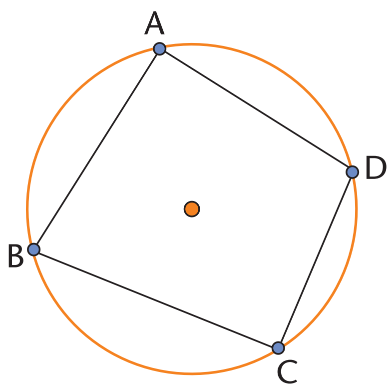
6.6 Cyclic quadrilaterals

Recall, a quadrilateral is a two-dimensional figure that has four sides and four vertices, for example, *MNOP* on the right is a quadrilateral.

In this section we will investigate the properties of cyclic quadrilateral. The word cyclic comes from the Greek word kyklos meaning 'circle'.

Cyclic quadrilateral (or cyclic quad for short) is a four-sided polygon that is inscribed in a circle, that is, all four vertices lie on the circumference of a circle.

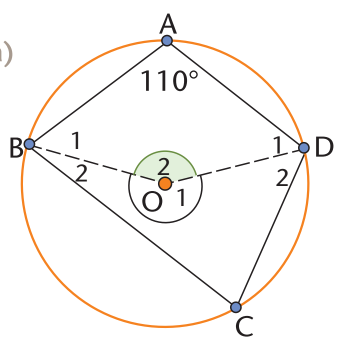


Cyclic quadrilateral *ABCD*

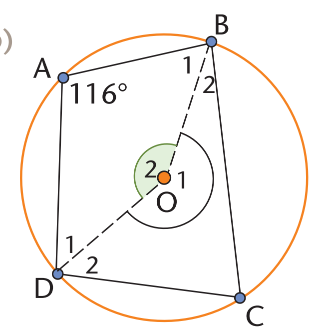
EXERCISE 6.17

1. For each of the following figures a) to e), calculate $\hat{O}_1$, $\hat{O}_2$ and $\hat{C}$. Then determine $\hat{A} + \hat{C}$.

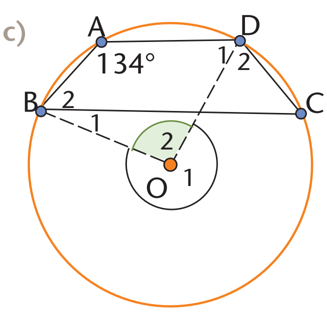
a)



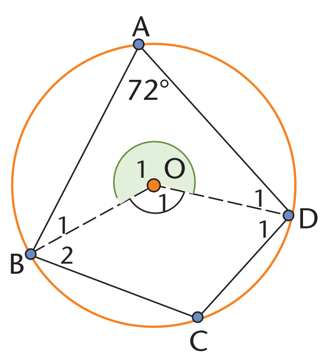
b)



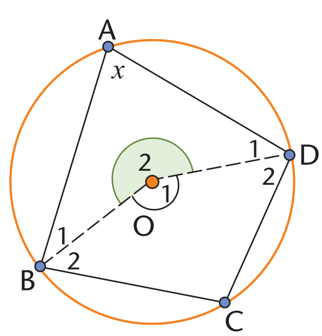
c)



d)



e)



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2. Copy and complete the table.

	$\hat{A}$	$\hat{O}_1$	$\hat{O}_2$	$\hat{C}$	$\hat{A} + \hat{C}$
(i)	110°				
(ii)	116°				
(iii)	134°				
(iv)	72°				
(v)	x				

a) What do you observe? Make a general statement.

b) Do you think the same observation can be made for $\hat{B}$ and $\hat{D}$? Investigate.

What is learnt from exercise 35 can be stated as:

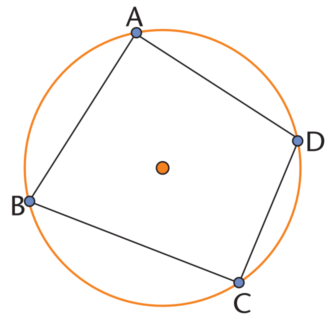
Theorem

Opposite angles of a cyclic quadrilateral are supplementary.

$\hat{A} + \hat{C} = 180°$

and

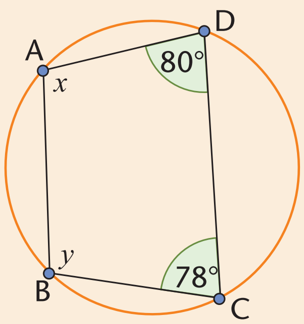
$\hat{B} + \hat{D} = 180°$



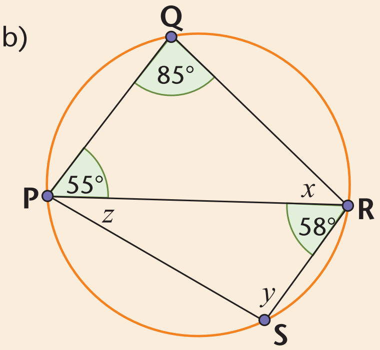
Worked example 6.10

Calculate the value of x and y.

a)



b)



Solution

a) $x = 102°$ (opp ∠ of cyclic quad)

$y = 100°$ (opp ∠ of cyclic quad)

b) $x = 180° - 85° - 55°$ (sum of ∠s of Δ)

$\therefore x = 40°$

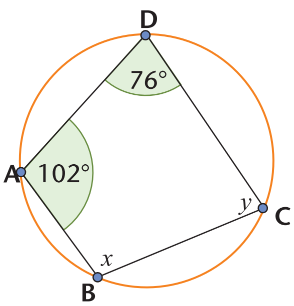
$x + 58° + 55° + z = 180°$ (opp ∠ of cyclic quad)

$\therefore z = 27°$

EXERCISE 6.18

Find the size of each unknown angle.

a)



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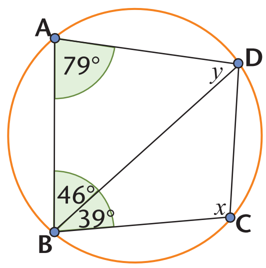
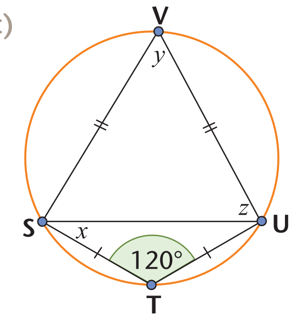
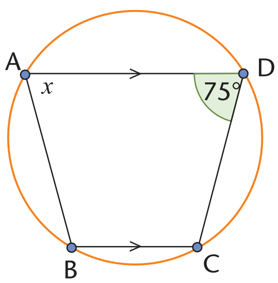


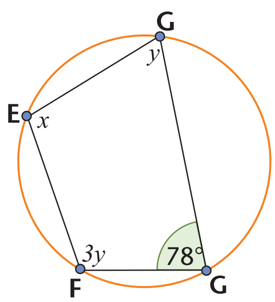
c)



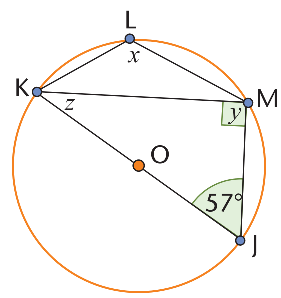
d)



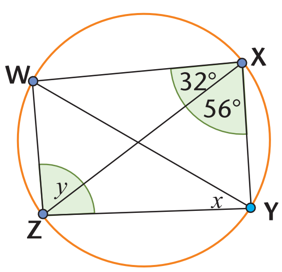
e)



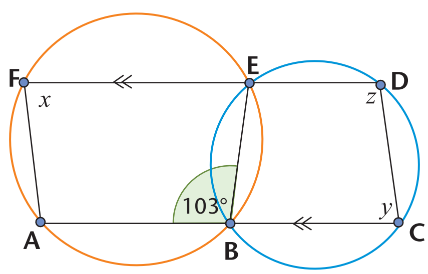
f)



g)



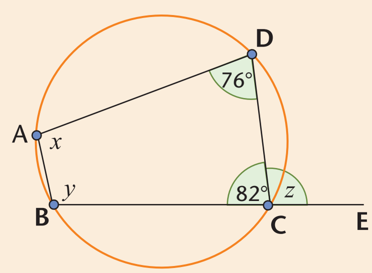
h)



Worked example 6.11

Calculate the value of x, y and z.

What do you notice about the value of x and z?



Solution

a) $z = 180° - 82° = 98°$ ($\angle$ on str line)

$y = 180° - 76° = 104°$ (opp $\angle$ cyclic quad)

$x = 360° - (76° + 82° + 104°) = 98°$

(int $\angle$ of cyclic quad)

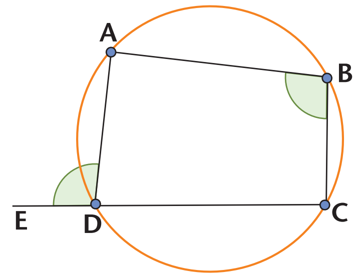
b) $x = z = 98°$

In the worked example above, z is called an exterior angle of the cyclic quad $ABCD$ obtained by extending BC to E. The following theorem generalizes the relationship between the exterior angle and the interior opposite angle of any cyclic quadrilateral.

Theorem

The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle.

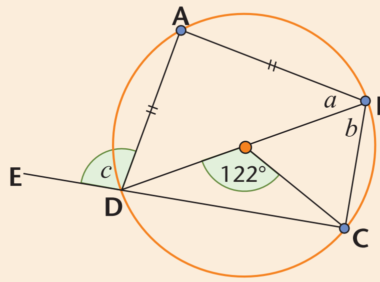
$A\hat{D}E = A\hat{B}D$



Worked example 6.12

O is the centre of the circle, $CDC = 122°$ and $AD = DC$.

Calculate the value of a, b and c.



Solution

$B\hat{A}D = 90°$ ($\angle$ in a semicircle)

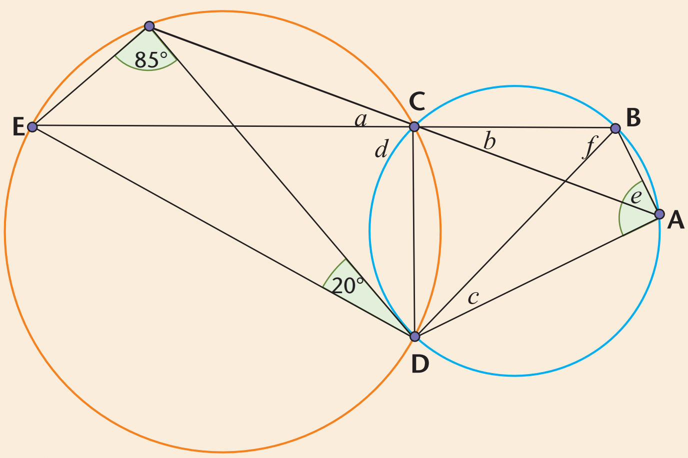
$\therefore b = 45°$ ($AD = AB$)

$\therefore c = 61° + 45°$ (ext of cyclic quad $ABCD$)

$\therefore c = 106°$

Worked example 6.13

Calculate the values of the unknown angles, given that $E\hat{F}D = 85°$ and $E\hat{D}F = 20°$



Solution

$a = 20°$ (arc EF subtends $=$ $\angle$s)

$b = 20°$ (vert. opp $\angle$s =)

$c = 20°$ (arc AB subtends $=$ $\angle$s)

$d = 85°$ (chord ED subtends $=$ $\angle$s)

$e = 85°$ (ext. $\angle$ = int. opp. $\angle$ of cyclic quad. $ABCD$)

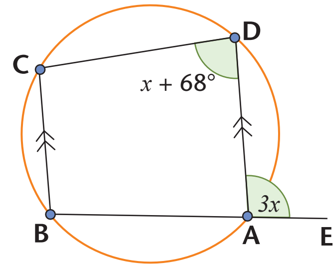
$f = 180°$ (sum of $\angle$s of Δ)

$f = 180° - (85° + 20°) = 75°$

EXERCISE 6.19

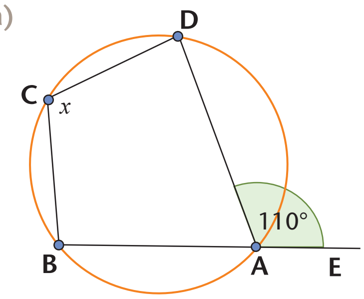
1. $ABCD$ is a cyclic quadrilateral such that $BC \parallel AD$. $D\hat{A}E = 3x$ and $\hat{D} = x + 68°$.

   Find the value of x.

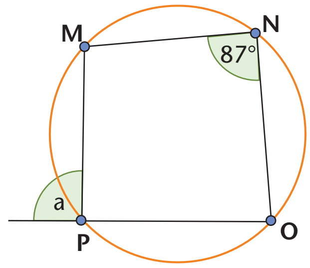


2. Calculate the size of unknown angles in each of the following questions (*a–f*)

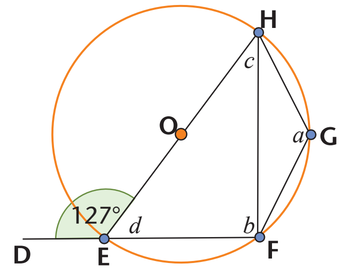
a)



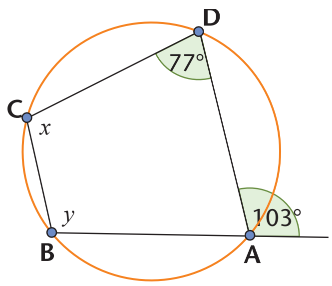
b)



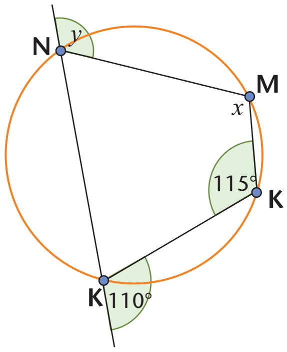
c)



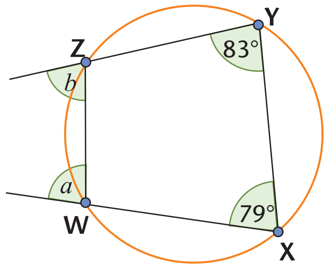
d)



e)



f)

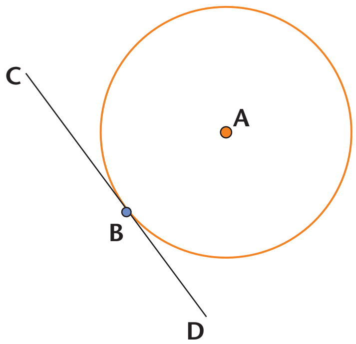


6.7 Tangents, secants and chords

Recall, a tangent to a circle is a line that intersects the circle at just one point. The point where the tangent touches the circle is the point of tangency. In this section you will learn about the tangent properties.

Line CD is a tangent to a circle A

Point B is a point of tangency

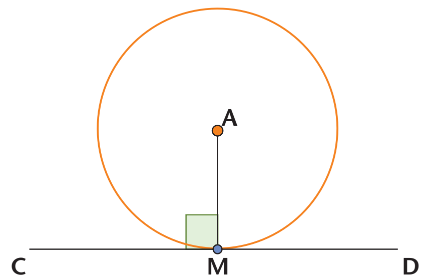


*An **axiom** is a stement that we accept as true without proving it.*

Tangent, secant and radius relationship

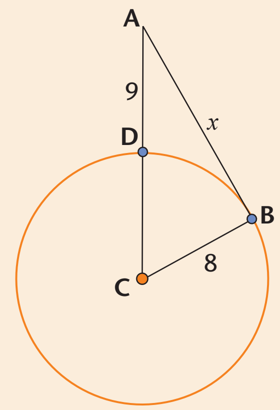
Axiom: A tangent to a circle is perpendicular to the radius drawn from the point of contact.

We will use the axiom to establish and explain other relationships that involves tangents to circles.



Worked example 6.14

Find the value of x if AB is tangent to $\odot C$ and $AC = 9$ units.



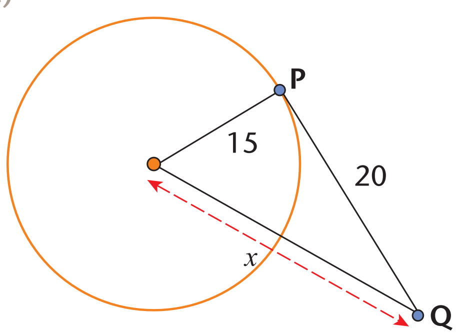
Solution

$AB \perp BC$ (tangent $\perp$ radius)
$AC = AD + DC$ (AC is a straight line)
$\therefore AC = 9 + 8 = 17$ units ($DC = CB$)
$AB^2 + BC^2 = AC^2$ (Pythagoras thm)
$x^2 + 8^2 = 17^2$ (sum of $\angle$s of a Δ)
$x^2 + 64 = 289$
$x^2 = 225$
$x = \sqrt{225}$
$x = 15$ units

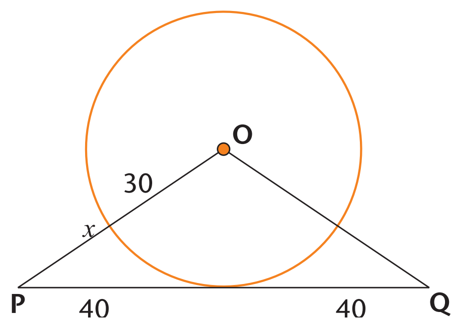
EXERCISE 6.20

1. For each $\odot O$, find the value of x, given that PQ is a tangent.

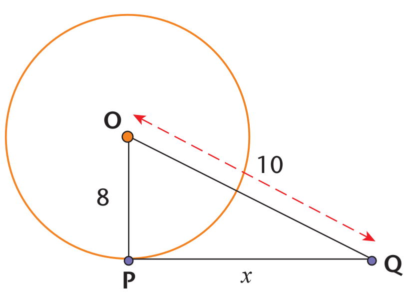
a)



b)

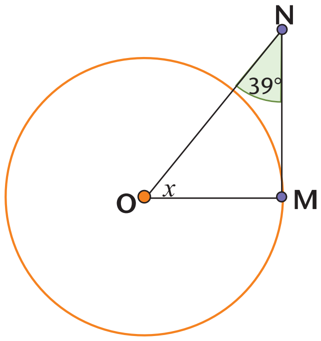


c)

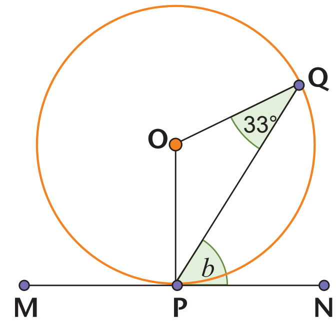


2. Calculate the size of a, b and c. O is the centre and MN is a tangent in each case.

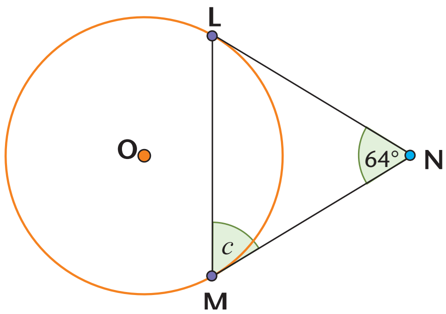
a)



b)



c)



Many theorems in geometry have relationships with one another. For example, in the above exercise, question 2 c), we can use tangent radius axiom and congruence of triangles to find the value of c.

From the example above we see that from congruent triangles, tangents MN and LN are equal. This discovery can be generalised and is stated as follows:

Worked example 6.15

In ΔOMN and ΔOLN

ON is common

$OM = OL$ (radii)

$O\hat{M}N = O\hat{L}N$ tan $\perp$ radiu

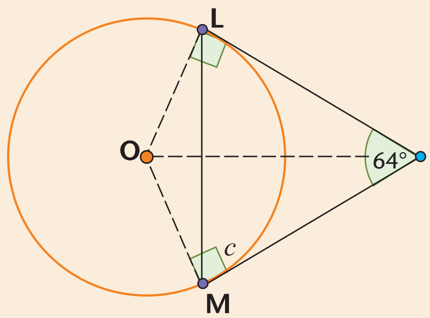
Hence ΔOMN ΔOLN (90° angle, hypotenuse, side [RHS])

$\therefore MN = LN$ (congruent Δs)

$M\hat{N}O = L\hat{N}O$ (congruent Δs)

thus $M\hat{N}O = 32°$ (ON bisects $M\hat{N}L$; proved)

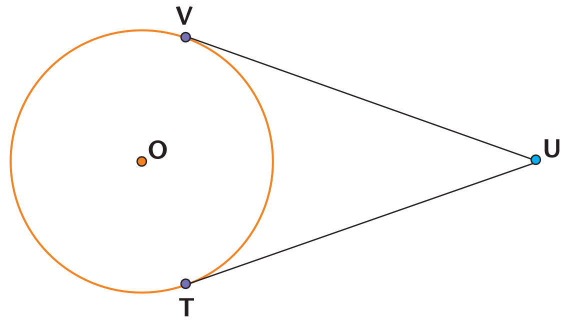
$c = 58°$



Theorem

Two tangents drawn from the same point outside the circle are equal in length.

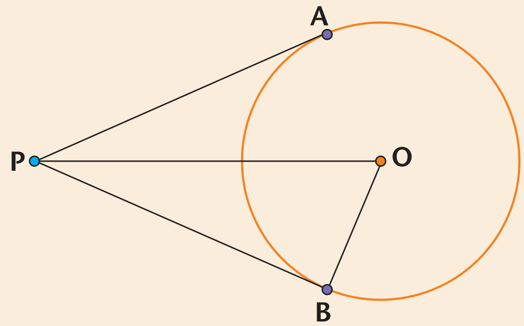
$UT = UV$



Worked example 6.16

PA and PB are tangents from P.
$PO = 17$, $OA = 8$ and $A\hat{P}O = 28°$.

Determine: $P\hat{O}A$ and the length of PB



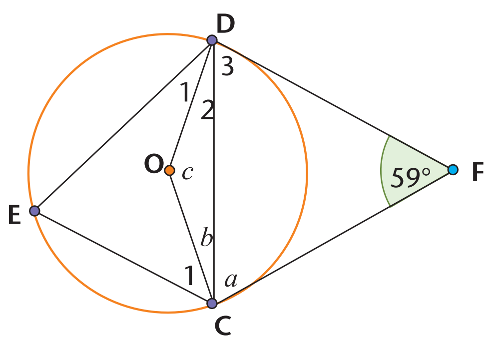
Solution

$P\hat{O}A = 90 - 28 = 62$ (tan ⊥ rad)
$OA^2 + PA^2 = PO^2$ (Pythagoras theorem)
$8^2 + PA^2 = 17^2$
$PA^2 = 17^2 - 8^2$
$\therefore PA = 15$ units
$PA = PB$ (tangents drawn from the same point)
$\therefore PB = 15$ units

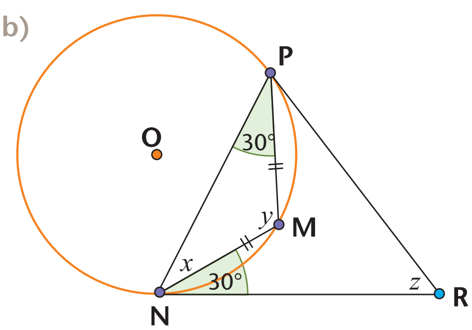
EXERCISE 6.21

Calculate the value of the unknown. O is the centre in each circle.

a)

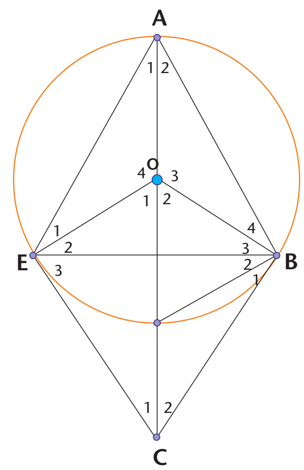


b)



EXERCISE 6.22

In the diagram, BC is a tangent to a circle with centre O. AD is a diameter of the circle and AC is a straight line. E is a point on the circle. BD bisects $E\hat{B}C$ and $D\hat{B}C = 22°$



Calculate the following with reasons:

a) $\hat{B}_2$

b) $\hat{A}_2$

c) $\hat{B}_3 + \hat{B}_4$

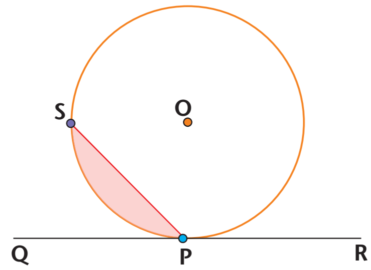
d) $\hat{A}_1$

e) $\hat{O}_1$

f) $\hat{C}_2$

Tangent and chord relationship

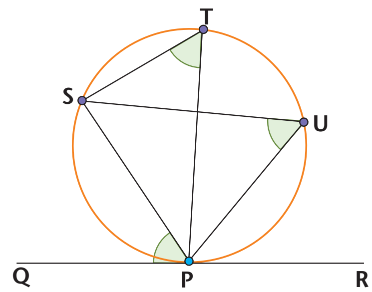
Consider the figure shown on the right. Line QR is a tangent to the circle at P. A chord is drawn from P to point S on the circumference. $Q\hat{P}S$ defines a segment, the area shaded. The unshaded part of the circle is called the alternate segment to $Q\hat{P}S$. Chord PS subtends equal angles in the alternate segment. The tangent and chord relationship theorem states that:



Theorem

The angle between a tangent and a chord is equal to the angle in the alternate segment.

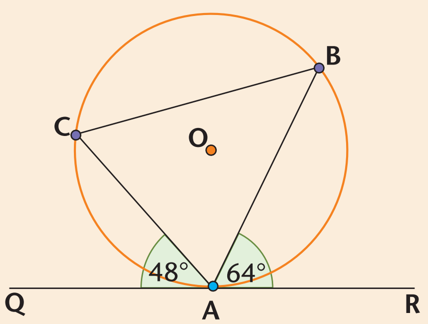
$Q\hat{P}S = S\hat{T}P$ and $Q\hat{P}S = S\hat{U}P$



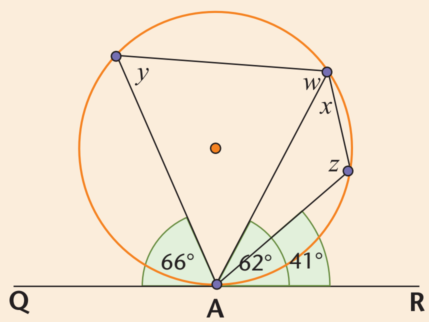
Worked example 6.17

O is the centre and QAR is a tangent to the circle. Calculate the values of the unknown angles

a)



b)



Solution

a) $x = 64°$ (Tan chord time)

$y = 48°$ (Tan chord thm)

b) $w = 66°$ (Tan chord time)

$x = 41°$ (Tan chord thm)

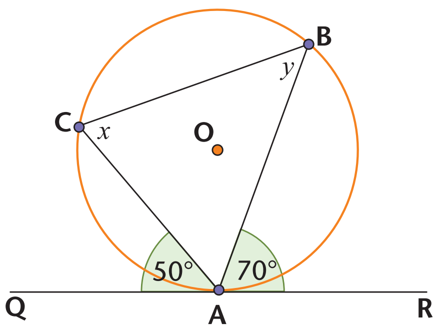
$y = 62°$ (tan – chord time)

$z = 118°$ (opp ∠s of cyclic quad are supplied)

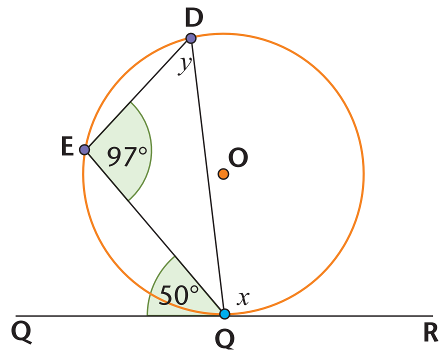
EXERCISE 6.23

1. O is the centre of the circle and PQR is the tangent. Calculate the values of the unknown angles.

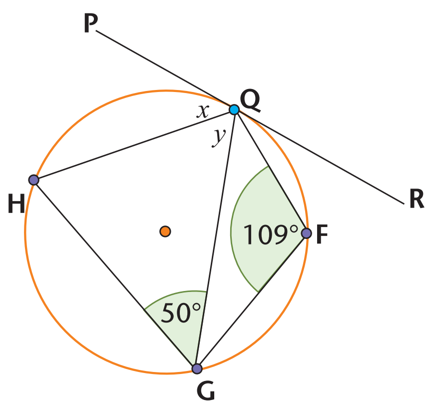
a)



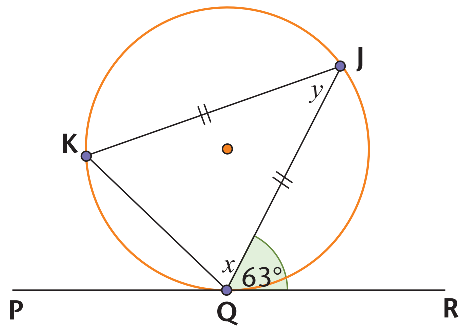
b)



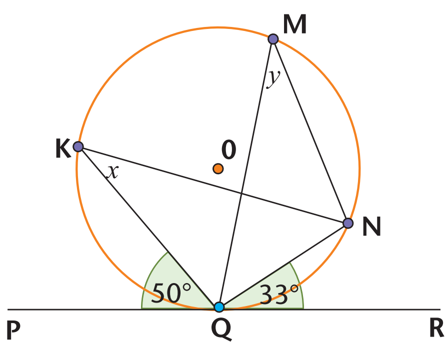
c)



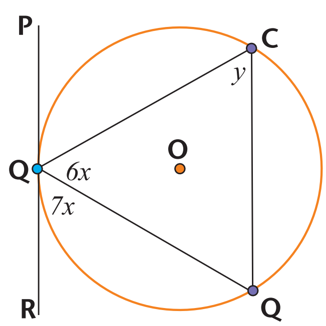
d)



e)

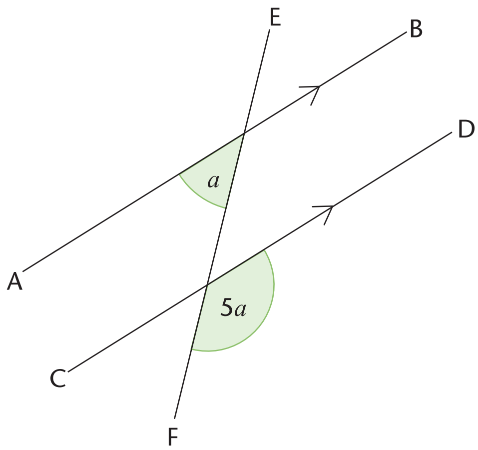


f)



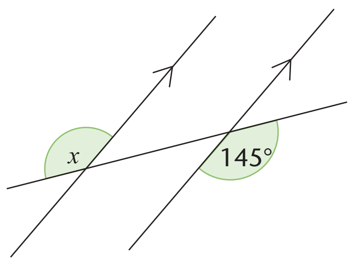
CONSOLIDATION EXERCISE

1. In the diagram, AB is parallel to CD. Calculate the value of a.

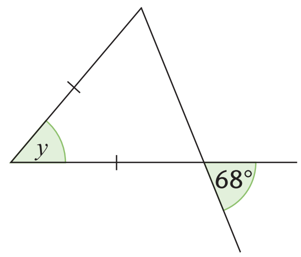


2. Find the values of x and y in the diagrams below.

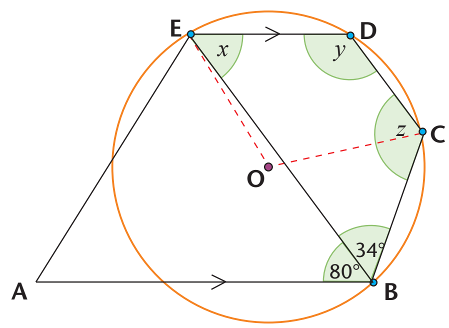
a)



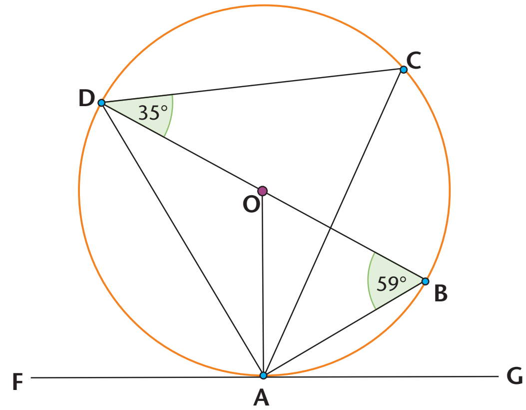
b)



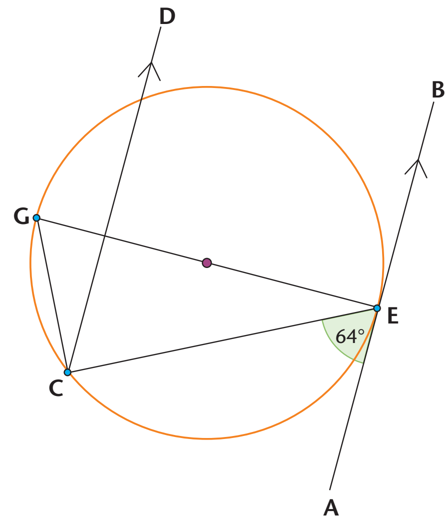
3. A circle, centre O passes through points B, C, D and E. $E\hat{B}C = 34^\circ$, $E\hat{B}A = 80^\circ$ and AB is parallel to ED.
   a) Find the values of x, y and z, giving a reason for each.
   b) Explain why EB is not parallel to DC.
   c) Find the value of $E\hat{O}C$.
   d) Given that $AB = BE$, find the value of $B\hat{A}E$.



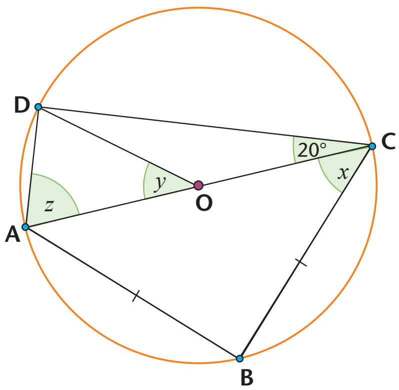
4. A, B, C and D lie on the circle; centre O. BD is a diameter and FAG is the tangent at A. $A\hat{B}D = 59^\circ$ and $C\hat{D}B = 35^\circ$. Find:
   a) $A\hat{C}D$
   b) $A\hat{D}B$
   c) $D\hat{A}G$
   d) $C\hat{A}O$



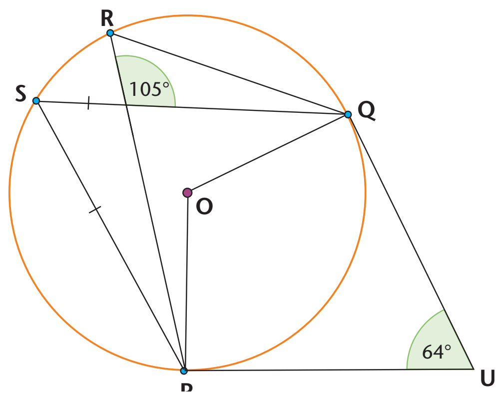
5. EG is a diameter of the circle through E, C and G. The tangent AEB is parallel to CD and $A\hat{E}C = 68^\circ$. Calculate the size of the following angles and give a reason for each answer:
   a) $C\hat{E}G$
   b) $E\hat{C}G$
   c) $C\hat{G}E$
   d) $E\hat{C}D$



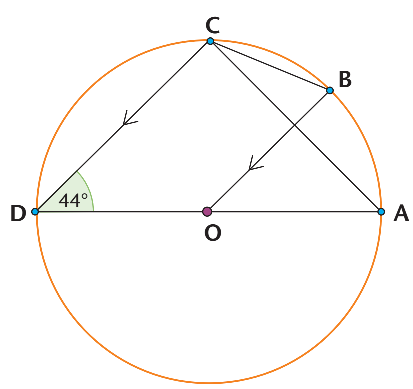
6. A, B, C and D lie on a circle centre O. AC is a diameter and $A\hat{C}D = 20^\circ$. $AB = BC$. Find the values of x, y and z.



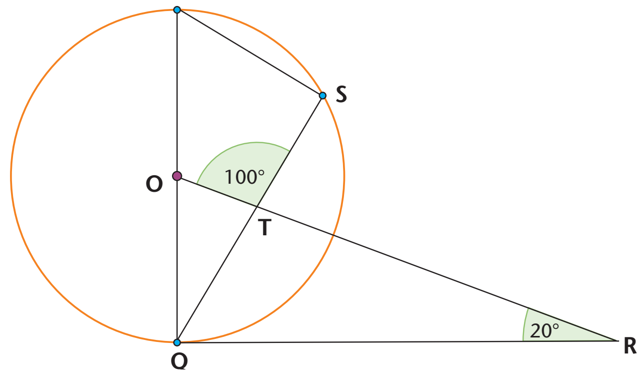
7. In the diagram, P, Q, R and S lie on a the circle, centre O. UP and UQ are tangents at P and Q. The lines PR and QS cross at T. $PS = QS$, $R\hat{T}Q = 105^\circ$.
   a) Calculate:
      (i) $P\hat{O}Q$
      (ii) $O\hat{P}Q$
      (iii) $Q\hat{P}S$
      (iv) $R\hat{P}Q$.
   b) Explain why $OPUQ$ is a cyclic quadrilateral.



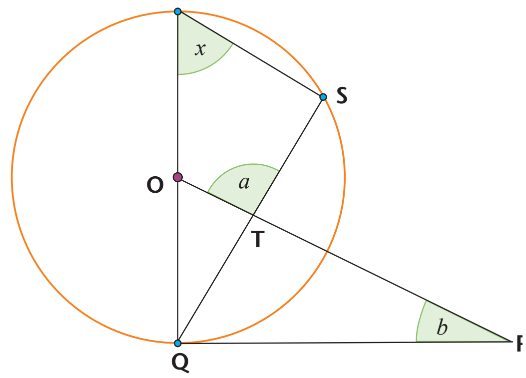
8. The diagram shows a circle, centre O, passing through A, B, C and D. AOD is a straight line, BO is parallel to CD and $C\hat{D}A = 44^\circ$. Find:
   a) $B\hat{O}A$
   b) $B\hat{C}A$
   c) $D\hat{C}B$
   d) $O\hat{B}C$.



9. a) In the diagram, find the value of x given that QR is a tangent, $S\hat{T}O = 100^\circ$ and $T\hat{R}Q = 20^\circ$.



b) In the diagram, PQ is a diameter of the circle centre O. Express x in terms of a and b.



Summary

In this chapter we learned about:

- Circular objects around us
- Revised concepts and theorems learned in earlier grades
  - The sum of adjacent angles on a straight line is 180°
  - Vertically opposite angles formed by intersecting straight lines are equal
  - If two parallel lines are cut by a transversal, then corresponding angles are equal
  - If two parallel lines are cut by a transversal, then alternate angles are equal
  - If two parallel lines are cut by a transversal, then the pair of co-interior angles are supplementary
  - The sum of interior angles of a triangle is 180°
  - If three sides in one triangle are respectively equal to the three sides in another triangle, the two triangles are congruent
  - If two angles and a side in one triangle are respectively equal to two angles and the corresponding side in another triangle, the two triangles are congruent
  - If a triangle has two sides equal, the angles opposite the equal sides are also equal

- In a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides
- The line segment joining the mid-points of two sides of a triangle is parallel to the third side and equal to half its length

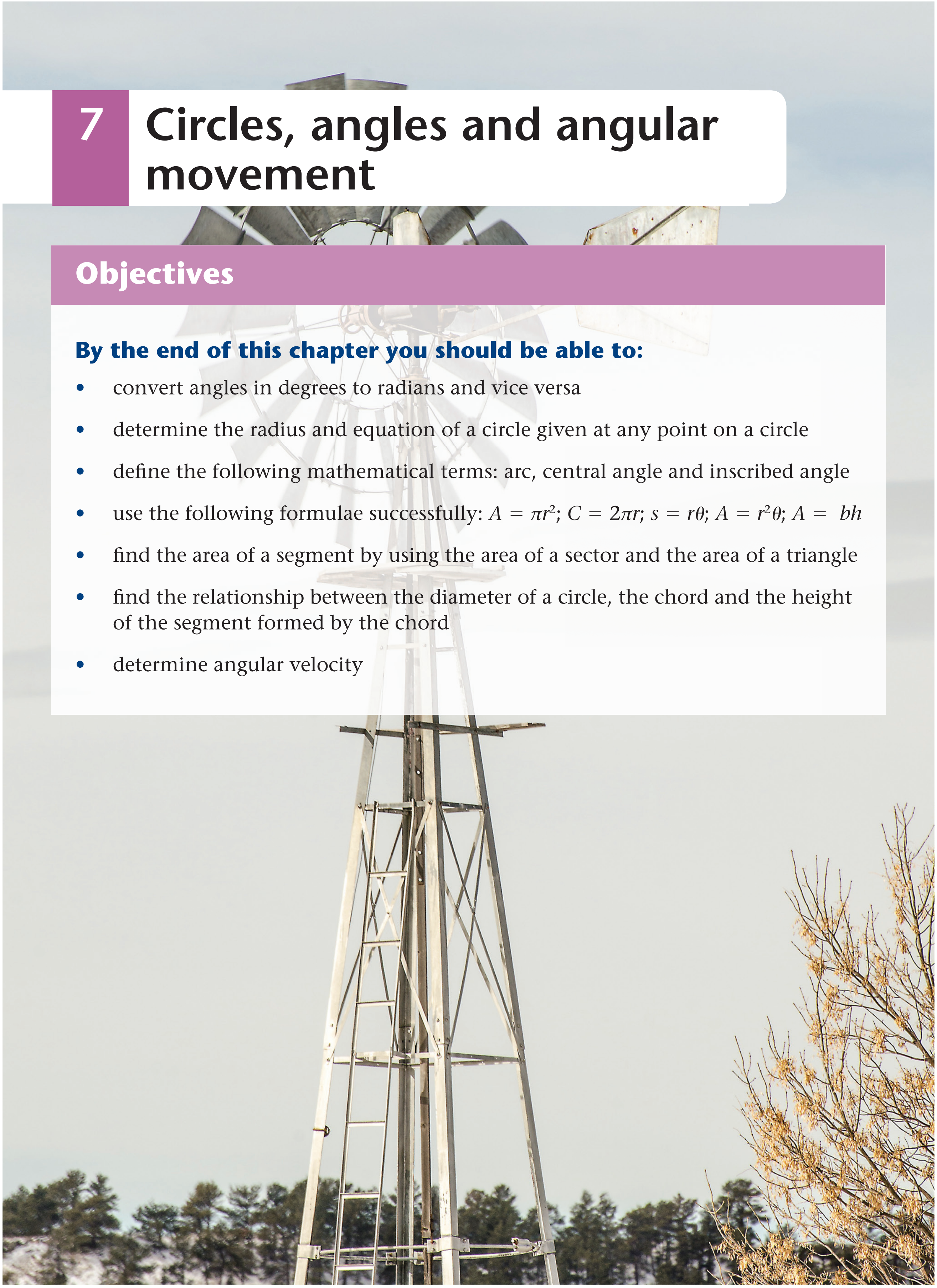
- Chords and radii/radiuses theorems
  - The line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord
  - The line from the centre of a circle, perpendicular to a chord, bisects the chord
- Angles in a circle theorem
  - The angle subtended at the centre of a circle by an arc is two times the angle subtended by the same arc at the circumference of a circle
  - The angle subtended at the circumference of a circle by the diameter is a right angle
  - The angles subtended by the chord or arc at the circumference of a circle are equal
- Cyclic quadrilaterals theorems
  - Opposite angles of a cyclic quadrilateral are supplementary
  - The exterior angle of a cyclic quadrilateral is equal to the interior opposite angle
- Tangents, secants and chords theorems
  - Axiom: A tangent to a circle is perpendicular to the radius drawn from the point of contact
  - Two tangents drawn from the same point outside the circle are equal in length
  - The angle between a tangent and a chord is equal to the angle in the alternate segment

7 Circles, angles and angular movement

Objectives

By the end of this chapter you should be able to:

- convert angles in degrees to radians and vice versa
- determine the radius and equation of a circle given at any point on a circle
- define the following mathematical terms: arc, central angle and inscribed angle
- use the following formulae successfully: $A = \pi r^2$; $C = 2\pi r$; $s = r\theta$; $A = r^2\theta$; $A = \ bh$
- find the area of a segment by using the area of a sector and the area of a triangle
- find the relationship between the diameter of a circle, the chord and the height of the segment formed by the chord
- determine angular velocity



A windmill converts wind energy into rotational energy.

7.1 Parts of a circle

When working with circles, there is some terminology (words) that help us to identify different parts of the circle. These are:

- arc – A portion of the circumference
- centre of a circle – the point from all points on a circle that is the same distance apart
- chord – a segment whose endpoints are on the circumference of the circle
- circumference – The length of the boundary. It is the circle's perimeter
- secant – a line that intersects the circle at two points
- diameter – a chord that passes through the centre of the circle
- radius – a line segment from the centre of a circle to the circumference of the circle
- vertex – is the point where two or more line segments (or edges) meet.

7.2 Circular measure

Let's revise some formulae that we have used when working with circles.

To measure the area of a circle, we use the formula $A = \pi r^2$, where r is the radius.

To measure the circumference of a circle, we use the formula $C = 2\pi r$, where r is the radius or $C = \pi d$, where d is the diameter).

Worked example 7.1

A bicycle has a rev counter on its front wheel that counts the number of revolutions made from a starting point. A cyclist resets the rev counter to zero before a trial race of 4 km. At the end of the trial, the rev counter reads 1 852.

What is the diameter of the front cycle wheel in cm? (Use the π key on your calculator).

Solution

To calculate the diameter:

$$4 \text{ km} = 400\,000 \text{ cm}$$

$$1\,852 \text{ revolutions} = 400\,000 \text{ cm}$$

$$\therefore 1 \text{ revolution} = \frac{400\,000}{1\,852} \text{ cm} = 215{,}98 \text{ cm}$$

$$1 \text{ revolution} = 1 \times \text{circumference}$$

$$C = \pi \times d$$

$$\therefore d = \frac{C}{\pi} = \frac{215{,}98}{\pi} = 69 \text{ cm}$$

Worked example 7.2

A circular reservoir has a diameter of 38 m.

Calculate the area and the circumference of the reservoir.

Solution

To calculate the radius:

$$\text{Radius} = \frac{D}{2} = \frac{38}{2} = 19 \text{ m}$$

$$\text{Area of a circle} = \pi r^2 = \pi \times 19^2$$

$$= 361\pi = 1\,134{,}1 \text{ m}^2$$

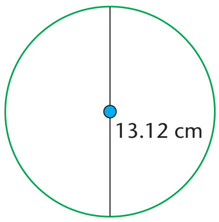
To calculate the circumference of the reservoir:

$$\text{Circumference} = 2\pi r = 2 \times \pi \times 19 = 119{,}38 \text{ m}$$

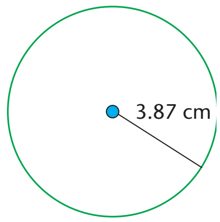
EXERCISE 7.1

1. Calculate the area and the circumference of each circle. Round your answer to two decimal places. (Use $\pi = 3{,}14$).

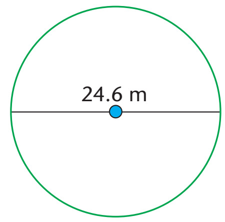
a)



b)



c)



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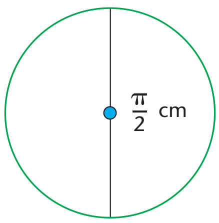
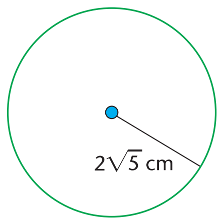


e)



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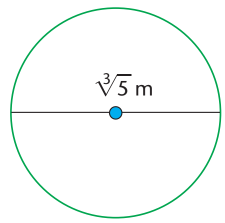


2. The diameter of the cake is 38,72 cm.

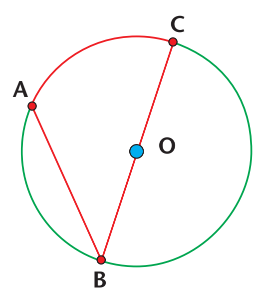
   What is the maximum area available for toppings?

7.3 Inscribed angle

An inscribed angle is an angle whose vertex is on a circle and whose sides contain chords of the circle.

An intercepted arc contains endpoints that lie on the sides on an inscribed angle and all the points of the circle between them.

A chord or arc subtends an angle if its endpoints lie on the sides of the angle.



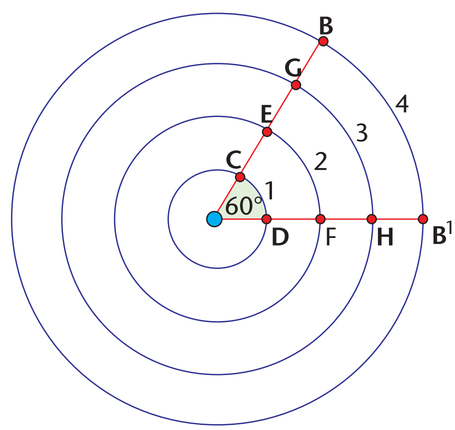
Arcs and their measure

Radian measures and arc length

To measure an angle, we mostly use degrees. The circumference of the circle is divided into 360 equal parts. We call each of these equal parts a degree.

Another common measure to measure an angle is the radian. The radian uses the relationship between the radius of a circle and the length of an arc of the circle.

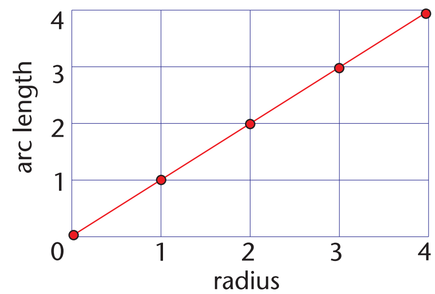
The radian measure, θ, of a central angle is the ratio of the arc length, s, the angle subtended by the radius of the circle, r.



$\theta = \frac{s}{r}$, where θ is a central angle, s is an arc length and r is the length of the radius.

If $s = r$, then $\theta = 1$ radian.

The relationship between the radius and arc length is linear, with a slope (gradient) of $2\pi r\left(\frac{60^\circ}{360^\circ}\right) = \frac{\pi}{3}$, or about 1,05.



The slope represents the ratio of an arc length to radius. This ratio is the radian measure of the angle, so 60° is the same as $\frac{\pi}{3}$ radians.

If the central angle θ of radius r intercepts an arc of length r, then the measure of θ is defined as 1 radian.

Since the circumference of a circle of radius r is $2\pi r$, an angle representing one complete revolution measure is 2π radians, or 360°.

Relationship between degrees and radians

$1 = \frac{180^\circ}{\pi \text{ radians}}$ or $1 \textit{ radian} = \frac{180^\circ}{\pi}$

$1 = \frac{\pi \text{ radians}}{180^\circ}$ or $1^\circ = \frac{\pi \text{ radians}}{180}$

7.4 Converting angle measure

Sometimes we may need to convert between degrees and radians.

To convert from degrees to radians:

- multiply the number of degrees by $\frac{\pi \text{ radians}}{180^\circ}$.

To convert from radians to degrees:

- multiply the number of radians by $\frac{180^\circ}{\pi \text{ radians}}$.

Worked example 7.3

Convert 45° to radians.

Solution

$45^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{4} \text{ radians}$

Worked example 7.4

Convert 75° to radians.

Solution

$75^\circ \times \frac{\pi \text{ radians}}{180^\circ} = \frac{5}{12}\pi \text{ radians}$

Worked example 7.5

Convert $\frac{\pi}{6}$ to degrees.

Solution

$\frac{\pi}{6} \times \frac{180^\circ}{\pi} = 30^\circ$

Worked example 7.6

Convert $\frac{2\pi}{9}$ to degrees.

Solution

$\frac{2\pi}{9} \times \frac{180^\circ}{\pi} = 40^\circ$

EXERCISE 7.2

1. Convert each measure to radians.

   a) 30° c) 90° e) 330°

   b) 60° d) 180° f) 270°

2. Convert each measure to degrees.

   a) $\frac{7\pi}{12}$ c) $\frac{7\pi}{30}$ e) $\frac{19\pi}{36}$

   b) $\frac{11\pi}{36}$ d) $\frac{7\pi}{60}$ f) $\frac{16\pi}{45}$

7.5 Length of a circular arc

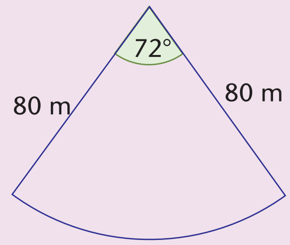
An angle whose radian measure is θ is subtended by an arc that is the fraction, $\frac{\theta}{2r}$, of the circumference of a circle. Thus, in a circle of radius r, the length s of an arc that subtends the angle θ, (see the diagram below) is:

$$\begin{aligned} s &= \frac{\theta}{2\pi} \times \text{ circumference of a circle} \\ &= \frac{\theta}{2\pi}(2\pi r) \\ &= r\theta \end{aligned}$$

Worked example 7.7

An enclosure is in the shape of a 72° sector of a circle with a radius of 80 m. Calculate the cost of fencing the enclosure, if fencing costs R360 per metre.

Solution



To calculate the cost of fencing, first calculate the perimeter.

Perimeter = twice the radius plus the arc length.

Arc length $= r\theta = 80 \times 72^\circ \frac{\pi}{180^\circ} = 100{,}53$ m

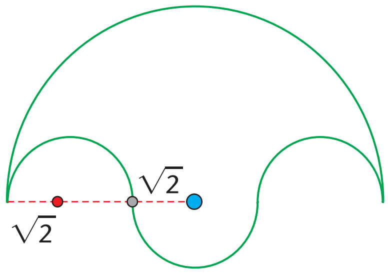
Perimeter $= 2 \times 80 + 100{,}53 = 261$ m

Cost of fencing $=$ R360 $\times$ 261

$=$ R93 600,00

EXERCISE 7.3

1. Greek mathematicians studied the salinon, which is a figure bounded by four semicircles. What is the perimeter of this salinon to the nearest tenth of a metre?



2. The pedals of a penny-farthing bicycle are directly connected to the front wheel.
   a) Suppose a penny-farthing bicycle has front wheels with diameter of 2,5 m.

      How far does the bike move when you turn the pedals through an angle of 90°? Write your answer correct to two decimal places.
   b) Through what angle should you turn the pedals to move forward by the distance of 4,5 m? Round your answer to the nearest degree?

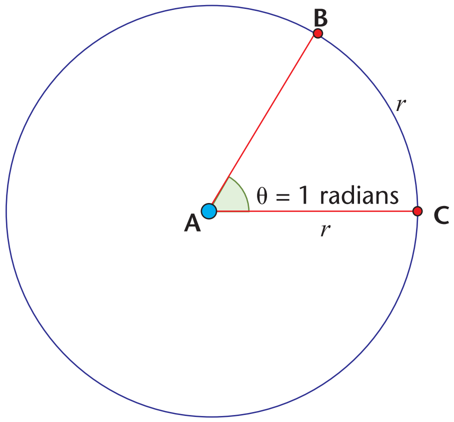
7.6 Sector area and arc length

A sector of a circle is a region bounded by two radii of the circle and their intercepted arc. The area of a sector is a fraction of the circle containing the sector.

To find the area of a sector whose central angle θ is measured in radians, multiply arc length and radius divided by two.

Formula for calculating area is given by:

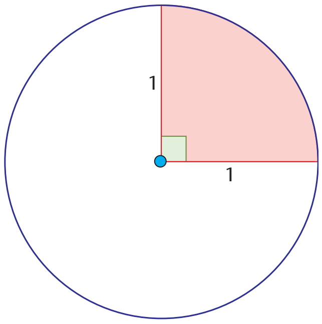
$$A = \frac{rs}{2} = \frac{r^2\theta}{2}$$



EXERCISE 7.4

1. Calculate the arc length and the area of each shaded sector.

a)



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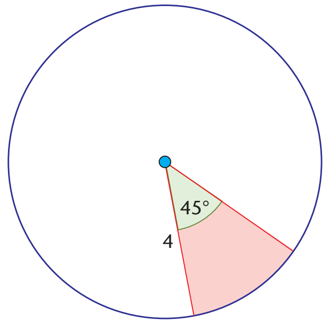

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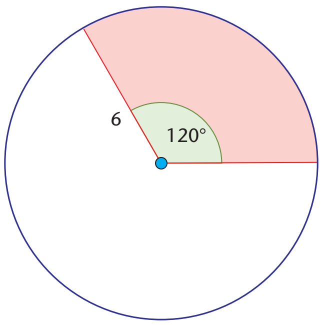

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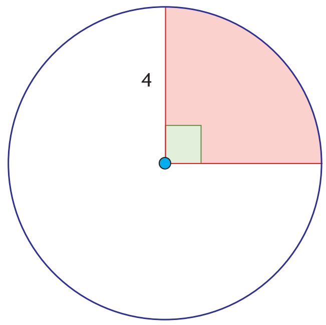
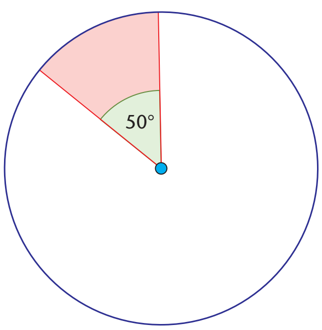
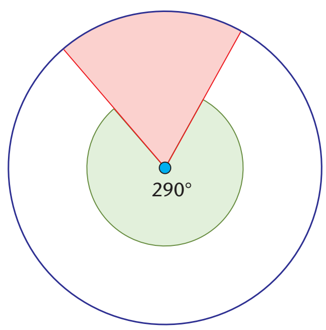


2. In a circle with radius 6 cm, AÔB = 60°. Make a sketch and calculate the area of the region bounded by arc AB and 2 radii of the circle.
3. A circle has an area of 160π cm². If a sector of the circle has an area of 40π cm², calculate the length of the arc of the sector to the nearest degree.
4. The area of sector AOB is $\frac{7\pi}{2}$ cm and AÔB = 315°. Calculate the length of the radius of circle O.
5. Use the given area of the sector and the angle bounded by an arc to calculate the radius of each circle.

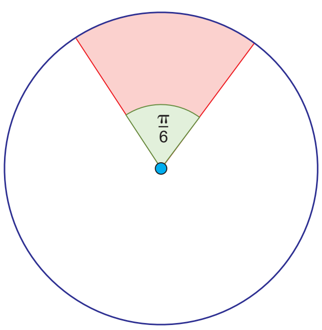
a) A = 15 π cm²



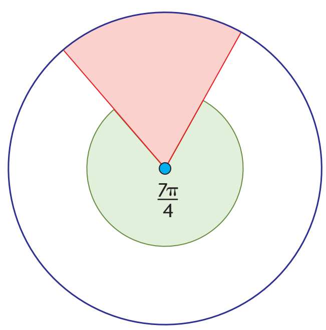
b) A = 20 π cm²



c) A = 15 π cm²

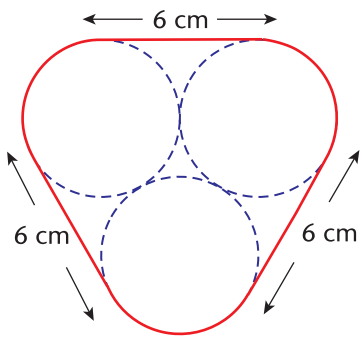


d) A = 20 π cm²

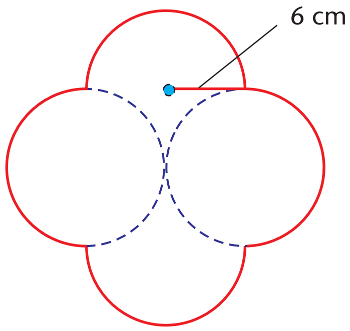


6. Calculate the perimeter of each object.

a)



b)



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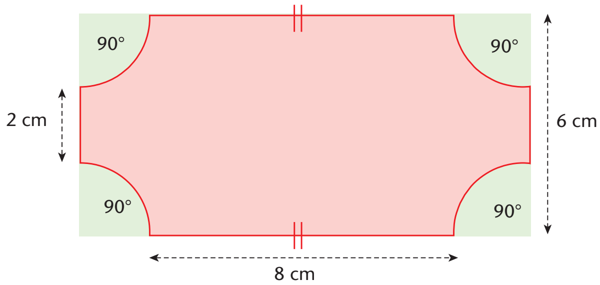

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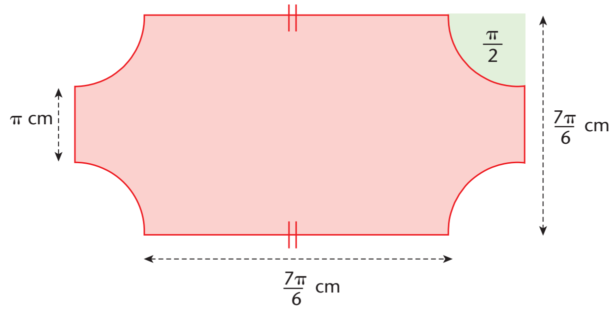
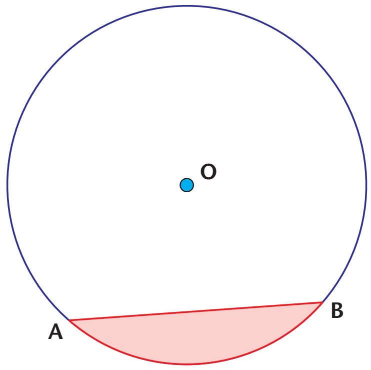


7.7 Segment

A segment is a part of a circle bounded by a chord and an arc. In the diagram on the right, the line AB is the chord, and the curve AB is the corresponding arc.

(The chord is not a diameter, since it does not pass through the centre O.)

The shaded region is a minor segment because its area is less than half that of the circle. The unshaded region is a major segment since its area is more than half that of the circle.

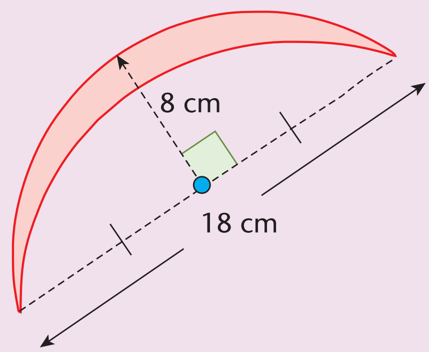


7.8 The relationship between a segment's height, diameter and its width

It is possible to derive the relationship between the diameter, the width and the height of the segment using the formula: $4h^2 - 4dh + x^2 = 0$; where h is the height of the segment, d is the diameter and x is the width of the segment.

Worked example 7.8

The art department was contracted to construct a wooden moon for a play.
One of the artists created a sketch of what it should to look like by drawing a chord and its perpendicular bisector.
Find the diameter of the circle used to draw the outer edge of the moon.



Solution

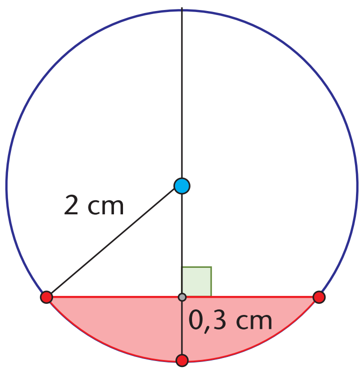
The width and the height of the segments are given.
We need to apply the formula: $4h^2 - 4dh + x^2 = 0$
Substituting $h = 8$ cm and $x = 18$ cm

We get: $4(8)^2 - 4d(8) + 18^2 = 0$

Solve for d:
$-32d = -580$
$d = 18{,}13$ cm

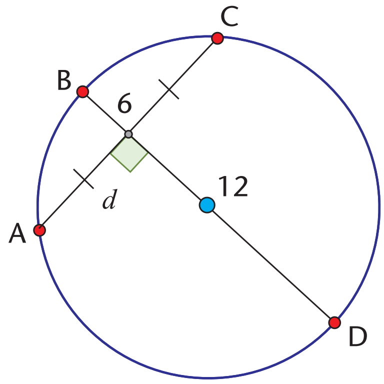
EXERCISE 7.5

1. Determine the width of the shaded segment using the dimensions below:



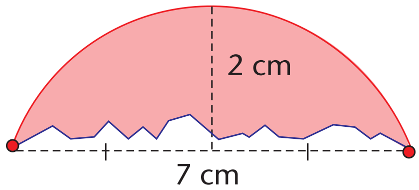
2. Calculate the height of the segment (called a sagitta) with a diameter of 10 m and width of 7 m.

   Use a rough sketch to visualise the problem.
3. Determine the value of d and the length of chord AC.



4. The diagram below shows a broken plate.

   Use the given information to calculate the diameter of the original plate.



7.9 Angular and circumferential velocity

A particle's motion can be measured using its angular velocity (also known as its linear velocity). When we calculate angular or linear velocity we assume that the object is moving along a circular path at a constant speed that is neither increasing nor decreasing.

Angular velocity ω (omega) can be determined by multiplying the radians in one revolution ($2\pi r$) with the revolutions per seconds (n). Thus we can say: $\omega = 2\pi n$

The angular velocity of an object can be defined as the object's angular displacement with respect to time, and it is measured as revolution per unit time or radians per second.

Circumferential velocity is the linear velocity of a point on the circumference. Thus we say: $v = \pi Dn$, D is the diameter and n is the rotational frequency.

The relationship between angular velocity and circumferential (linear) velocity is given below:

$$\omega = 2\pi n$$

$$\Rightarrow n = \frac{\omega}{2\pi} \ldots (1)$$

$$v = \pi Dn$$

$$\Rightarrow n = \frac{v}{\pi D} \ldots (2)$$

Equating (1) and (2):

$$\therefore \frac{\omega}{2\pi} = \frac{v}{\pi D}$$

$$\therefore \omega = \frac{v \times 2\pi}{\pi D} = \frac{2v}{D} = \frac{2v}{2r} = \frac{v}{r}$$

$$\therefore v = \omega \times r$$

The unit for measuring linear velocity is metre per second.

Worked example 7.9

The Earth rotates on its own axis once every 24 hours. Calculate the Earth's angular velocity of rotation in degrees per hour.

Solution

The complete circle is 2π radians, therefore, the angular velocity of the Earth's rotation equals $\frac{2\pi}{24}$ radians per hour.

$= \frac{2\pi}{24}\left(\frac{180}{\pi}\right)$

$= 15^{\circ}$ per hour

Worked example 7.10

A ceiling fan rotates 30 times per minute. Calculate the angular velocity of the fan in degrees per hour.

Solution

The angular velocity $= 2\pi r \times 30$ radian per minute

$= \frac{60\pi \text{ radians}}{\text{min}} \times 60 \times \frac{\text{min}}{\text{hour}}$

$= 3\ 600\frac{\text{radians}}{\text{min}} \times \frac{\text{min}}{\text{hour}}$

$= \frac{3\ 600\pi \text{ radians}}{\text{hour}}\left(\frac{180}{\pi}\right)$

$= 648^{\circ}$ per hour

EXERCISE 7.6

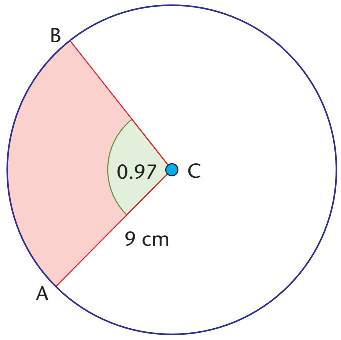
1. An automobile engine is described as operating at 5 000 rpm, meaning its crankshaft completes 5 000 revolutions per minute. Express this angular velocity in radians per second.
2. The second hand of a clock is 10,2 centimetre long. Calculate the linear speed of the tip of this second hand as it passes around the clock face.
3. A disk with a 12 cm diameter spins at the rate of 45 revolutions per minute.

   Calculate the angular and linear velocities of a point at the edge of the disk in radians per second and centimetre per second, respectively.
4. A Ferris wheel with a 50 m radius makes 1,5 revolutions per minute.
   a) Calculate the angular speed of the Ferris wheel in radians per minute.
   b) Calculate the linear speed of the Ferris wheel.

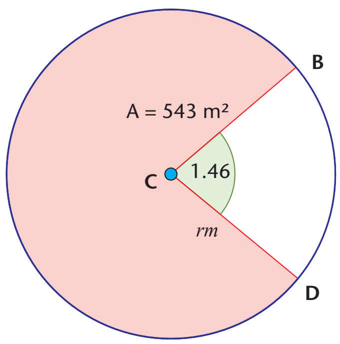
CONSOLIDATION EXERCISE

1. Write the value of an angle measuring $\frac{\pi}{15}$ radians in degrees.
2. Write 45° in radians. Give your answer as a multiple of π.
3. Use your calculator to calculate the value of $\sin\left(\frac{18\pi}{5}\right)$. Give your answer correct to three decimals.
4. An arc AB of a circle, centre O and radius, r, subtends and angle x radians at O. The length of AB is l.
   a) Calculate l given that $r = 9$ m and $x = \frac{3\pi}{5}$.
   b) Calculate r given $l = 14{,}9$ cm and $x = 2{,}98$ radians.
   c) Calculate x given $l = \frac{49\pi}{9}$ m and $r = 7$.

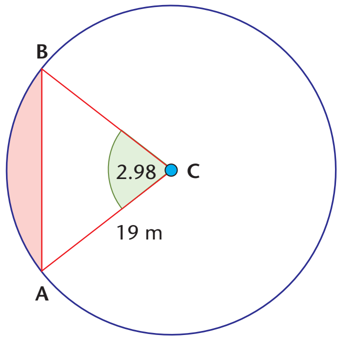
5. A minor arc CD of a circle, centre O and radius 12 m, subtends an angle $3x$ at O.
   The major arc CD subtends an angle $7x$ at O. Calculate, in terms of π the length of minor arc CD.
6. A sector of a circle of radius 17 cm contains an angle x radians.
   Given that the perimeter of the sector is 53 cm, calculate the value of x.
7. Calculate the area of the shaded sector in the following diagram.
   Give your answer correct to three decimal places.



8. In the diagram below, the area A of the shaded area and the angle are given.
   Calculate the length of the radius.



9. Find the area of the shaded segment in the following diagram.
   Give your answer correct to 3 decimals places.



10. The arc AB of a circle, centre O and radius r m is such that the angle AOB is 2,22 radians.
    Given that the perimeter of the minor sector AOB is 80 m, calculate the area of the segment enclosed by the chord AB and minor arc AB.
11. A woman is riding a bicycle whose wheels are 66 cm in diameter.
    If the wheels rotates at 125 revolutions per minute (rpm), find the speed (in km/h) at which she is travelling.

Summary

- angular velocity refers to how fast an angle is changing with respect to time. It can also be described as the change in angle over the change in time. The symbol for angular velocity is ω (omega) and the unit of measurement is radians per minute.

> **Note:**
> For calculations radians per minute must be converted to radians per seconds ($rad.s^{-1}$) since the standard unit of time is seconds.

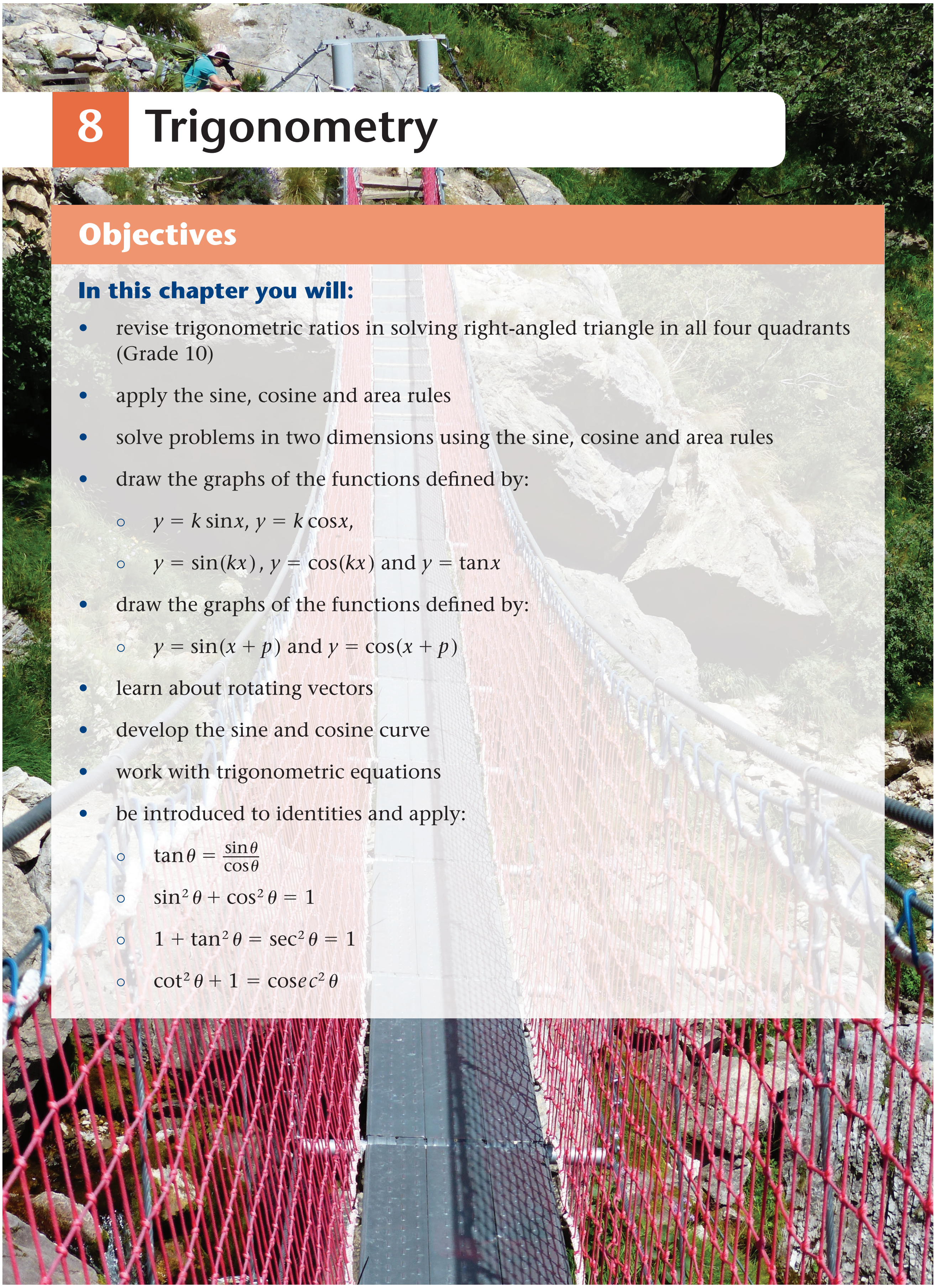
- the formula for calculating Angular velocity is: $\omega = \frac{\theta}{t}$
- rotational frequency is the number of revolutions per second (rev/s). The symbol for rotational frequency is n. The formula for rotational frequency is: $n = \frac{\text{no of revolutions}}{t}$
- in one revolution per second, the angular velocity is: $\omega = \frac{\theta}{t} = \frac{2\theta \text{ radians}}{1s} = 2\pi\text{rad}.s^{-1}$
- for n number of revolution: $\omega = n \times 2\pi = 2n\pi$, hence $\omega = 2n\pi$
- circumferential/peripheral velocity refers to how fast the object is travelling with respect to time. Hence the formula for calculating circumferential velocity is $v = \frac{s \text{ (metres)}}{t \text{ (seconds)}} = \text{m}/s$.
- in one revolution per second, the Circumferential velocity is: $v = \frac{s}{t} = \frac{\pi D}{1s} = \pi D \text{ m}/s$
- for n number of revolution $v = n \times \pi D$ or $v = \pi Dn$
- if we further explore the above formula $v = \pi Dn = \pi \times 2r \times n = 2\pi nr = \omega r$ or $v = \omega r$.

8 Trigonometry

Objectives

In this chapter you will:

- revise trigonometric ratios in solving right-angled triangle in all four quadrants (Grade 10)
- apply the sine, cosine and area rules
- solve problems in two dimensions using the sine, cosine and area rules
- draw the graphs of the functions defined by:
  - $y = k\sin x$, $y = k\cos x$,
  - $y = \sin(kx)$, $y = \cos(kx)$ and $y = \tan x$
- draw the graphs of the functions defined by:
  - $y = \sin(x + p)$ and $y = \cos(x + p)$
- learn about rotating vectors
- develop the sine and cosine curve
- work with trigonometric equations
- be introduced to identities and apply:
  - $\tan\theta = \frac{\sin\theta}{\cos\theta}$
  - $\sin^2\theta + \cos^2\theta = 1$
  - $1 + \tan^2\theta = \sec^2\theta = 1$
  - $\cot^2\theta + 1 = \text{cosec}^2\theta$

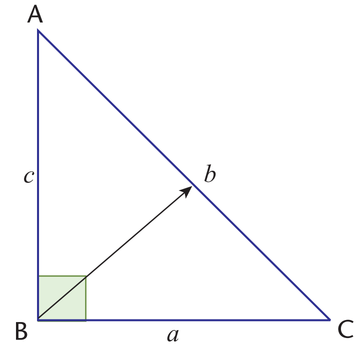


*In a **bridge** construction project, **trigonometry** is essential for determining just how long a **bridge** needs to be.*

8.1 Revision: Trigonometric ratios

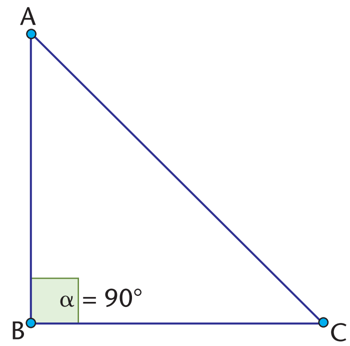
Labelling sides of a right-angled triangle

Consider the right-angled triangle below.



There is an agreed upon way (convention) for naming and labelling the parts of a right-angled triangle. A reference angle is used to decide the position of a side in a triangle. Sides are named as being opposite to or adjacent to the reference angle.

- The **hypotenuse** is identified by its position in relation to the 90˚ angle. It is the side opposite the right angle and it is the longest side of the three sides of the triangle.
- The **opposite side** is opposite to a reference angle.
- The **adjacent side** is adjacent to a reference angle.



Three trigonometric ratios

For a right-angled triangle ABC, the sine, cosine and tangent ratios of angle C are defined as:

a) $\sin\hat{C} = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{AB}{AC}$

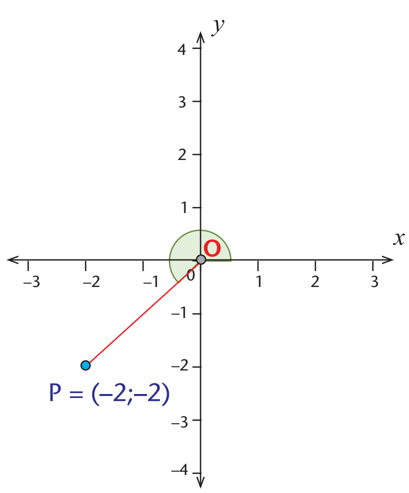
b) $\cos\hat{C} = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{BC}{AC}$

c) $\tan\hat{C} = \frac{\text{opposite}}{\text{adjacent}} = \frac{AB}{BC}$

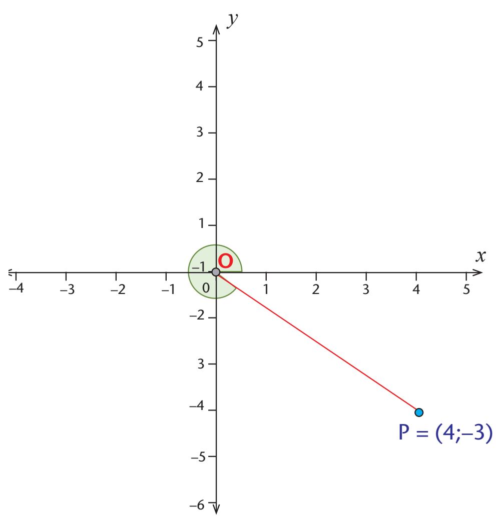
EXERCISE 8.1

1. If $\sin\alpha = t$, calculate:
   a) $\cos\alpha$
   b) $\tan\alpha$
2. For each of the figures below, calculate:
   a) the length of OP
   b) all three trigonometric ratios for θ. (Leave answers in simplified surd form where necessary).

   (i)



   (ii)



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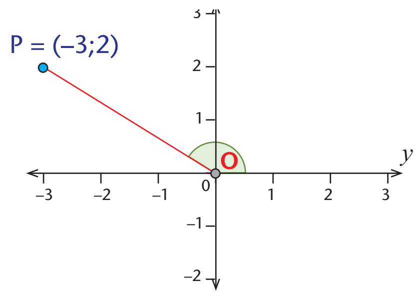
(iii)



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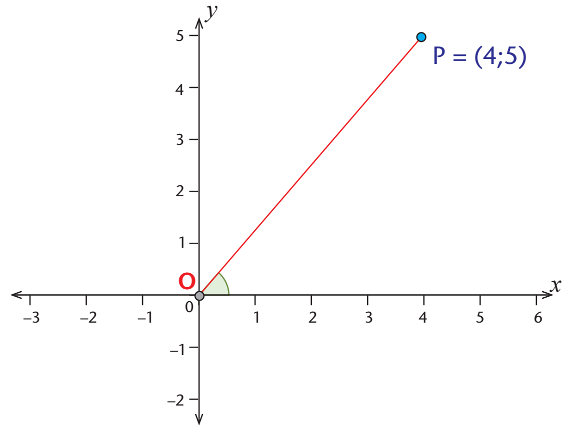
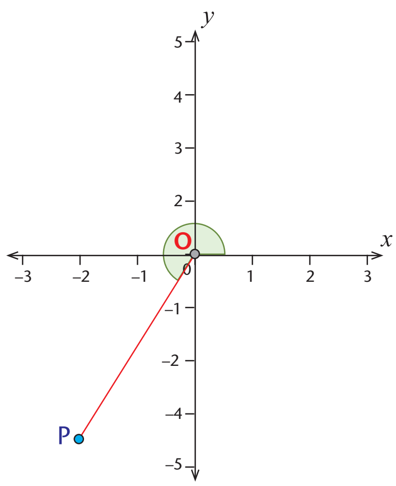


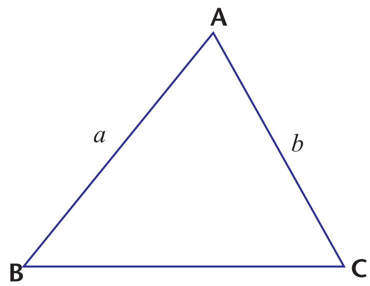
3. If $25\sin\theta + 7 = 0$, and $0^\circ < \theta < 270^\circ$:
   a) calculate $\cos\theta$
   b) calculate $\tan\theta$
   c) calculate the value of $\sin^2\theta + \cos^2\theta$ (without using a calculator)
   d) $\dfrac{\sin\theta}{\cos\theta}$
4. If $5\tan\theta = 3$, calculate the value of:
   a) $\cos\theta$
   b) $\sin\theta$
   c) $\dfrac{\sin\theta}{\cos\theta}$
   d) $\dfrac{\cos\theta}{\sin\theta}$
   e) $\cos\theta\sin\theta\ \cos\theta\ \sin\theta$
   f) $(\sin\theta + \cos\theta)(\cos\theta - \cos\theta)$
   g) $\sin^2\theta + \cos^2\theta$
4. If $P(-2; -\sqrt{12})$, without a calculator, find:
   a) the length of OP
   b) find
   c) $\sin\dfrac{\theta}{4}$
   d) $\dfrac{1}{4}\sin\theta$



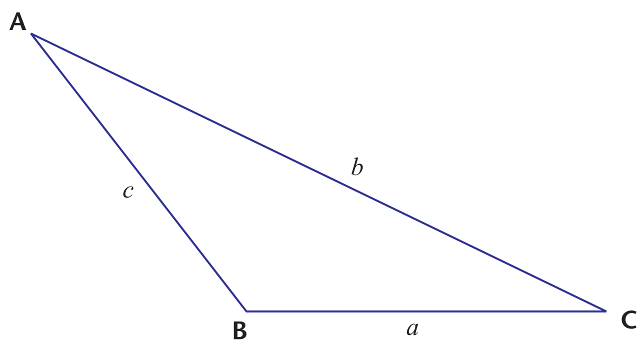
8.2 Solving triangles

We have used trigonometric ratios (sine, cosine and tangent ratios) to solve right-angled triangles. However, not all triangles are right-angled triangles. We also have triangles that do not have a right angle. Such triangles are called oblique triangles.

In some oblique triangles each angle of the triangle has a value less than 90˚. We say such triangles are acute oblique triangles. ΔABC below is an example of an oblique triangle that is acute.

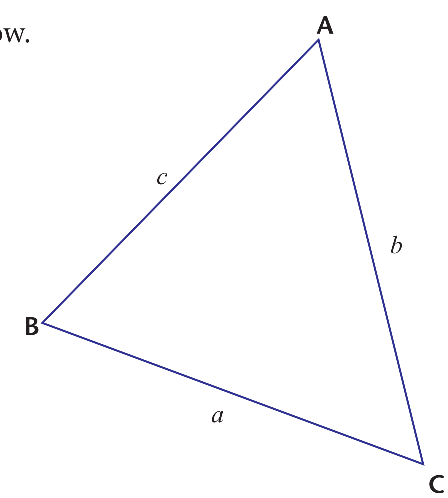


In other cases, one of the angles of a triangle has a value that is more than 90˚. Such a triangle are said to be an obtuse oblique triangles.



8.3 The sine rule

Consider the triangle given below.



The sine rule states that in any triangle where a is the length of the side opposite angle A, b the length of the side opposite angle B and c the length of the side opposite angle C

$$\frac{\text{the length of any side}}{\text{the sine of the angle opposite the side}} = \frac{\text{the length of any other side}}{\text{the sine of the angle opposite this side}}$$

We can state the above in symbolic form as follows: In any ΔABC:

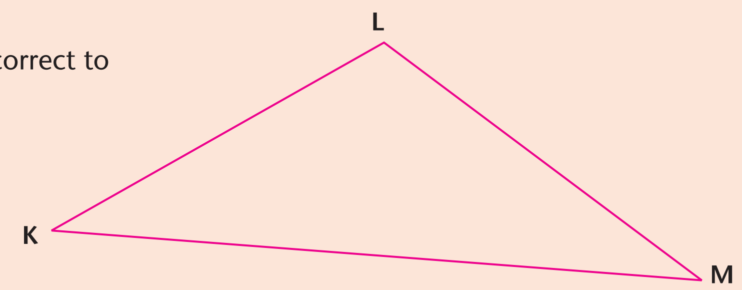
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \text{ or } \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The sine rule is helpful when we have the following information about a triangle:

- the lengths of the two sides of the triangle and the size of an angle opposite one of the sides or
- the values of two angles and the length of one side

Worked example 8.1

Solve the triangle alongside. Give answer correct to 2 decimal places.



Note:

To solve a triangle means to calculate the lengths of all the sides and all the angles of the triangle.

Solution

$$\hat{L} = 180° - (\hat{K} + \hat{M})$$
$$= 180° - (34{,}45° + 32{,}29°)$$
$$= 113{,}26°$$

$$\frac{LM}{\sin 34{,}45°} = \frac{3{,}44}{\sin 32{,}29°}$$

$$LM = \frac{3{,}44 \ \sin 34{,}45°}{\sin 32{,}29°}$$
$$= 3{,}64 \text{ units}$$

$$\frac{KM}{\sin 113{,}26°} = \frac{3{,}44}{\sin 32{,}29°}$$

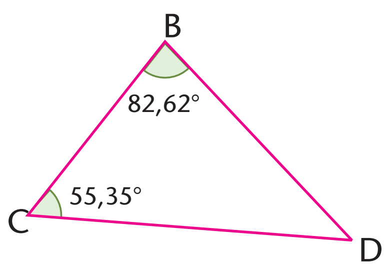
$$KM = \frac{3{,}44 \sin 113{,}26°}{\sin 32{,}29°}$$

$$KM = 5{,}92 \text{ units}$$

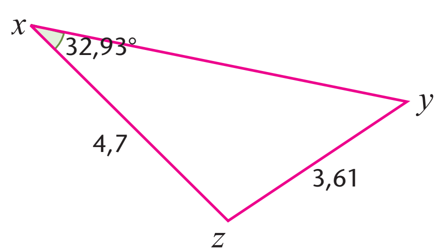
EXERCISE 8.2

Solve the triangles given below. Give your answers correct to two decimal places.

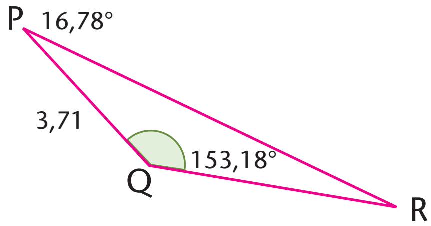
a)



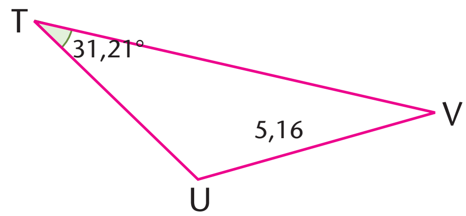
b)



c)



d)



More than one possibility: The impossible case and the ambiguous case

You need a compass, a protractor and a ruler to do the task below.

1. Draw $\hat{A} = 30°$, the arms forming the angle may be of any length greater than 10 cm.
2. On one of the arms of $\hat{A}$, measure AC = 10 cm.
3. From point C draw three arcs with the following measurements:

   From C draw an arc that measures 4 cm.

   From C draw an arc that measures 5 cm.

   From C draw an arc that measures 6 cm.

   Note whether any of the arcs crosses the other arm of $\hat{A}$. If it does, note in how many points.

   The above activity demonstrates that if we are given the lengths of two sides of a triangle and a non-included angle, we have three possibilities:

   A. no triangle at all
   B. one triangle possible
   C. two triangles possible. This possibility is referred to as the ambiguous case.

EXERCISE 8.3

In ΔABC calculate the value of

a) $\hat{A} = 30°$, $b = 10$ cm; $a = 4$ cm
b) $\hat{A} = 30°$, $b = 10$ cm; $a = 5$ cm
c) $\hat{A} = 30°$, $b = 10$ cm; $a = 6$ cm

Worked example 8.2

We can use the sine rule to show that it is not possible to draw a triangle with the following measurements: $\hat{A} = 60°$, $a = 16$ cm and $c = 20$ cm

Solution

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

$$\frac{\sin 60°}{16} = \frac{\sin C}{20}$$

$$\sin C = \frac{20 \sin 60°}{16}$$

$$\sin C = 1{,}082531755$$

We know there is no angle for which the sine is greater than 1. So it is not possible to draw such a triangle.

Worked example 8.3

Solve the triangle given the following dimensions: AB = 8 cm, BC = 10 cm and $\hat{C} = 50°$

Solution

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

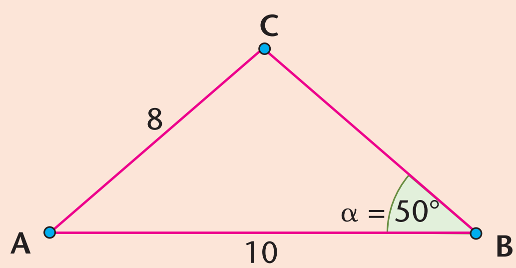
$$\frac{\sin A}{10} = \frac{\sin 50°}{8}$$

$$\sin A = \frac{10 \sin 50°}{8}$$

$$= 0{,}9575555539$$

$$A = \sin^{-1}(0{,}9575555539)$$

$$= 73{,}2° \text{ or } 180° - 73{,}2° = 106{,}8°$$



Worked example 8.4

Solve the triangle given the following dimensions: AB = 10 cm, BC = 8 cm and $\hat{C} = 50°$

Solution

$$\frac{\sin A}{a} = \frac{\sin C}{c}$$

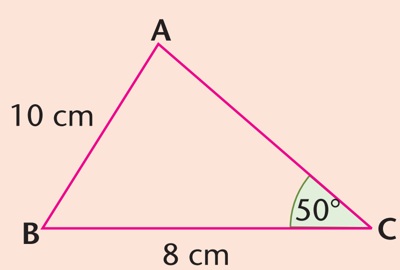
$$\frac{\sin A}{8} = \frac{\sin 50°}{10}$$

$$\sin A = \frac{8 \sin 50°}{10}$$

$$= 0{,}6128355545$$

$$A = \sin^{-1}(0{,}6128355545)$$

$$= 37{,}8° \text{ or } 180° - 37{,}8° = 142{,}2°$$



However, a 142,2° angle is not possible because if we add all three angles of ΔABC we will get more than 180°. So only one angle is possible, that is, $\hat{A} = 37{,}8°$ and there is no ambiguity in this case.

EXERCISE 8.4

1. If p = 51,25 cm, q = 43,8 cm and $\hat{P}$ = 130°, determine the remaining side and angles of the triangle.
2. A radio antenna, which is 25 metres high, is on top of a building. At a distance d, from the foot of the building, the angle of elevation to the top of the antenna is 25°, and the angle of elevation to the bottom of the antenna is 15°. How high is the building? (i.e. find h). Give the answer correct to one correct decimal place.
3. A pole is supported by two wires on each side creating an angle of 75° between the wires. The ends of the wires are 10 meters apart on the ground. One of the wires forms an angle of 30° with the ground. How long are the wires? Give the answer correct to one decimal place.
4. Bongani measures the angle of elevation of the highest point of a mountain to be 38°. Amantha, who is 750 metres closer on *a* level ground, measures the angle of elevation of to be 45°. How high is the mountain?
5. One of the base angles of an isosceles triangle is 75°. The length of one of the base sides is 10 cm. Calculate the perimeter of the triangle. Give the answer correct to one decimal place.

8.4 The cosine rule

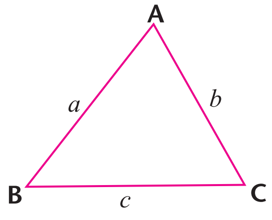
In some instances, we know the:

- lengths of all the three sides of an oblique triangle or
- lengths of two sides and the value of the angle between the two sides whose lengths are given.

In cases such as the above we use an alternative rule called the cosine rule.

The cosine rule states that:

In any triangle, the square of the length of one side of any triangle is equal to the sum of the squares of the lengths of the other two sides minus twice the product of the lengths of these two sides, multiplied by the cosine of the angle included between them.



We can state the Cosine rule symbolically as follows:

$a^2 = b^2 + c^2 - 2bc\cos A$ or $b^2 = a^2 + c^2 - 2ac\cos B$ or $c^2 = a^2 + b^2 - 2ab\cos C$

We can rewrite the rule in terms of the angle as follows:

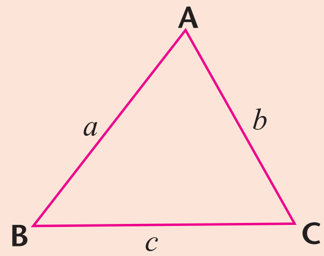
$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$ or $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$

This formula is useful when we want to calculate the measure of one of the angles when we know the lengths of three sides of a triangle.

Worked example 8.5

Solve the triangle given below. Give the answer correct to two decimal places.

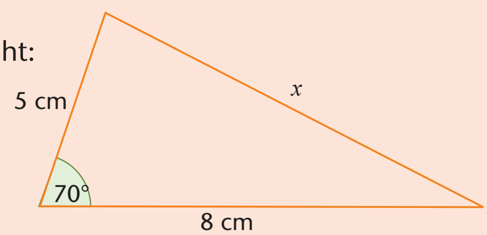
Solution



$$\cos A = \frac{(4.7)^2 + (6.49)^2 - (3.61)^2}{2(4.7)(6.49)}$$
$$= 0,9288906009$$
$$A = \cos^{-1}(0,9288906009)$$
$$A = 21.74°$$
$$\cos B = \frac{(6,49)^2 + (3,61)^2 - (4,7)^2}{2(6,49)(3,61)}$$
$$= 0,7055858363$$
$$B = \cos^{-1}(0,7055858363)$$
$$= 45,12°$$
$$\hat{C} = 180° - (21,74° + 45,12°)$$
$$= 113,14°$$

Worked example 8.6

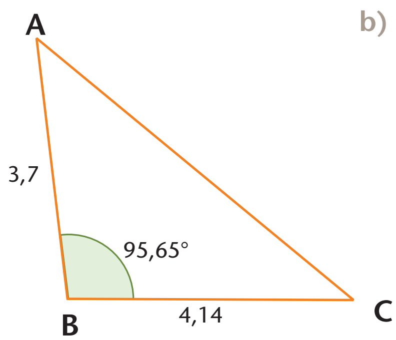
Calculate the length of the third side of the triangle on the right:

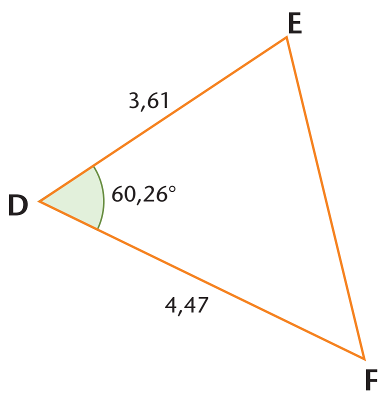


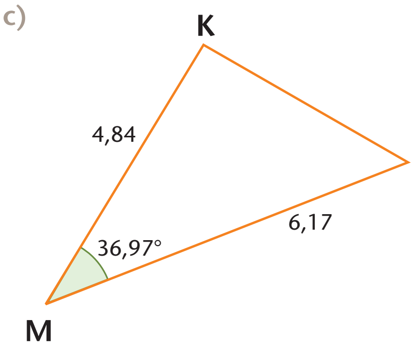
$$x^2 = 5^2 + 8^2 - 2(5)(8)\cos 70°$$
$$= 61.63838853$$
$$x = 7,851011943$$
$$= 7,85 \text{ correct to two decimal places}$$

EXERCISE 8.5

1. Solve the triangles below.

a)


b)


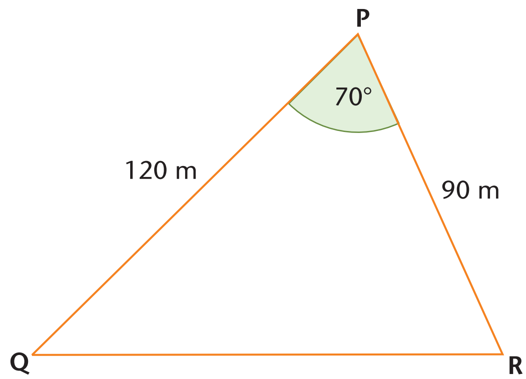
c)


2. In ΔABC, solve the triangle if:

$a = 4$, $b = 7$, $c = 5$

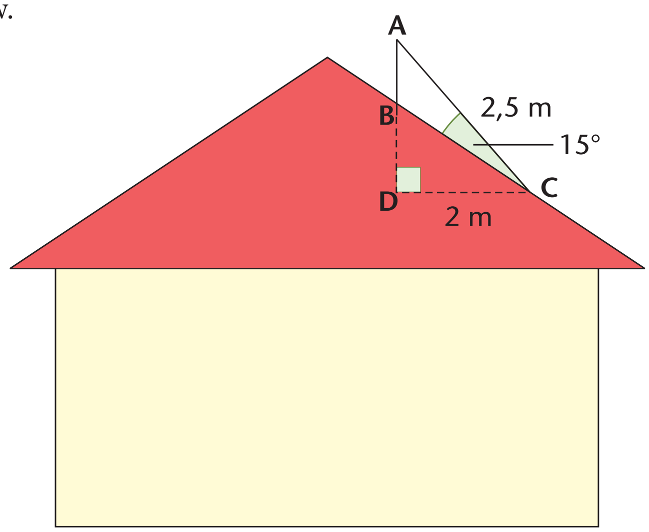
$a = 15,73$ m, $\hat{B} = 121°$, $c = 23,15$ m

3. A paddock is in the form of a triangle as shown



   a) Calculate QR
   b) How much will it cost to fence the paddock at R80,50 per metre?

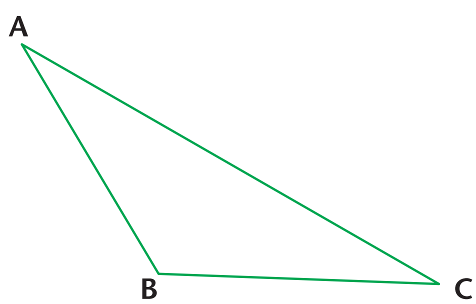
4. The sides of a parallelogram are 6 cm and 10 cm, and the angle between them is 70°. How long are the diagonals of the parallelogram?
5. The Dlamini family wants to install a solar collecting panel on the roof of their house as shown in the figure below.



AC = 2,5 m; DC = 2 m and $A\hat{C}B = 15°$. Calculate the length of the vertical brace AD.

8.5 The area rule

The area rule states that if we know the lengths of two sides and the angle included between them in any triangle, then the area of the triangle is given by half the product of the lengths of the two sides multiplied by the sine of the angle between them.

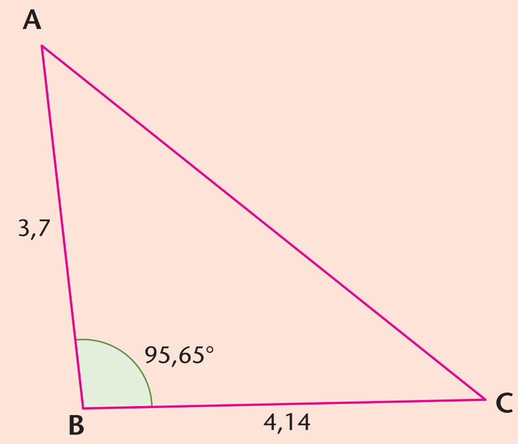


We can state the area rule symbolically as follows:

Area of $\Delta ABC = \frac{1}{2}bc\sin A$ or Area of $\Delta ABC = \frac{1}{2}ac\sin B$ or Area of $\Delta ABC = \frac{1}{2}ab\sin C$

Worked example 8.7

Calculate the area of ΔABC.



Solution

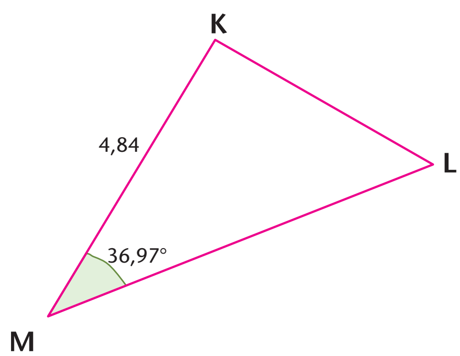
Area of $\Delta ABC = \frac{1}{2}(3,7)(4,14)\sin 95,65°$

$= 7,61$ square units

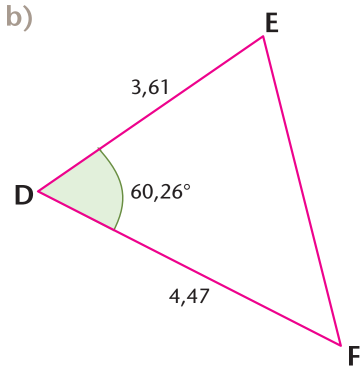
EXERCISE 8.6

1. Calculate the area of the given triangle. The answer should be given correct to two decimal places.

a)

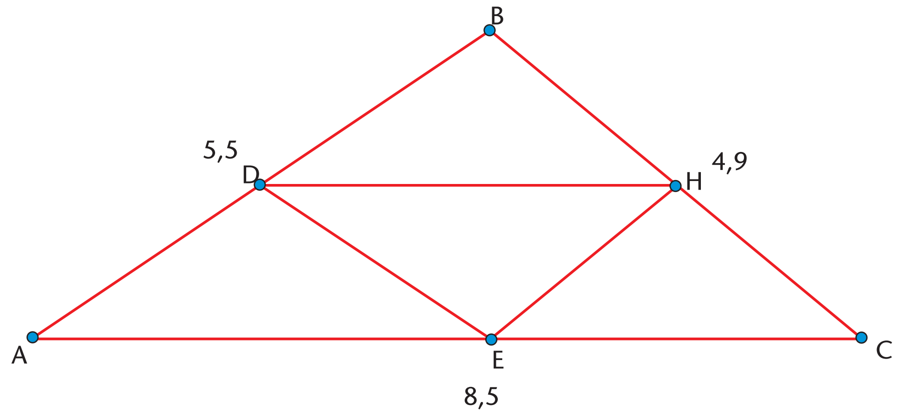


b)

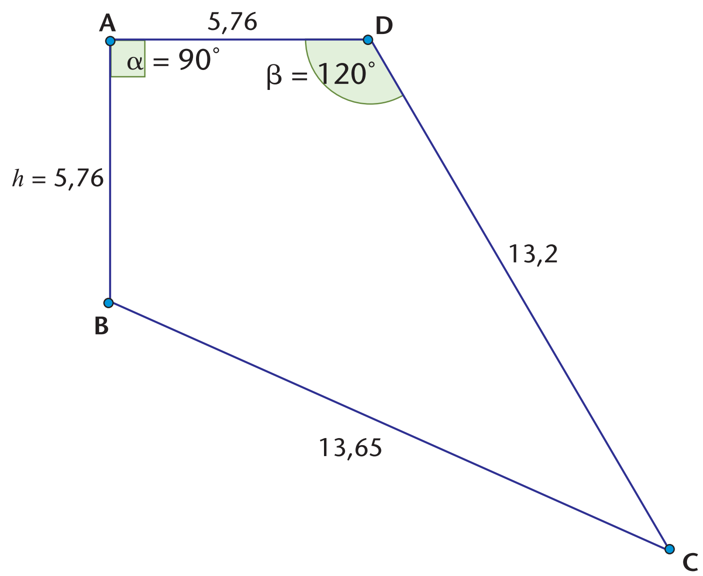


2. Calculate the area of the triangle with the given measurements. Give the answer correct to two decimal places. You may leave your answer in surd form where necessary.
   a) $\hat{L} = 80°$, $m = 5$ cm and $n = 3$ cm
   b) $\hat{M} = 45°$, $l = 8$ cm and $n = 6$ cm
   c) $\hat{N} = 150$, $m = 10$ cm and $n = 5$ cm
3. Calculate the area of ΔLMN with the given measurements. Give your answers rounded off to two decimal places.
   a) $\hat{E} = 127°$, $d = 32$ cm, $f = 25$ cm
   b) $\hat{F} = 152°$, $d = 6$ cm, $e = 5$ cm
   c) $\hat{D} = 115°$, $e = 10$ cm, $f = 5$ cm

4. Roof trusses are used to form the stability and core for a roof of a dwelling or a commercial building. Below is a figure showing the truss that is used for the roof of a new house. The length of rafter AB is 5,5 metres, rafter BC is 4,9 m long and rafter AC is 8,5 metres long. Calculate the area of triangle ABC.



5. The figure below shows the dimensions of a stand as it is drawn on a map. Calculate the area of the stand. (The lengths are in metres.)



8.6 Trigonometric graphs

Developing the sine and cosine curve through rotation of vectors

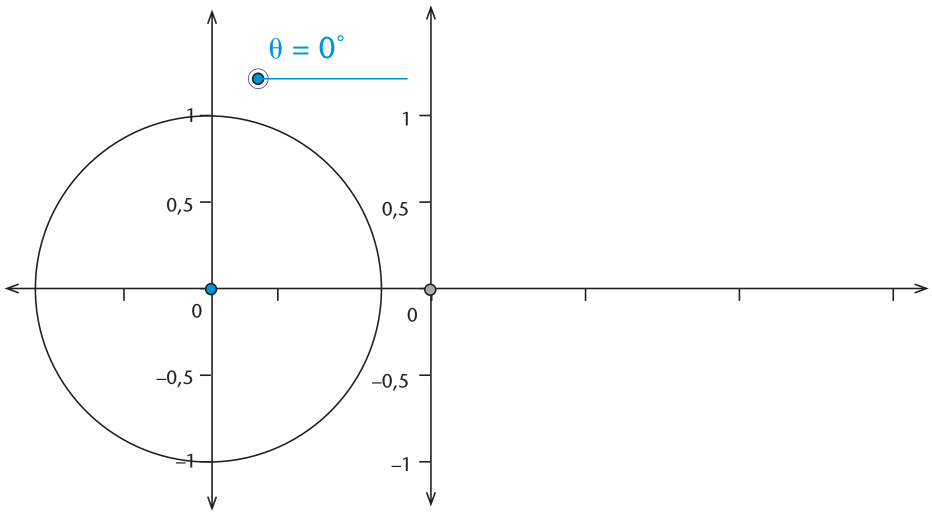
We will introduce this section through the context of alternating current; the most common form of electrical current used in our homes and by industry. Alternating current is usually created by a generator consisting of a rotating coil of wire (conductor) between two magnets. When the coil rotates, a current is induced by the magnetic field. The amount of the electromotive force (emf) or voltage generated is dependent on the position of the coil in relation to the magnets.

A vector is a quantity or phenomenon with two independent properties: magnitude and direction. The magnitude of a vector is represented by the length of a line segment. Direction can be represented by measuring the angle formed by the direction of the quantity and a reference axis. A coil rotating between two magnets can be considered a vector because it has both length (magnitude) and direction (rotation).

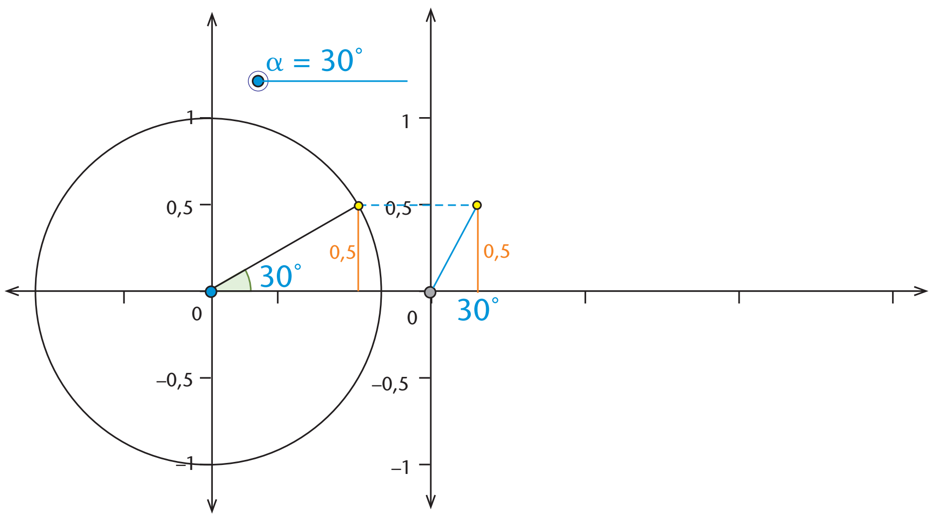
The relationship between the electromotive force generated by a rotating coil and the angle of rotation of the coil (conductor) can be represented using a graph. The graphs on the following pages show 12 different positions of the coil as it rotates at intervals of 30° and the amount of emf generated as the coil rotates. The electromotive force (emf) is plotted along the y-axes because it depends on the angle of rotation of the coil (it is the dependent variable), and the angle of rotation of the coil is plotted along the x-axes (it is the independent variable).

Developing the sine graph

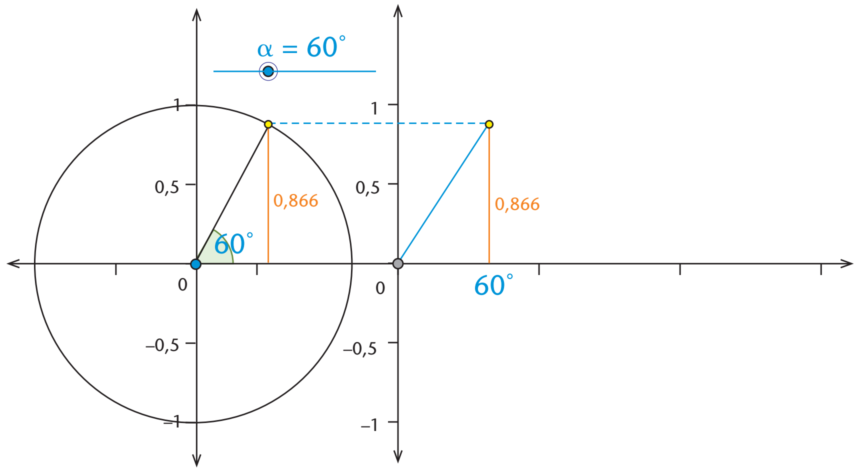
At 0° the conductor is lying parallel to the lines of the magnetic field between the pole magnets. In this position the conductor is generating no emf (electricity) because it is not interfering with the lines of the magnetic field.



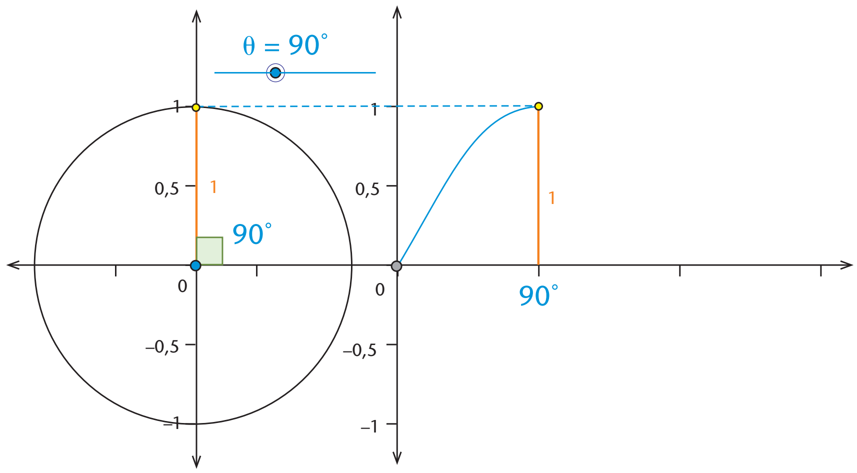
When the conductor is rotated through 30°, the magnitude of emf is 0,5 units. We assume that the maximum emf generated in this case is 1 unit. The coordinates of the point on the Cartesian plane corresponding to the magnitude of the generated emf are (30°; 0,50).



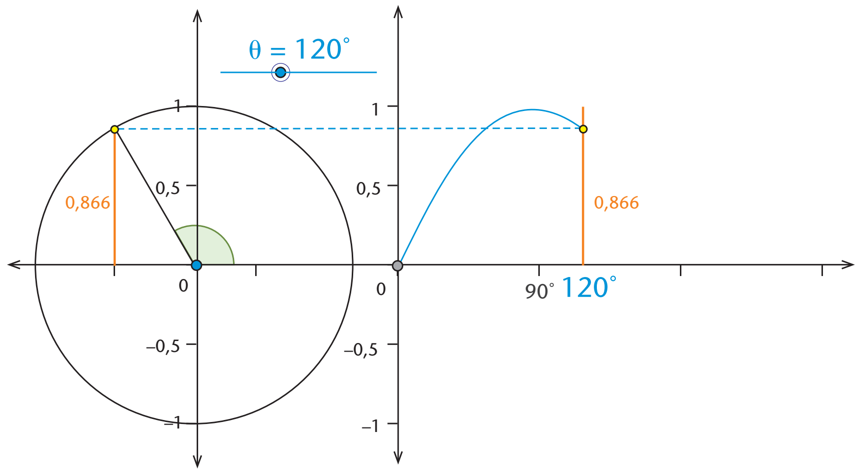
When the vector is rotated through 60°, the magnitude of emf is 0,866 units. The coordinates of the point on the Cartesian plane are (60°; 0,866).



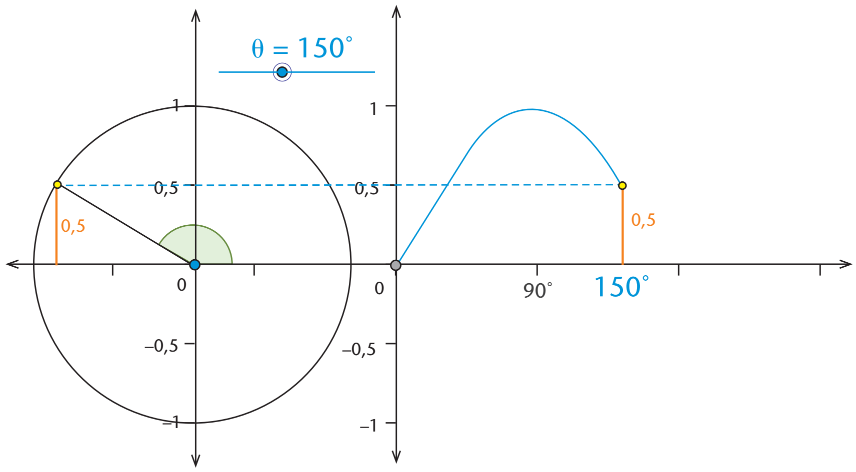
At 90°, the magnitude of emf reaches the maximum (1 unit). The coordinates of the point on the Cartesian plane are (90°; 1).



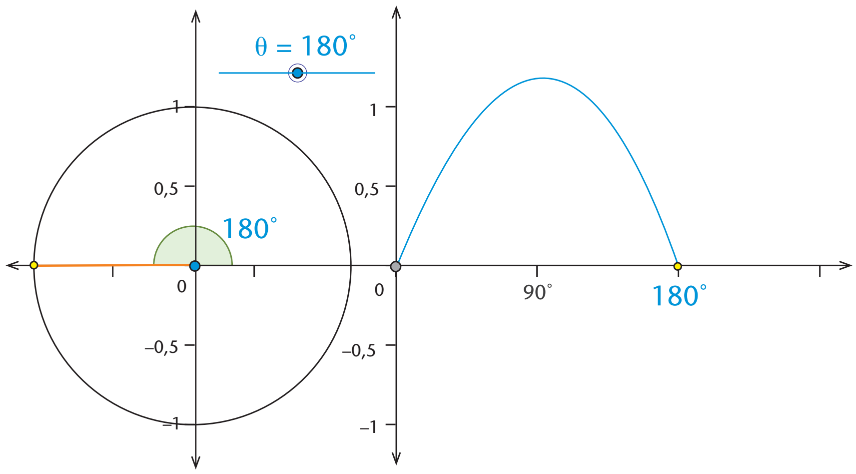
At 120°, the magnitude of emf starts to decrease and is now 0,866 units. The coordinates of the point on the Cartesian plane are (120°; 0,866).



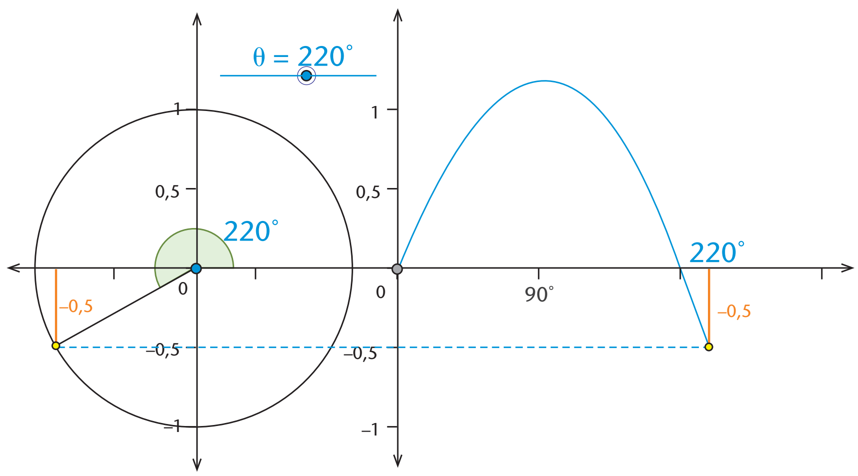
At 150°, the magnitude of emf decreases by 0,5 units. The coordinates of the point on the Cartesian plane are (150°; 0,5).



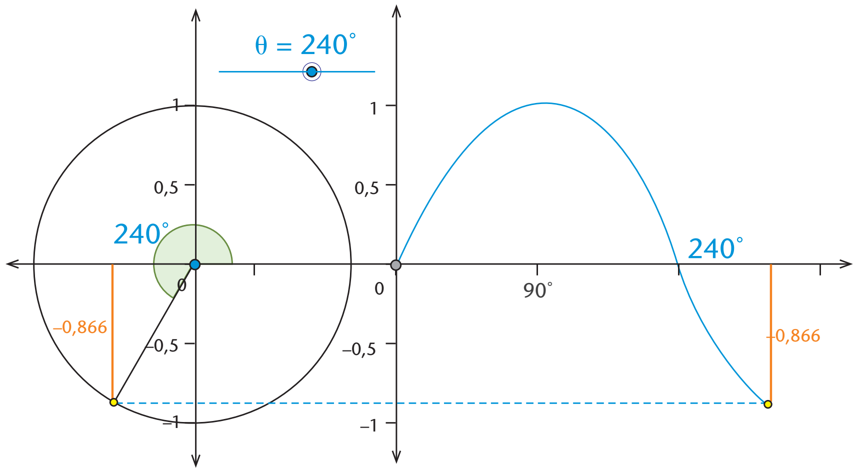
At 180°, the magnitude of emf is 0 units, because the vector is in parallel with lines of magnetic field. The coordinates of the point are (180°; 0).



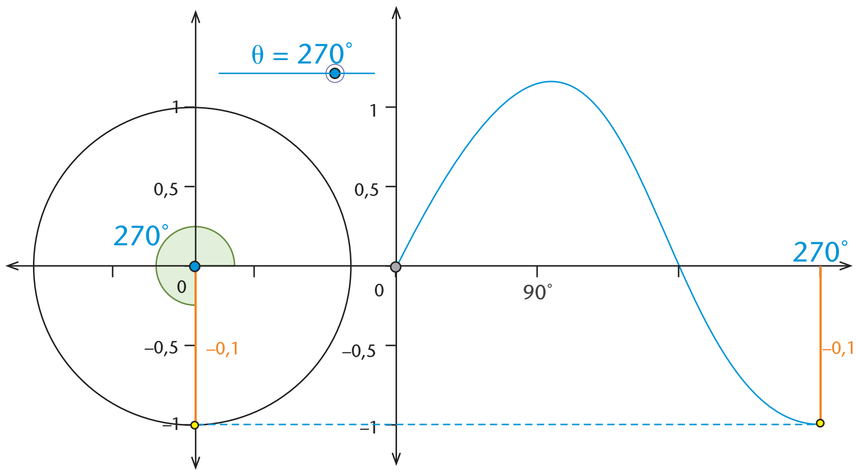
At 210°, the magnitude of emf rises in opposite direction to –0,5 units. The coordinates of the point on Cartesian plane is (210°; –0,5). NB: the magnitude of emf rises in the negative direction from 0° to 90° and decreases to zero at 180°.



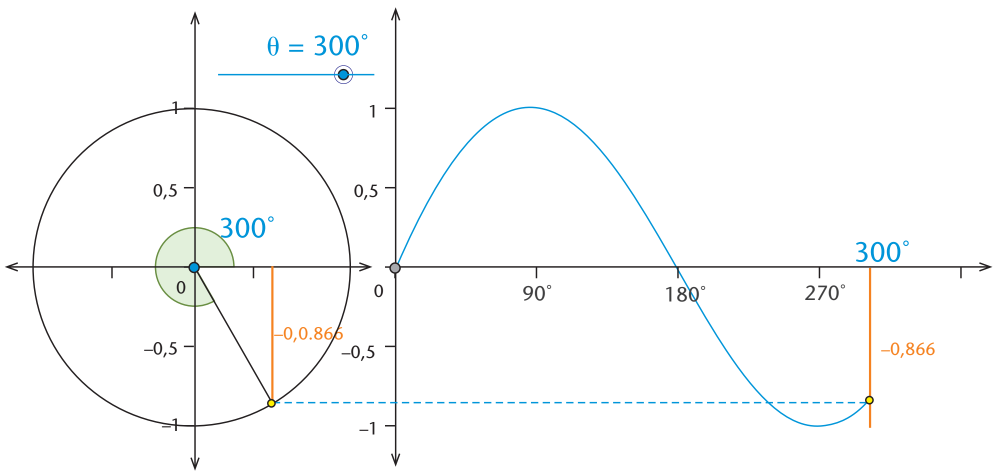
At 240°, the magnitude of emf rises to –0,866 units. The coordinates of the point on the Cartesian plane is (240°; –0,866).



At 270°, the magnitude of emf reaches maximum (–1 unit) in the negative direction. The coordinates of the point on the Cartesian plane is (270°; –1).



At 300°, the magnitude of emf decreases to –0,866 units. The coordinates of the point on the Cartesian plane is (300°; –0,866).



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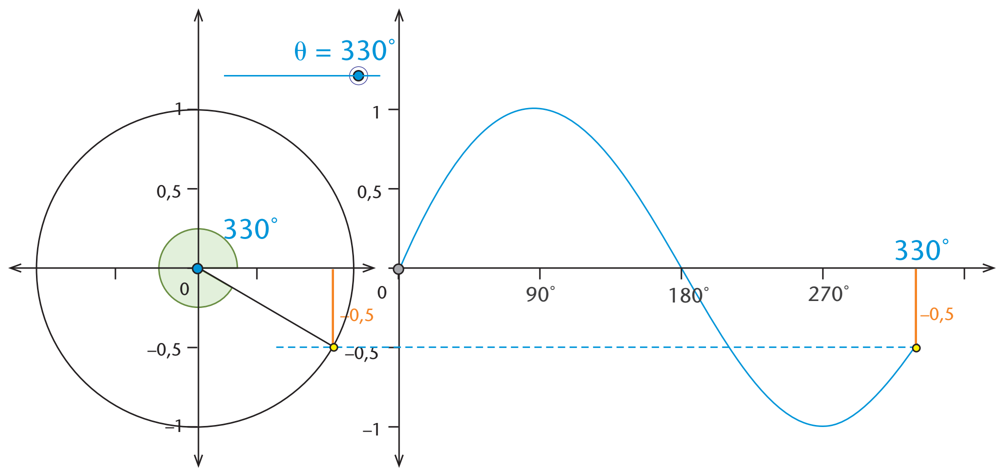
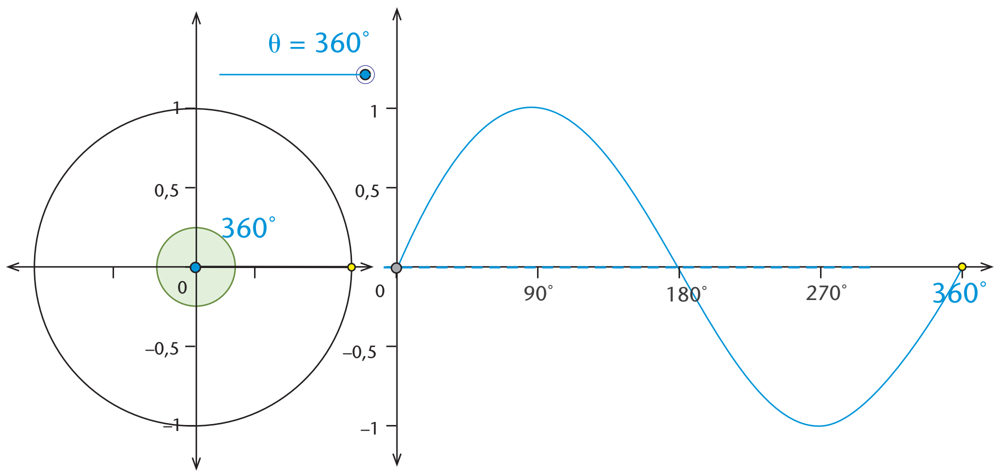
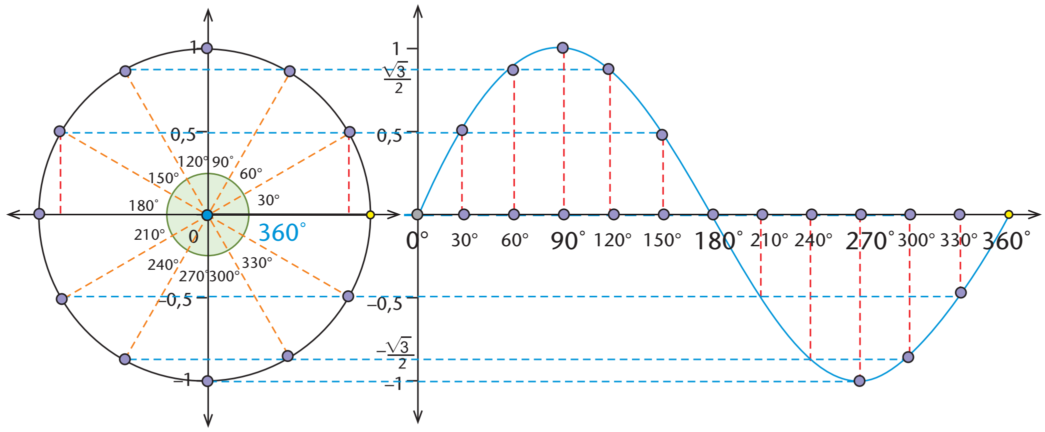
At 330°, the magnitude of emf decreases further to –0,5 units. The coordinates of the point on the Cartesian plane is (330°; –0,5).



At 360°, the magnitude of emf decreases further to zero. The coordinates of the point on the Cartesian plane is (360°; 0).



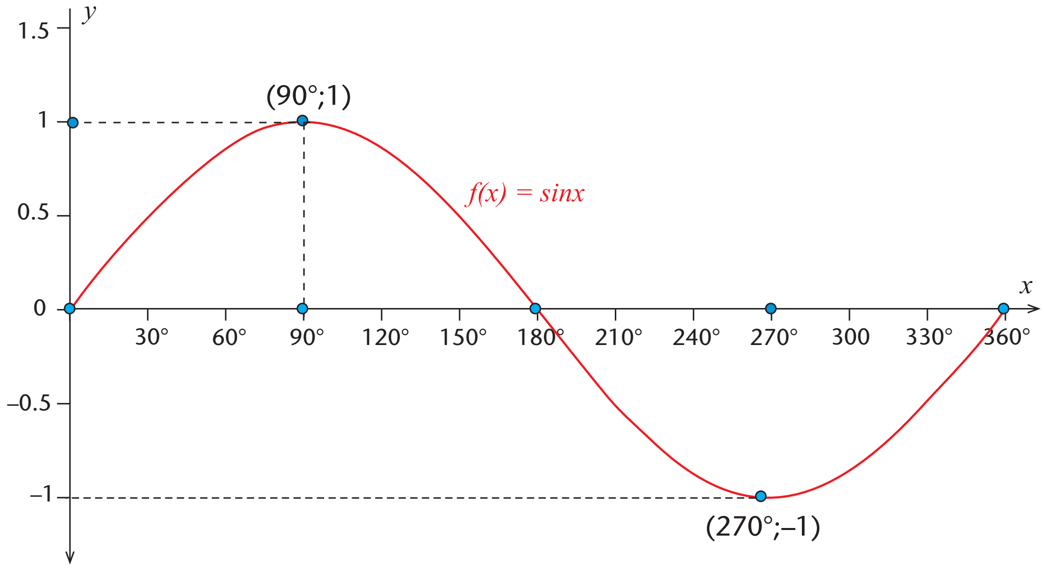
In a nutshell we have the following graph. This graph is referred to as the sine wave.



The sine graph: Revision

$y = \sin x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	0	0,5	0,87	1	0,87	0,5	0	–0,5	–0,87	–1	–0,87	–0,5	0



Properties of $y = \sin x; x \in [0^\circ; 360]$

1. Domain: $[0°; 360°]$
2. Range: $[-1; 1]$
3. x-intercepts at 0°,180° and 270°
4. y-intercept at 0
5. Maximum value of 1 at 90°
6. Minimum value of –1 at 270°
7. Period: 360°
8. Amplitude: 1

The effect of k: $y = k \sin x$ and $y = k \cos x$

We will draw the graphs of $y = \sin x$; $y = 2\sin x$ and $y = 3\sin x$ on the same system of x-y axis. We want to see which property of the sine function is affected by k.

EXERCISE 8.7

1. Copy and complete the table below.

   a) $y = 2 \sin x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y													

b) $y = 3\sin x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y													

2. Draw the graphs of $y = \sin x$, $y = 2\sin x$, and $y = 3\sin x$ on the same system of axes.
3. Copy and complete the table below.

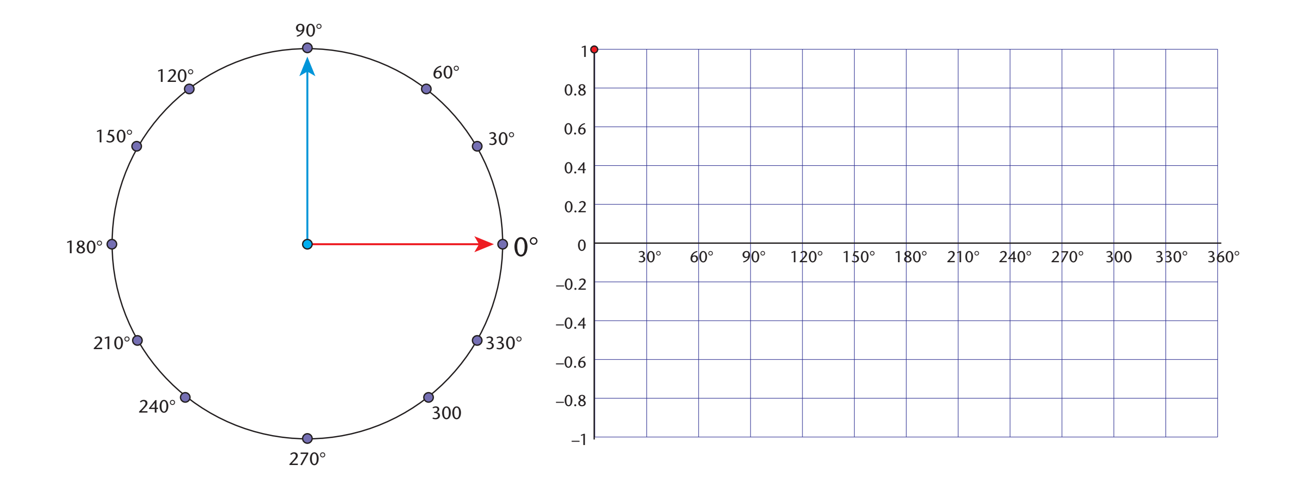
			Intercepts					
Function	**Domain**	**Range**	x	y	**Max point(s)**	**Min point(s)**	**Period**	**Amplitude**
$y = \sin x$								
$y = 2\sin x$								
$y = 3\sin x$								

4. In what respects are the graphs of $y = 2\sin x$ and $y = 3\sin x$ different from that of $y = \sin x$?
5. Copy and complete the table below.

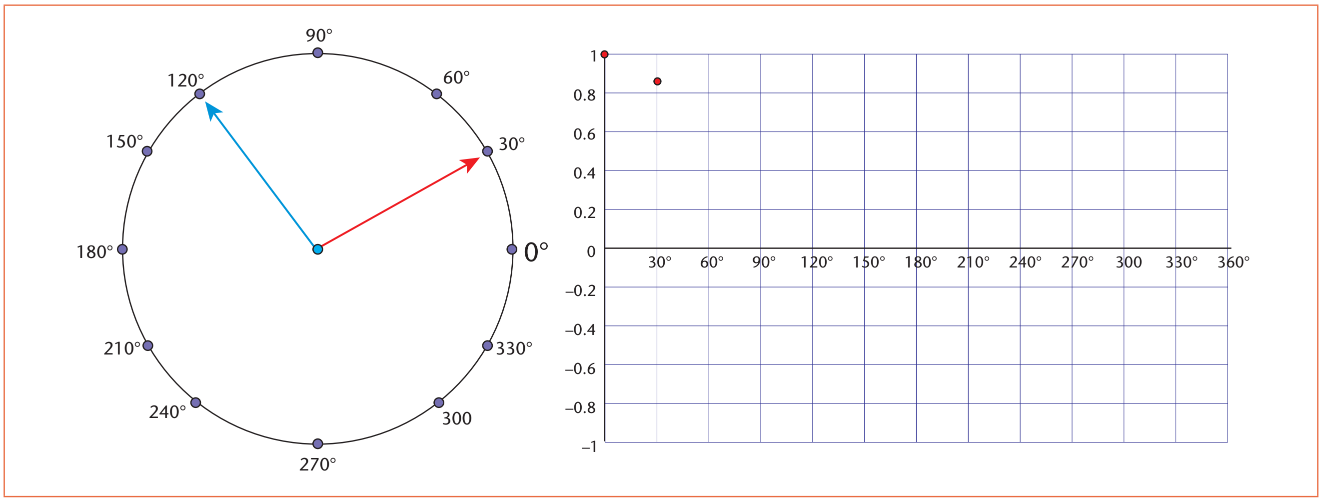
Function	$y = \frac{1}{4}\sin x$	$y = \frac{1}{3}\sin x$	$y = 4\sin x$	$y = 6\sin x$	$y = 7{,}5\sin x$
Maximum values					
Minimum Value					
x**-intercepts**					

Developing the cos graph

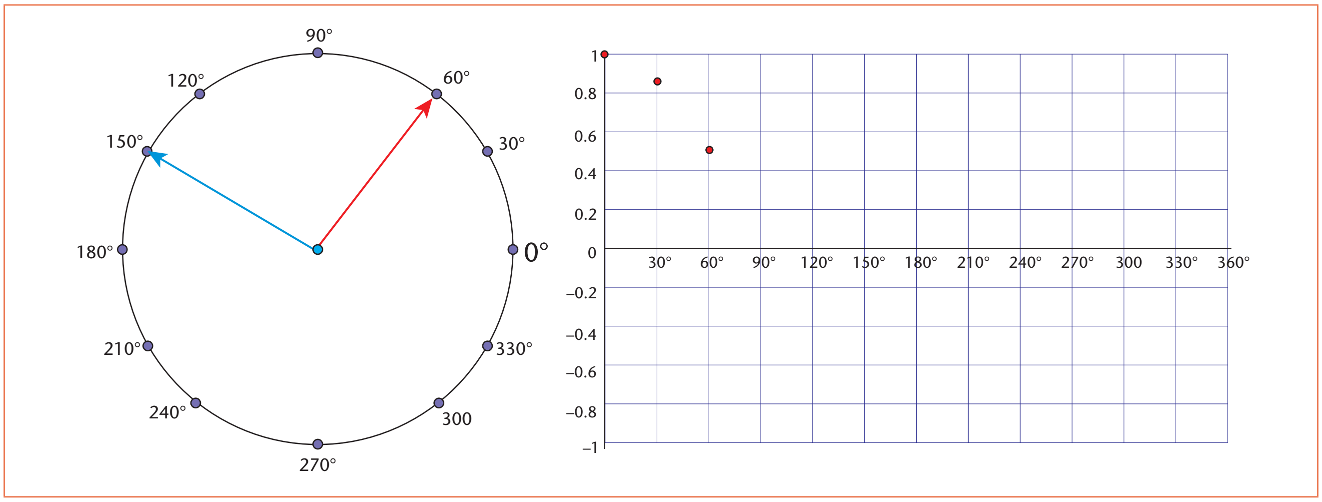
Consider the two vectors (coils) shown in the sketch below. The blue vector (coil) is always displaced at 90° to the red vector (coil), and at this position it generates maximum emf (1 unit). The co-ordinates representing the relationship between the position of the blue vector and output voltage is (0°;1). To determine the position of generation we use the position of the red vector. When the blue vector is positioned at 90°, the position of generation is 0°.



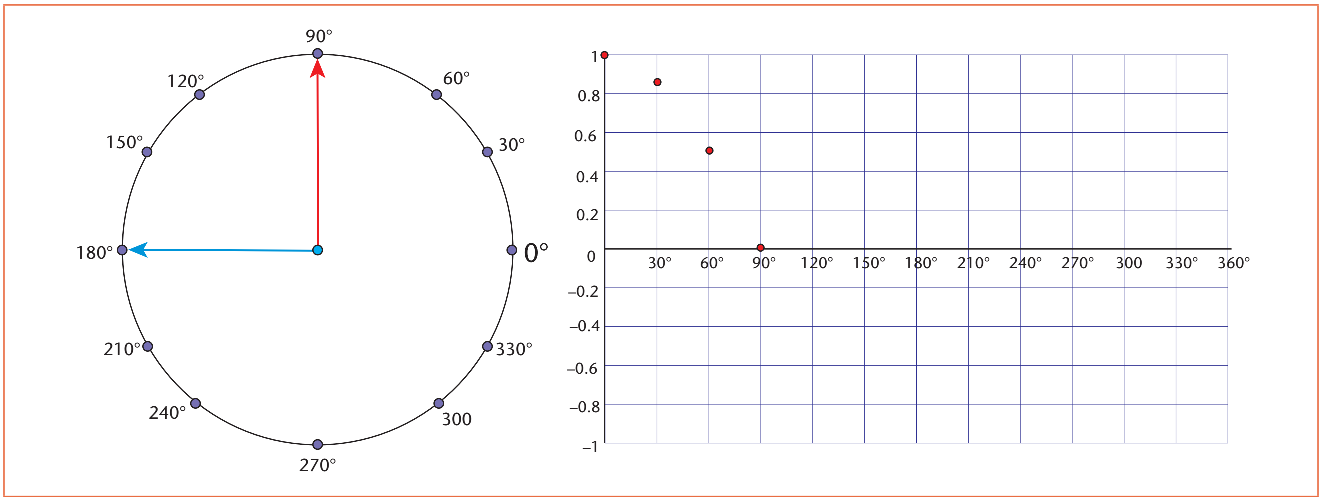
At 120°, the magnitude of emf decreases to 0,866 units.



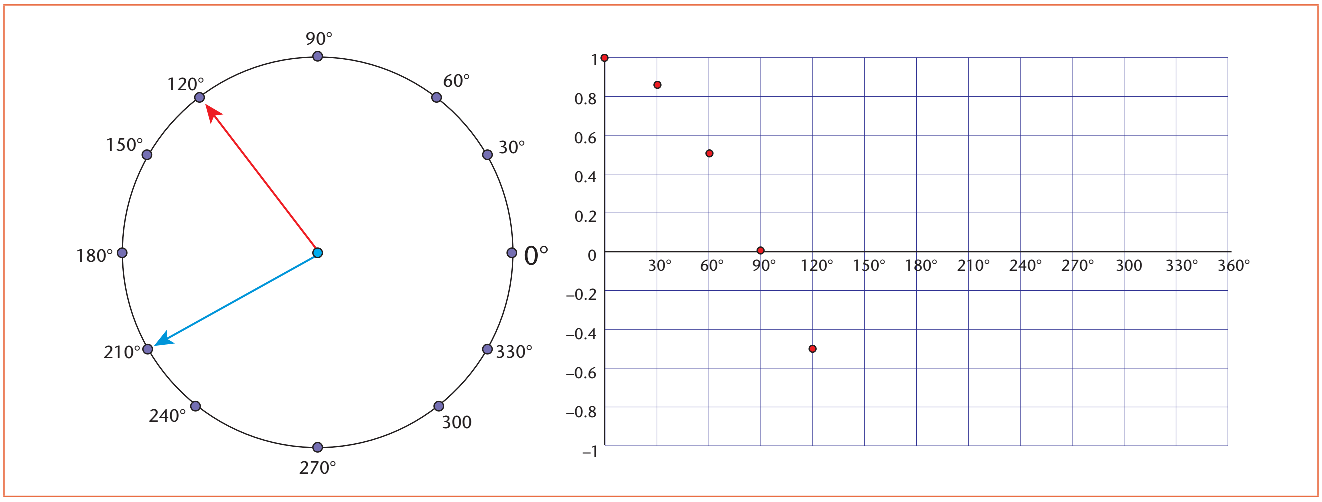
At 150°, the magnitude of emf decreases further to 0,5 units.



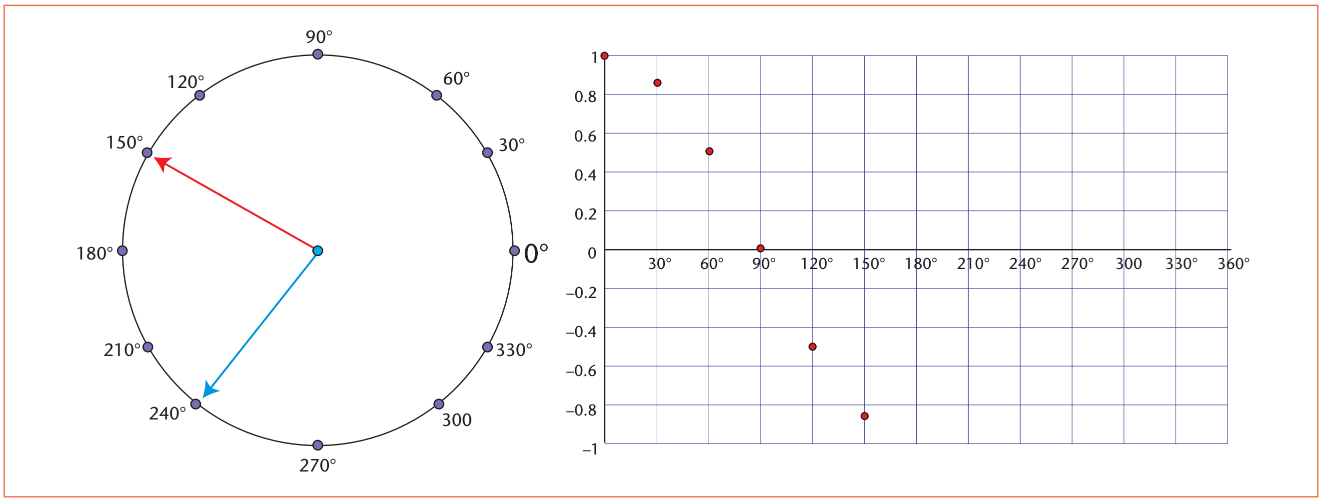
At 180°, the magnitude of emf decreases to zero.



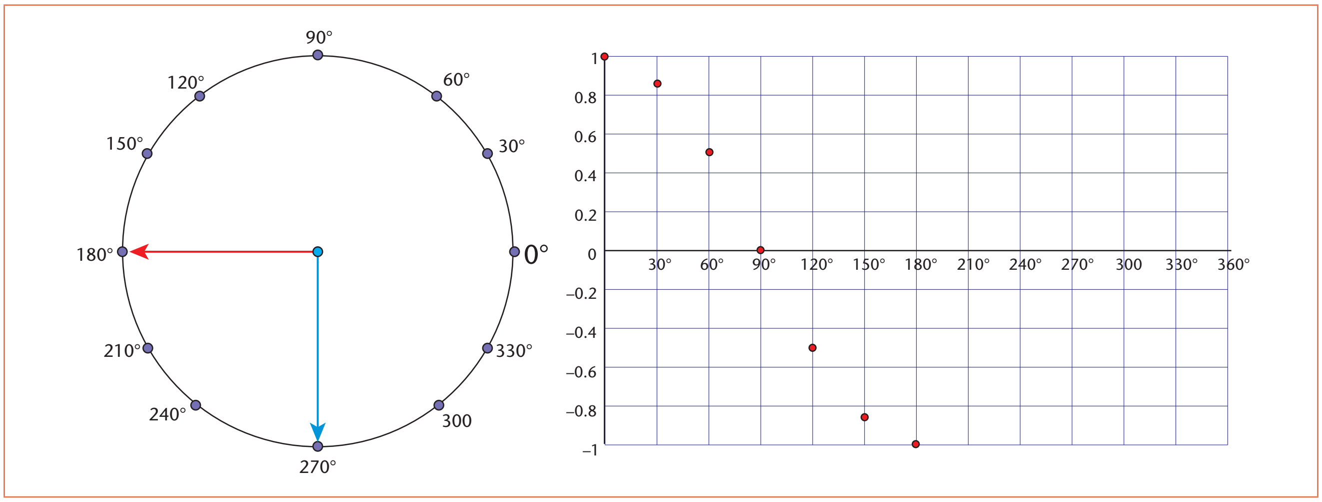
At 210°, the magnitude of emf increases in a negative direction to 0,5 units.



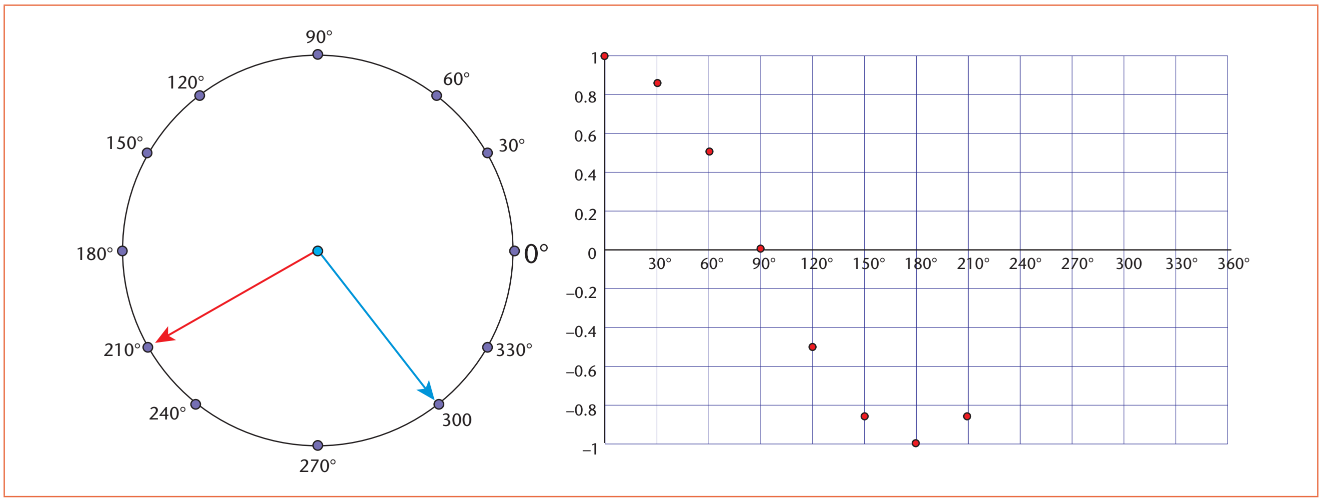
At 240°, the magnitude of emf increases further to –0,866 units.



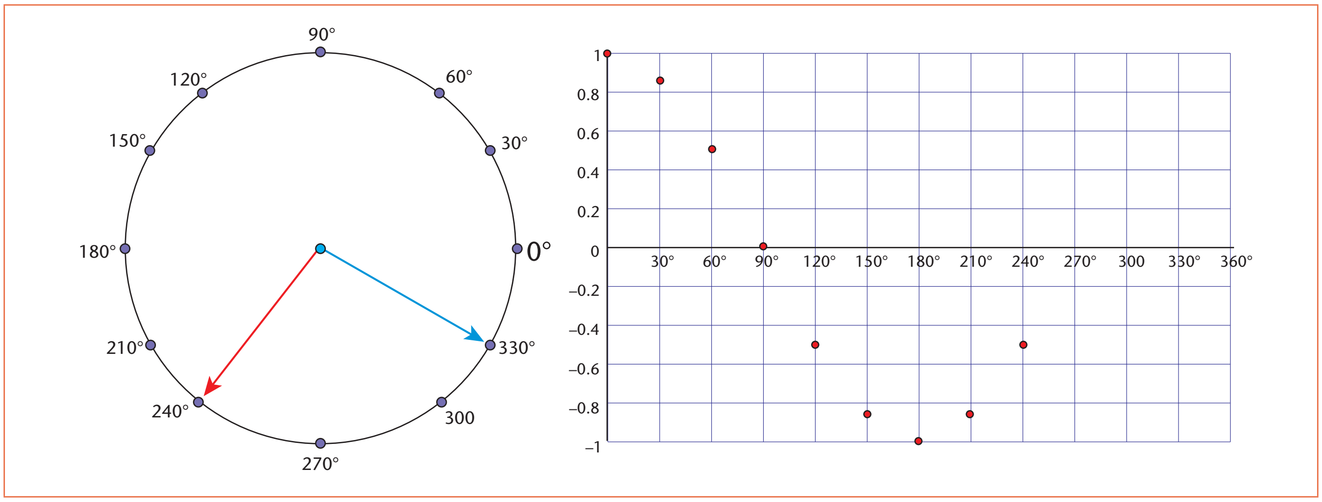
At 270°, the magnitude of emf is maximum (–1 unit).



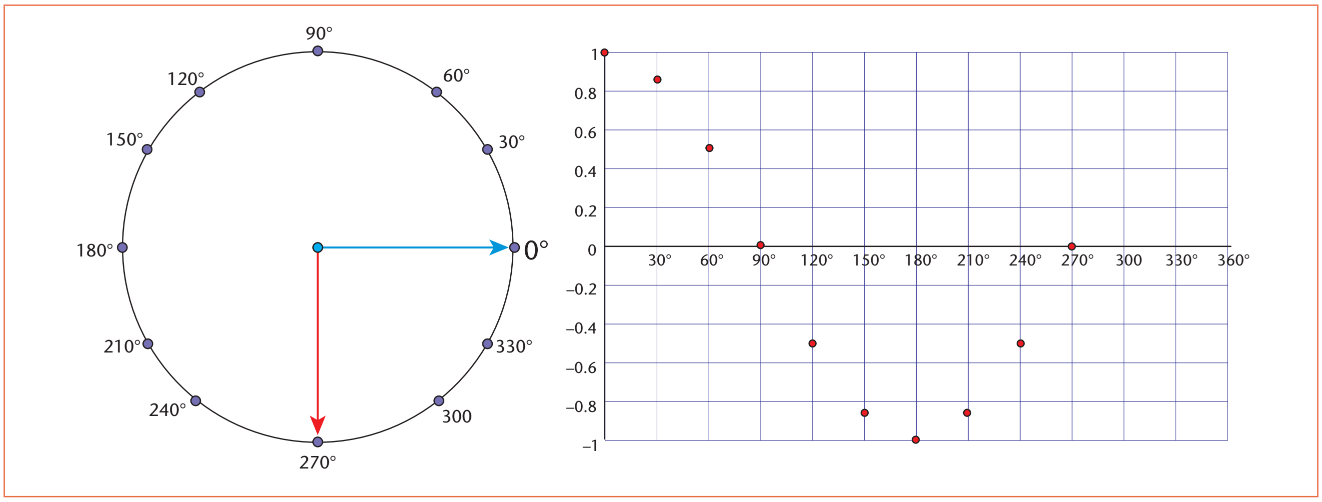
At 300°, the magnitude of emf decreases further to –0,866 units.



At 330°, the magnitude of emf decreases further to 0,5 units.



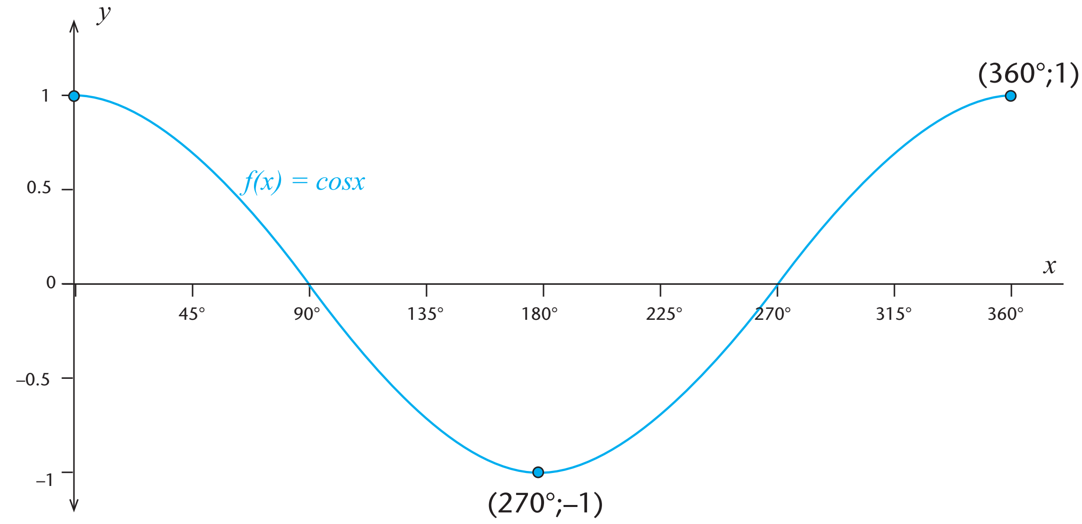
At 360°, the magnitude of emf is zero.



The cosine graph

$y = \cos x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y	1	0.87	0.5	0	–0.5	–0.87	–1	–0.87	–0.5	0	0.5	0.87	1



Properties of $y = \cos x$, $x \in [0°; 360°]$

1. Domain: [0°; 360°]
2. Range: $[-1; 1]$
3. x-intercepts at 90° and 270°
4. y-intercept at 1
5. Maximum value of 1 at 0° and 360°.
6. Minimum value of –1 at 270°
7. Period: 360°
8. Amplitude: 1

The effect of k: $y = k\cos x$

We will draw the graphs of $y = \cos x$; $y = 2\cos x$ and $y = 3\cos x$ on the same system of x-y axis. We want to see which property of the cos function is affected by k.

EXERCISE 8.8

1. Copy and complete the table below.

   a) $y = \frac{1}{2}\cos x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y													

b) $y = 2\cos x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y													

c) $y = 3\cos x$

x	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
y													

2. Draw the graphs of $y = \cos x$, $y = \frac{1}{2}\cos x$, $y = 2\cos x$, and $y = 3\cos x$ on the same system of axes.
3. Copy and complete the table below.

			Intercepts					
Function	**Domain**	**Range**	x	y	**Max point(s)**	**Min point(s)**	**Period**	**Amplitude**
$y = \cos x$								
$y = \frac{1}{2}\cos x$								
$y = 2\cos x$								
$y = 3\cos x$								

4. In what respects are the graphs of $y = \frac{1}{2}\cos x$, $y = 2\cos x$ and $y = 3\cos x$ different from that of $y = \cos x$?

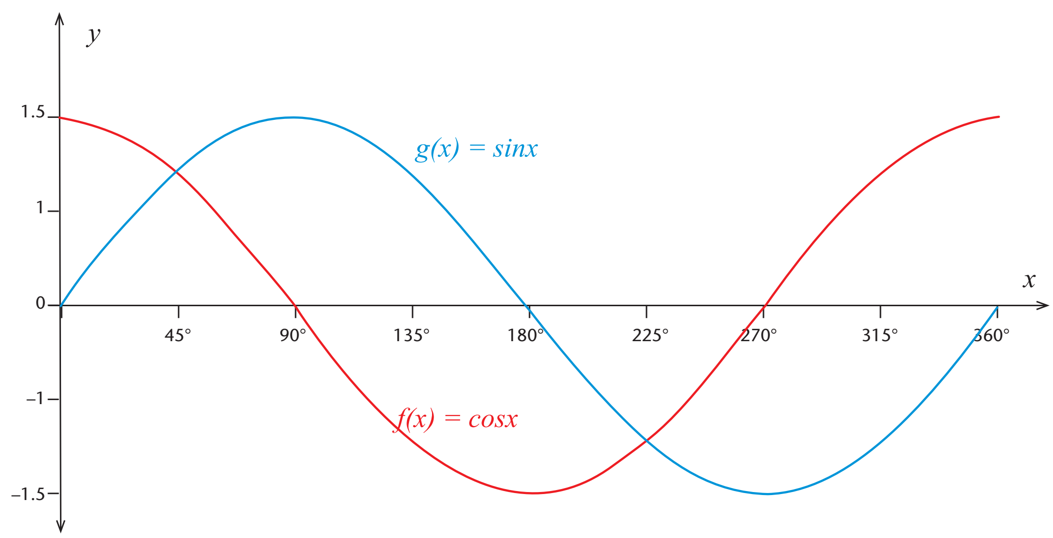
A summary of the properties of the sine and cosine functions

Both $\sin x$ and $\cos x$ will go through the same values every time the angle x completes a cycle. We refer to such functions as **periodic** functions.

The cosine has its largest value at the beginning of the cycle, when $x = 0$ (since $\cos 0° = 1$); the sine function has its maximum value at $x = 90°$ (since $\sin 90° = 1$).

The period of the sine function $y = \sin x$ is defined as the value of x for which one whole cycle has been completed. The periods for both the sine and cosine functions is 360° or $2\pi rad$.

The maximum value of each function is 1 and the minimum value is −1.



The effect of k: $y = \sin ax$

$y = \sin 2x$, $x \in [0°; 360°]$

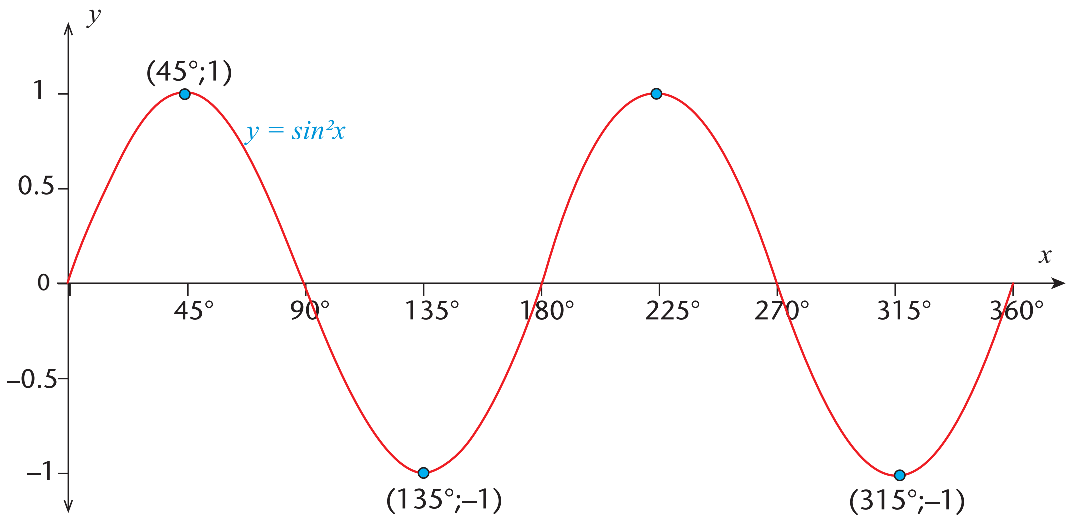
To complete the table, we calculate the y values:

$y = \sin 2(0°)$	$y = \sin 2(45°)$	$y = \sin 2(90°)$	$y = \sin 2(135°)$
$= \sin 0°$	$= \sin 90°$	$= \sin 180°$	$= \sin 270°$
$= 0$	$= 1$	$= 0$	$= -1$

x	0°	45°	90°	135°	180°	225°	270°	315°	360°
$2x$	0°	90°	180°	270°	360°	450°	540°	630°	720°
y	0	1	0	–1	0	1	0	–1	0

Properties of $y = \sin 2x$

1. Domain: [0°; 360°]
2. Range: [–1; 1]
3. x-intercepts; 0°, 180°,270°
4. y-intercept: 0
5. Maximum point(s): (90°; 1)
6. Minimum point(s): (270°; − 1)
7. Period: 180°
8. Amplitude: 1



EXERCISE 8.9

1. Copy and complete the table below.

			Intercepts					
Function	Domain	Range	x	y	Max point(s)	Min point(s)	Period	Amplitude
$y = \sin x$								
$y = \sin 2x$								

2. Draw the graphs of $y = \sin x$ and $y = \sin 2x$ on the same set of axes.
3. In what respects are the graphs of $y = \sin x$ and $y = \sin 2x$:
   a) the same
   b) different a

Formula for calculating the period of sine

The formula for calculating the period of $y = \sin kx$ is:

Period $= \frac{360°}{k}$, $k \in R$

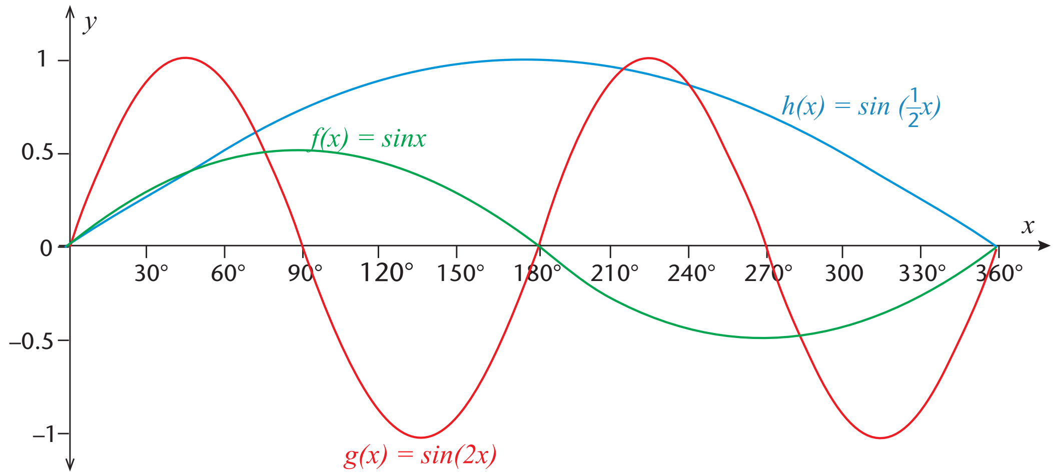
EXERCISE 8.10

1. Copy and complete the table below.

Function	$y = \sin\frac{1}{2}x$	$y = \sin\frac{1}{3}x$	$y = \sin\frac{1}{4}x$	$y = \sin x$	$y = \sin 2x$	$y = \sin 3x$	$y = \sin 4x$
Value of k							
Period							

2. Draw the graph of $y = \cos 2x \in [0°;\ 360°]$, show the following:
   a) the x-intercepts
   b) the amplitude
   c) the period
   d) the maximum and minimum values.
3. How is the graph of $y = \cos 2x$ different from that of $y = \cos x$?
4. Draw the following graphs on a separate system of axes.
   a) $y = \sin 3x$
   b) $y = \cos 3x$
   c) $y = \sin 5x$
   d) $y = \cos 5x$

5. Consider the graphs below [Enrichment].

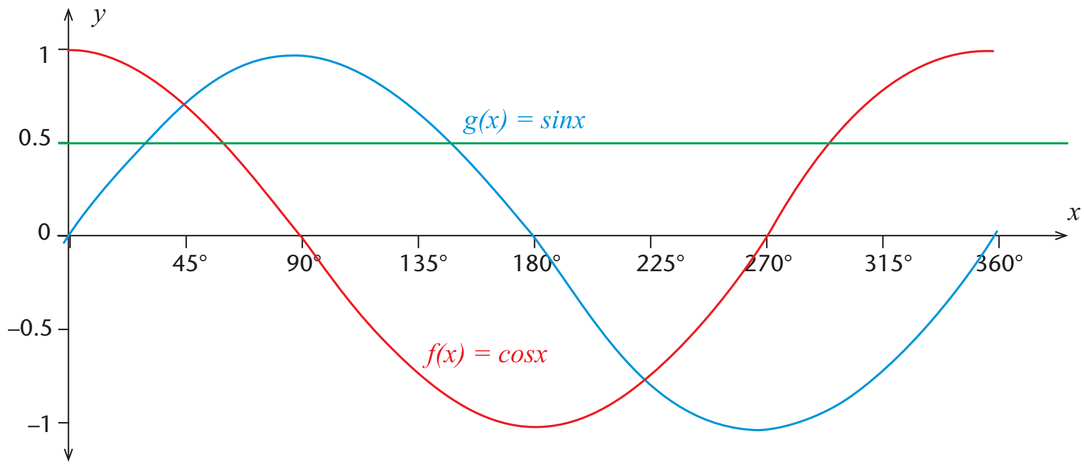


a) For which value(s) of x is $\sin\frac{1}{2}x = \sin x = \sin 2x$?

b) For which value(s) of x is $\sin x = \sin 2x$?

c) What is the period of $\sin\frac{1}{2}x$? What does that mean?

6. The questions below refer to the sine and cosine graphs drawn.



a) For which value(s) of x is $\sin x = \cos x$?

b) What is the value of $\sin 360°$?

c) What is the value of $\cos 360°$?

d) Is the statement $\cos 360° = 1 + \sin 360°$, true? Explain.

7. A green line, horizontal to the x-axis, is drawn and cuts both graphs at two different points.

a) At what points does the green line cut the graph of $g(x) = \sin x$? Estimate your answer as accurately as you can.

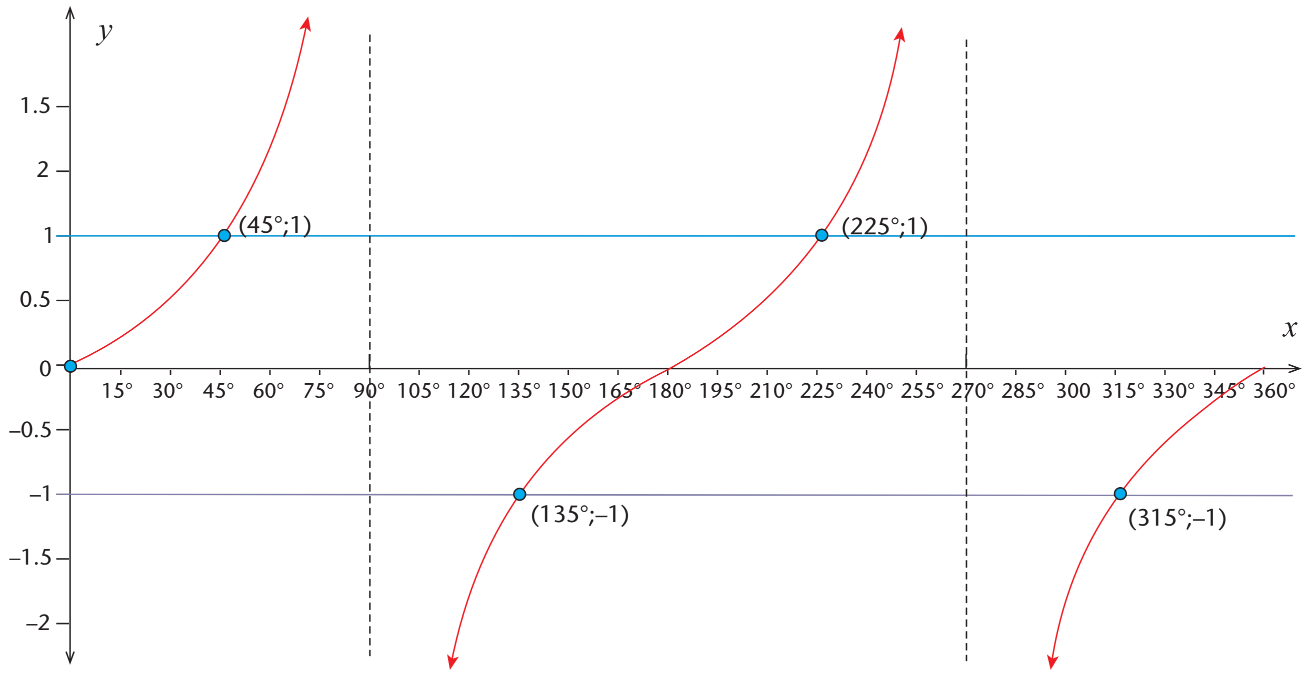
b) At what points does the green line cut graph of $f(x) = \cos x$? $g(x) = \sin x$? Estimate your answer as accurately as you can.

c) Write down the equation of the green line.

The tangent graph; $y = \tan x$

Consider the function $y = \tan x$, $x \in [0°;\ 360°]$

x	0°	30°	45°	60°	90°	135°	150°	180°	225°	270°	315°	360°
y	0	0,58	1	1.73	∞	–1	–0,58	0	1	∞	–1	0



Characteristics of $y = \tan x$

1. Domain: $[0°;\ 360°]$
2. Range: $y \in R$
3. Period: 180°
4. No maximum or minimum value
5. Asymptotes at $x = 90°$ and $270°$

The graph of $y = k \tan x$

EXERCISE 8.11

1. a) Copy and complete the table below for $y = 2 \tan x$

x	0°	30°	45°	60°	90°	135°	150°	180°	225°	270°	315°	360°
y												

b) On the same system of axes as $y = \tan x$ draw the graph of $y = 2 \tan x$ for $[0°;\ 360°]$.

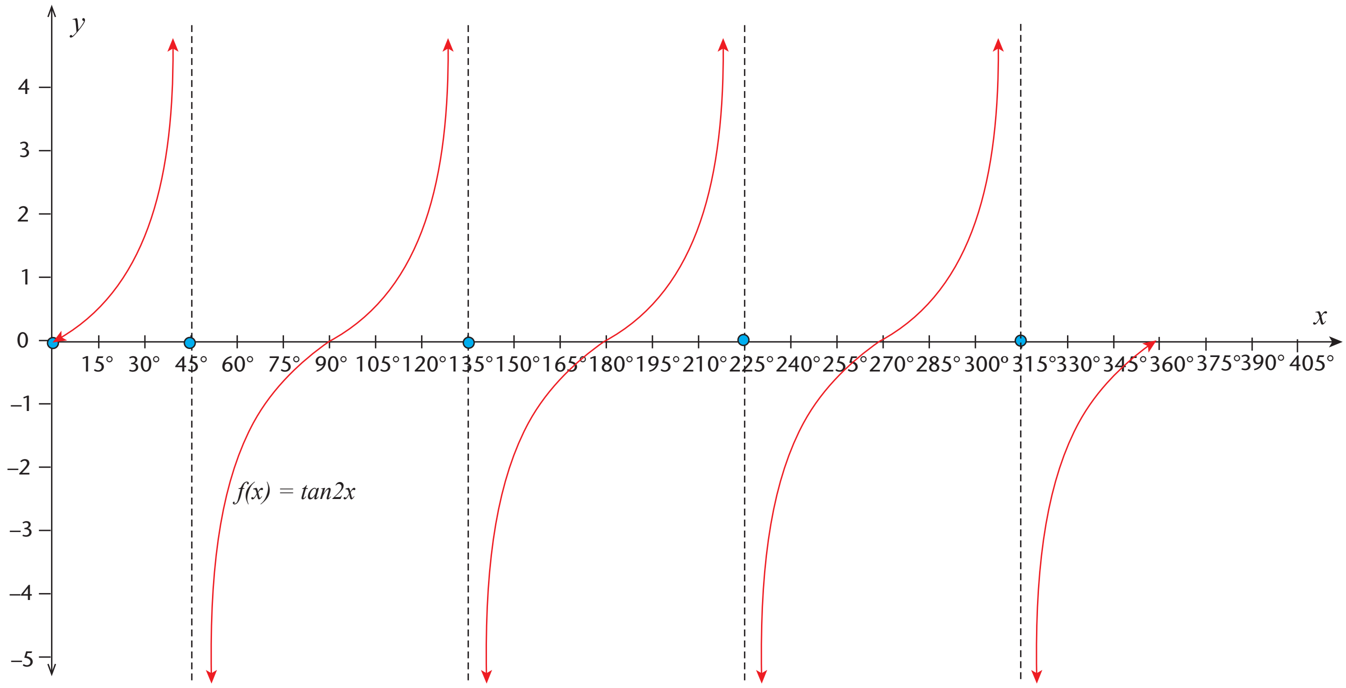
c) How are the two graphs: (i) the same (ii) different?

d) What is the effect of 2 on the graph of $y = \tan x$?

e) How will the graph of $y = 3 \tan x$ be different from the other two graphs?

The graph of $y = \tan kx$

Below is the graph of $y = \tan 2x$



EXERCISE 8.12

1. Study the graph and answer the questions below.
   a) Construct a table of values in intervals of 15° as shown in the graph. Read the values of y from the graph when you complete the table. Use your calculator to check how far off your reading may be.
   b) Complete the table below.

	Function	Period	Asymptotes	The co-ordinates $(x; 0)$	The co-ordinates $(x; 1)$	The co-ordinates $(x; -1)$
(i)	$y = \tan x$					
(ii)	$y = \tan 2x$					

2. Copy and complete the table below.

	Function	Period	Asymptotes	The co-ordinates $(x; 0)$	The co-ordinates $(x; 1)$	The co-ordinates $(x; -1)$
a)	$y = \tan 3x$					
b)	$y = \tan 4x$					
c)	$y = \tan 5x$					

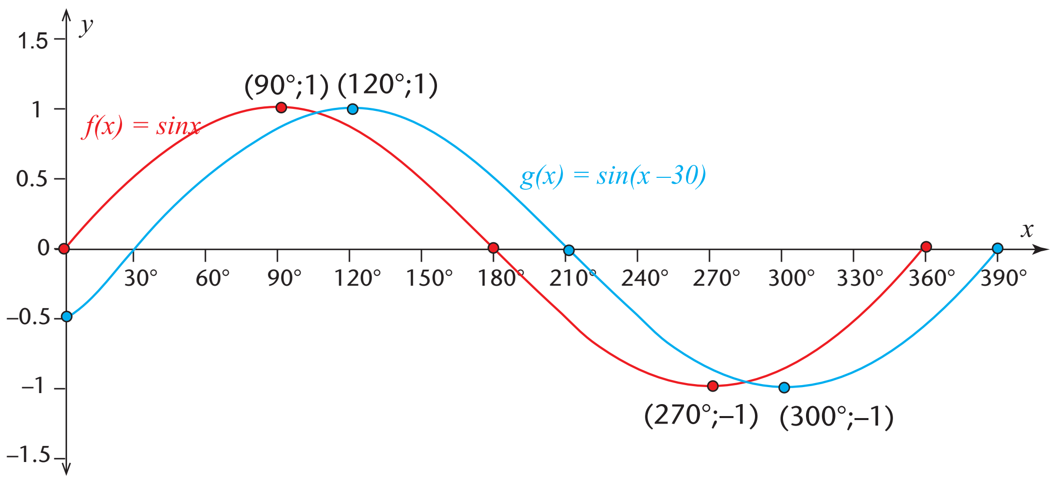
8.7 The graphs of $y = \sin(x \pm p)$ and $y = \cos(x \pm p)$

The graph of $y = \sin x$, $y = \sin(x - 30°)$

Consider the graphs of $y = \sin$ and $y = \sin(x - 30°)$for $x \in [0°; 360°]$ shown below.

x	0°	30°	60°	90°	120°	150°	180°	210°	240°
$y = \sin x$	0	0.5	0.87	1	0.87	0.5	0	–0.5	–0.87
$y = \sin(x - 30°)$	–0.5	0	0.5	0.87	1	0.87	0.5	0	–0.5

x	270°	300°	330°	360°	390°				
$y = \sin x$	–1	–0.87	–0.5	0	0.5				
$y = \sin(x - 30°)$	–0.87	–1	–0.87	–0.5	0				



The graph of $y = \sin(x - 30°)$ is shifted 30° to the right along the x-axis. Note the minus sign in $y = \sin(x - 30°)$.

The tables below summarise the key features of the two functions.

Function	x-intercepts	y-intercept	Min value	Max value	Range	Domain
$y = \sin x$	0°, 180°, 360°	0	–1 at 270°	1 at 90°	$-1 \leq y \leq 1$	$0° \leq x \leq 360°$
$y = \sin(x - 30°)$	30°, 210°, 390°	–0.5	–1 at 300°	1 at 120°	$-1 \leq y \leq 1$	$0° \leq x \leq 360°$

Function	Period	Amplitude				
$y = \sin x$	360°	1				
$y = \sin(x - 30°)$	360°	1				

EXERCISE 8.13

1. a) Draw the graphs of $y = \sin x$ and $y = \sin(x - 45°)$ on the same system of axis for $x \in [0°; 360°]$.

	0°	45°	90°	135°	180°	225°	270°	315°	360°
$y = \sin x$									
$y = \sin(x - 45°)$									

b) Copy and complete the table below.

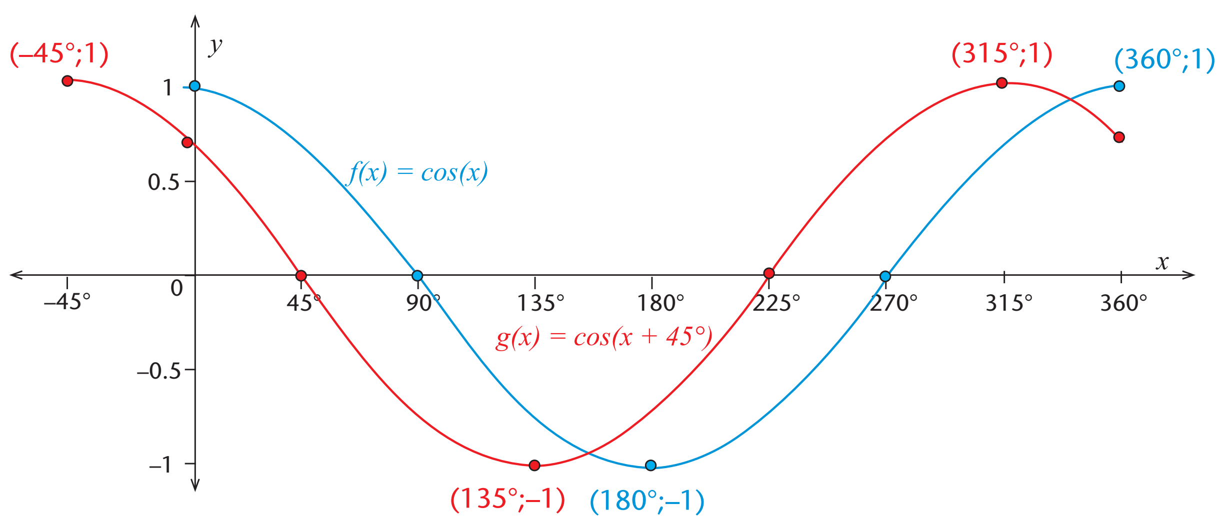
Function	x-intercepts	y-intercept	Min value	Max value	Range	Domain
$y = \sin x$						
$y = \sin(x - 45°)$						

Function	Period	Amplitude				
$y = \sin x$						
$y = \sin(x - 45°)$						

The graph of $y = \cos(x + 45°)$

Consider the graphs of $y = \cos x$ and $y = \cos(x + 45°)$ for $x \in [0°; 360°]$ shown below.

	–45°	0°	45°	90°	135°	180°	225°	270°	315°	360°
$y = \cos x$	0.71	1	0.71	0	–0.71	–1	–0.71	0	0.71	1
$y = \cos(x - 45°)$	1	0.71	1	–0.71	–1	–0.71	0	0.71	1	0.71



EXERCISE 8.14

1. Copy the table below. Use the graphs above to complete the table.

Function	x-intercepts	y-intercept	Min value	Max value	Range	Domain
$y = \cos x$						
$y = \cos(x + 45°)$						

Function	Period	Amplitude				
$y = \cos x$						
$y = \cos(x + 45°)$						

2. On the same system of axis draw the graphs of $y = \sin x$ $y = \sin(x + 45°)$ for $0° \leq x \leq 360°$.

	$-45°$	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$y = \sin x$										
$y = \sin(x + 45°)$										

3. Copy the table below. Use the graphs above to complete the table.

Function	x-intercepts	y-intercept	Min value	Max value	Range	Domain
$y = \sin x$						
$y = \sin(x + 45°)$						

Function	Period	Amplitude				
$y = \sin x$						
$y = \sin(x + 45°)$						

8.8 Summary: The graphs

$y = \sin(x \pm p)$ and $y = \cos(x \pm p)$

p determines the phase shift, or how the graph is shifted p units from left or right.

The basic functions $y = \sin x$ and $y = \cos x$

Each graph has an amplitude of 1.

The period of each graph is 360°.

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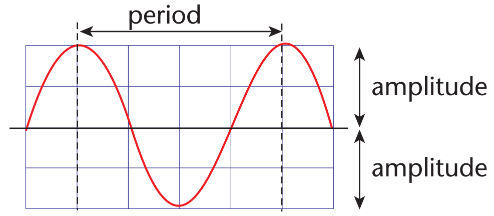
Terminology	Explanation	Example
Amplitude	Is half the distance of the vertical height from the minimum value to the maximum value of the function.	
Period	Is the horizontal length of one repeating patterns of the function?	
Interval	Is the domain of one cycle.	
Phase shift	Is the horizontal distance a function moves.	

Sketching the graphs

Use key points in one period of each graph to sketch the graphs of sine and cosine functions. These are:

- the maximum points and minimum points
- the intercepts

Calculating the phase shift

The number $\frac{p}{k}$ gives the phase shift in $y = \sin(kx \pm p)$ and $y = \cos(kx \pm p)$

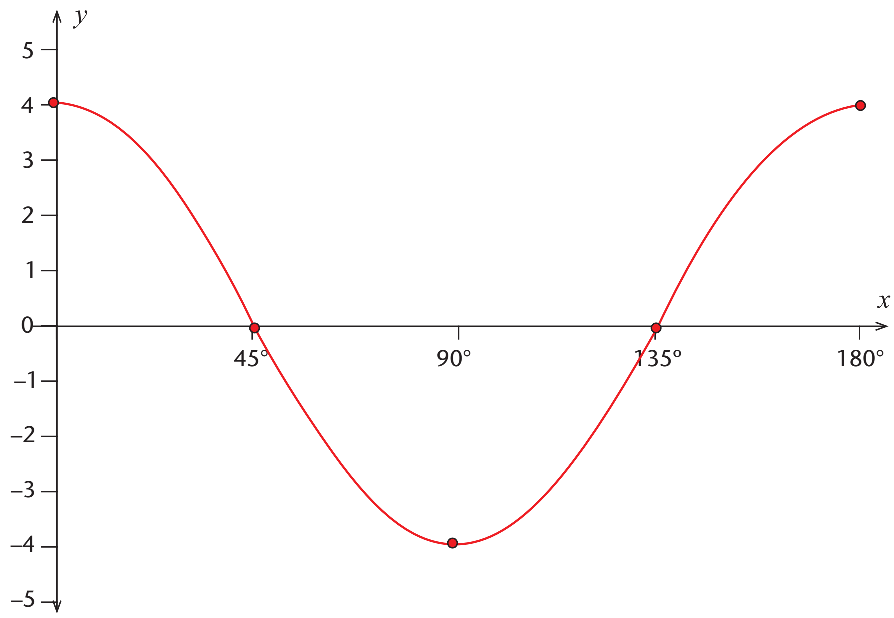
Calculating the period

Period $= \frac{360°}{k}$ in $y = \sin(kx \pm p)$ and $y = \cos(kx \pm p)$

EXERCISE 8.15

1. Calculate the amplitude, the period and the phase shift of:
   a) $y = 3\cos 2x$
   b) $y = 2\sin(x - 60°)$
   c) $y = -2\sin\frac{x}{2}$
   d) $y = \frac{3}{2}\cos(x + 30°)$

2. Consider the graph below given by $y = k\cos(bx + p)$



a) What is the value of each of the following parameters? Give a justification for each value you give.

(i) k

(ii) b

(iii) p

b) What is the equation for the function whose graph is drawn above?

c) What is the range of this function?

d) What is the domain of the function?

e) State the minimum value(s) of this function. Give the answer in coordinate form.

f) State the maximum value(s) of this function. Give the answer in coordinate form.

3. Sketch the graph of each of the following functions.

a) $y = \ 3\sin(x + 60°)$

b) $y = 2\cos(x - 15°)$

c) $y = \cos(2x - 30°)$

d) $y = 3\sin(2x + 45°)$

8.9 Solving trigonometric equations

We solve trigonometric equations using the algebraic techniques similar to those we use when solving equations such as:

$3x - 1 = x$ and/or $3x^2 + 5x - 2 = 0$

We consider the solution of trigonometric equations. The strategy we adopt is to find one solution using knowledge of special angles, and then use the symmetries in the graphs of the trigonometric functions to deduce additional solutions.

In this section we look at the solutions of trigonometric equations. We will refer to the graphs we have already drawn in solving these equations. The periodicity of the graphs is important.

Special angles

The values of the six trigonometric ratios cannot be calculated exactly for most angles. Nor can the exact value of an angle generally be found given the value of one of the ratios.

There are, however, three special angles that lend themselves nicely to ratio calculation. They are 30˚, 45˚ and 60˚.

Notice that 30˚ and 60˚ angles are complementary and that a 45˚ angle is its own complement.

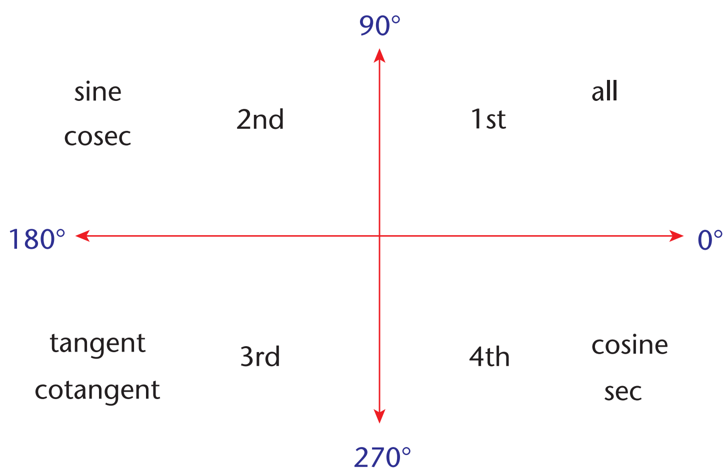
EXERCISE 8.16

Copy and complete the table below.

x	0°	30°	45°	60°	90°
$\sin x$					
$\cos x$					
$\tan x$					

The CAST diagram

Remember the CAST diagram from Grade 10? It was introduced to you as a diagram that shows the quadrants in which the trigonometric functions are positive.



Worked example 8.8

Solve for x: $\sin x = \frac{1}{2}$, $0° \leq x \leq 360°$

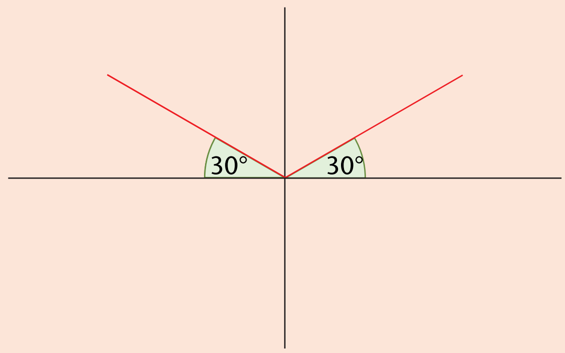
Solution

$x = \sin^{-1}\left(\frac{1}{2}\right)$

$x = 30°$

We get the Principal value $x = 30°$ using the calculator.

Next, using the CAST diagram, we consider the other quadrant where the sine function is also positive.



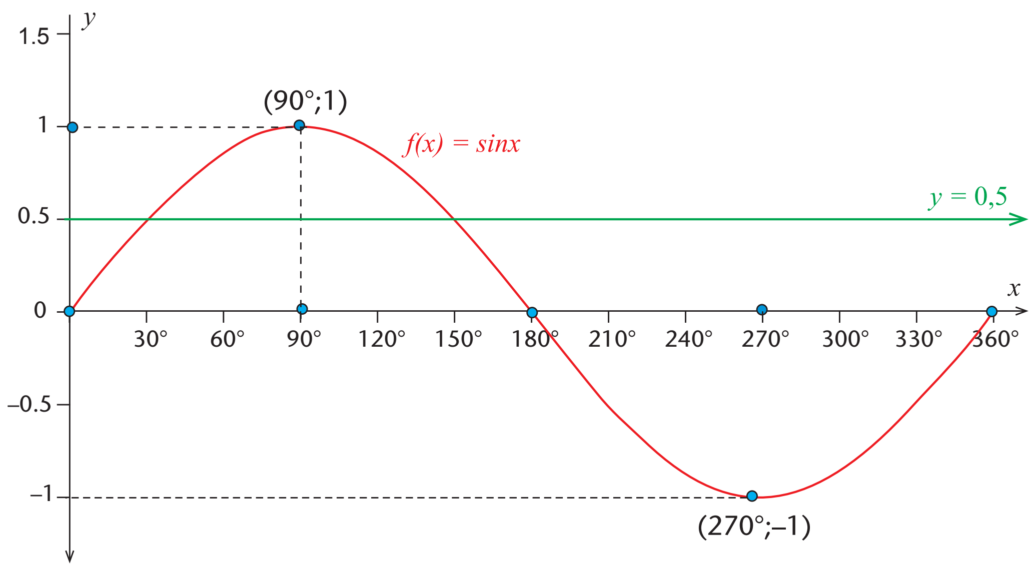
We then consider all solutions in that fall within $0° \leq x \leq 360°$

$x = 30°$ or $x = 180° - 30°$

$x = 150°$

Or using the sine graph can also sketch a graph of the function $y = \sin x$ over the interval $0° \leq x \leq 360°$.

The green line ($y = 0{,}5$) intersects the graph of $y = \sin x$ at 30° and 150°. The points where the graph of $y = 0{,}5$ intersect the graph of $y = \sin x$ are the solutions of the equation $\sin x = \frac{1}{2}$



EXERCISE 8.17

1. Solve for x: $x \in [0°; 360°]$

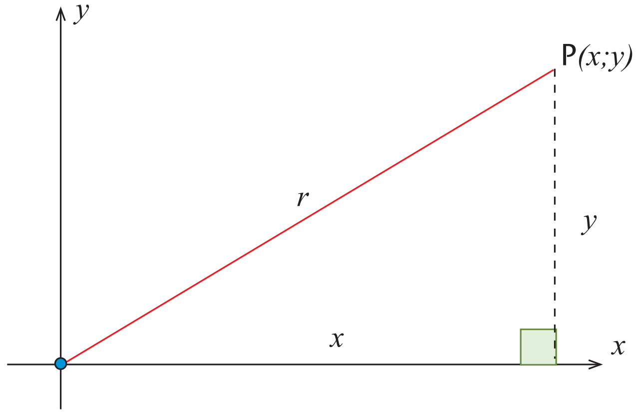
	Equation	Reference angle	Quadrants where the function has +/- values	Cast diagram or graph	All solutions
a)	$\cos x = \frac{1}{2}$				
b)	$\tan x = 1$				
c)	$\sin x = \frac{\sqrt{3}}{2}$				
d)	$\cos x = \frac{1}{\sqrt{2}}$				
e)	$\tan x = \sqrt{3}$				
f)	$\sin x = -\frac{1}{2}$				
g)	$\tan x = -\frac{1}{\sqrt{3}}$				
h)	$\cos x = -\frac{1}{2}$				

2. Solve for $\theta, x \in [0°; 360°]$. Give your answer correct to one decimal place.
   a) $\sin x = 0,7$
   b) $\tan x = -0,35$
   c) $\cos x = 0,43$
   d) $\tan x = 0,91$
   e) $\sin x = -0,80$
   f) $\cos x = -0,76$
   g) $2\sin\theta + 1 = 0$
   h) $\frac{1}{\sqrt{3}}\tan x + 1 = 0$
   i) $2\cos\theta + \sqrt{3} = 0$
   j) $\sqrt{2}\sin\theta = 1$
   k) $2\sin^2 x = 1$

8.10 Trigonometric identities

An equation which is true for all replacements of the variables for all the terms of the equation is called an identity. We are going to use the definitions of trigonometric ratios below to derive two very useful identities in trigonometry.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$



The identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

Let's consider the quotient of the sine and cosine ratios: $\dfrac{\sin\theta}{\cos\theta}$.

According to the definitions of the sine and cosine ratios: $\dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{r} \div \dfrac{x}{r}$

$$= \frac{y}{r} \times \frac{r}{x}$$

$$= \frac{y}{x}$$

According to one of the definitions of the trigonometric ratios $\dfrac{y}{x} = \tan\theta$

So we conclude that $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$.

Since division by zero is not defined, $\tan\theta$ is not defined when $\cos\theta = 0$.

Identities are useful when we have to simplify expressions.

Worked example 8.9

Simplify the expression $\cos^2\theta\tan^2\theta$

Solution

$$\cos^2\theta\tan^2\theta = \cos^2\theta\frac{\sin^2\theta}{\cos^2\theta}$$

$$= \sin^2\theta$$

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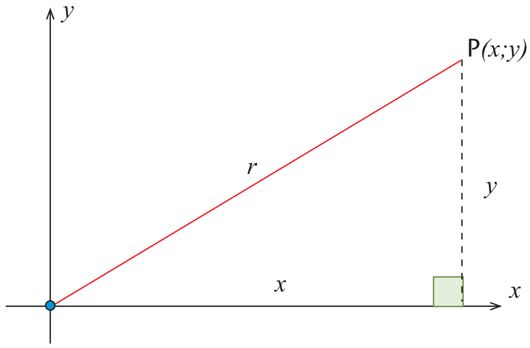
The identity $\sin^2\theta + \cos^2\theta = 1$

Copy and complete the table below. You may use your calculator.

θ	0°	30°	45°	60°	90°	180°	270°	360°
$\sin\theta$								
$\cos\theta$								
$\sin^2\theta + \cos^2\theta$								

We have shown for a few values of θ that $\sin^2\theta + \cos^2\theta = 1$. Let's show that $\sin^2\theta + \cos^2\theta = 1$ is true for any value of θ.



By definition $\sin\theta = \frac{y}{r}$ and $\cos\theta = \frac{x}{r}$.

$$\begin{aligned}
\text{Now } \sin^2\theta + \cos^2\theta &= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 \\
&= \frac{y^2}{r^2} + \frac{x^2}{r^2} \\
&= \frac{y^2 + x^2}{r^2} \\
&= \frac{r^2}{r^2} \text{ [Pythagoras Theorem]} \\
&= 1
\end{aligned}$$

Worked example 8.10

Simplify the given expression: $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$

Solution

$$\begin{aligned}
\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta} \\
&= \frac{1}{\cos\theta\sin\theta} \\
&= \frac{1}{\cos\theta} \times \frac{1}{\sin\theta} \\
&= \sec\theta\,\mathrm{cosec}\theta
\end{aligned}$$

The identity $1 + \tan^2\theta = \sec^2\theta$

We will use the identities we have already proven to derive the other two identities.

$$\begin{aligned}
1 + \tan^2\theta &= \frac{\cos\theta}{\cos\theta} + \frac{\sin^2\theta}{\cos^2\theta} \\
&= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} \\
&= \frac{1}{\cos^2\theta} \\
&= \sec^2\theta
\end{aligned}$$

The identity $1 + \cot^2 x = \operatorname{cosec}^2 x$

$$
\begin{aligned}
1 + \cot^2 x &= \frac{\sin x}{\sin x} + \frac{\cos^2 x}{\sin^2 x} \\
&= \frac{\sin^2 x + \cos^2 x}{\sin^2 x} \\
&= \frac{1}{\sin^2 x} \\
&= \operatorname{cosec}^2 x
\end{aligned}
$$

EXERCISE 8.18

1. Use your knowledge of identities to reduce the following to its simplest form
   a) $(1 + \cos x)(1 - \cos x)$
   b) $\dfrac{\sin\theta\cos\theta}{1 - \cos^2\theta}$
   c) $\dfrac{\cos\beta}{1 + \cos\beta} + \dfrac{1 + \cos\beta}{\cos\beta}$
   d) $\dfrac{1}{\cos^2\theta}(1 - \sin^2\theta)$
   e) $\dfrac{\tan^2\theta}{1 + \tan^2\theta}$
   f) $\dfrac{\sin\beta\cos\beta + \sin\beta}{\cos\beta + 1}$
   g) $\dfrac{\sin x\tan x + \tan x}{\tan x + \tan^2 x}\left(1 + \dfrac{\sin x}{\cos x}\right)$
   h) $(1 - \cos^2 A)(1 + \tan^2 A)$
   i) $\dfrac{\sin A}{\sqrt{1 - \sin^2 A}}$
   j) $\sqrt{1 - \sin A\cos A\tan A}$
2. Prove that:
   a) $(1 - \sin x)(1 + \sin x) = \cos^2 x$
   b) $\sin x\sec x = \tan x$
   c) $\cot^2 x\tan x = \cot x$
   d) $(1 + \tan^2 x)\cos^2 x - 1 = 0$
   e) $\operatorname{cosec}^2 x\cos x = \tan x\cos x$
   f) $\sin^2 x(1 + \cot^2 x) = 1$
   g) $\sin x\cot x + \sec x = \cos x + \sec x$
   h) $\dfrac{\sin\theta}{1 + \sin\theta} + \dfrac{\sin\theta}{1 - \sin\theta} = -2\tan\theta$

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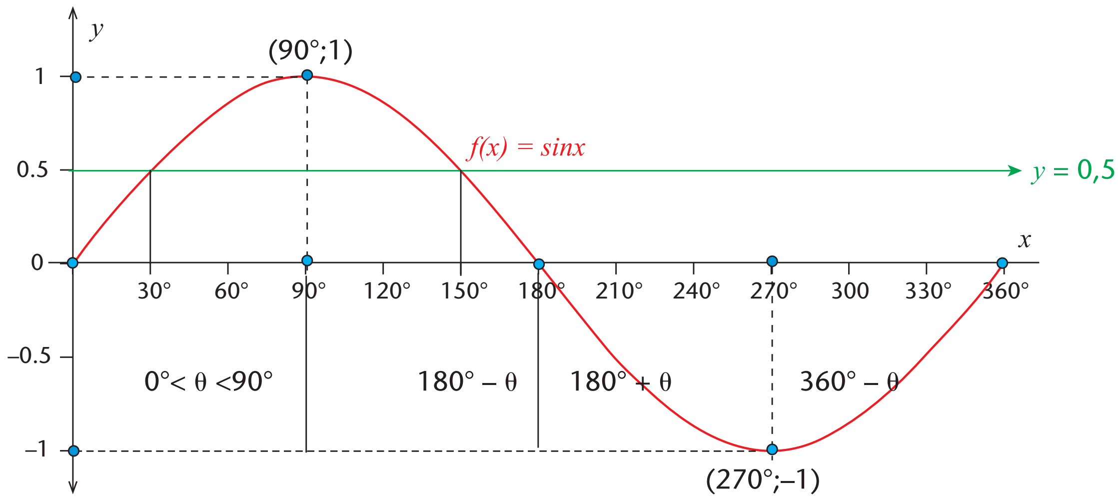
8.11 The reduction formula

The basic idea behind the reduction formula is that trigonometric functions of angles greater than 90°can be rewritten as functions of acute angles.

The idea behind the use of the reduction formula can be illustrated by means of a graph. Consider the graph of $y = \sin x$ and the graph of $y = 0,5$ drawn below. The two graphs intersect at the points $(30°; 0,5)$ and $(150°; 0,5)$. What this means is that $\sin 30°$and $\sin 150°$ have the same function value which is 0,5. In other words, instead of using $\sin 150°$ in calculations we can use $\sin 30°$. The sine of 30° ($\sin 30°$) is in the first quadrant where all ratios are positive, and $\sin 150°$ is in the second quadrant where only the sine ratio is positive.



In this unit we will discuss theorems (without proving them) that will help us to express the sine, cosine or tangent of any angle as equal to, or the negative of, some acute angle.

Reduction formula for $(180° - \theta)$

The following theorems are used to reduce trigonometric functions of angles in the second quadrant. Angles in the second quadrant are written as $(180° - \theta)$.

a) $\sin(180° - \theta) = \sin\theta;\ 0° \leq \theta < 90°$

b) $\cos(180° - \theta) = -\cos\theta;\ 0° \leq \theta < 90°$

c) $\tan(180° - \theta) = -\tan\theta;\ 0° \leq \theta < 90°$

Worked example 8.11

Rewrite $\sin 168°$ as functions of acute angles. Leave the answer in the simplest form where possible, without the use of a calculator. $0° \leq \theta < 90°$.

Solution

$$\begin{aligned}\sin 168° &= \sin(180° - 12°)\\ &= \sin 12°\end{aligned}$$

Worked example 8.12

Rewrite $\sin 158°$ as functions of acute angles. Leave the answer in the simplest form where possible, without the use of a calculator. $0° \leq \theta < 90°$.

Solution

$$\begin{aligned}\sin 158° &= \sin(180° - 22°)\\ &= \sin 22°\end{aligned}$$

Worked example 8.13

Rewrite $\cos 145°$ as functions of acute angles. Leave the answer in the simplest form where possible, without the use of a calculator. $0° \leq \theta < 90°$.

Solution

$$\begin{aligned}\cos 145° &= \cos(180° - 35°)\\ &= -\cos 35°\end{aligned}$$

Only the sine ratio is positive in the second quadrant.

EXERCISE 8.19

Use the reduction formulae to write the following as functions of acute angles. Give the answer in simplified form where necessary. Do not use a calculator. $0° \leq \theta < 90°$.

a) $\cos(180° - \beta)$
b) $\tan(180° - 70°)$
c) $\sin(180° - 50°)$
d) $\sin 145°$
e) $\tan 105°$
f) $\cos 179°$
g) $\cos 120°$
h) $\tan 135°$
i) $\sin 150°$
j) $\sin 120°$
k) $\cos 135°$
l) $\tan 150°$
m) $\tan 120°$
n) $\sin 135°$
o) $\cos 150°$

Reduction formula for $(180° + \theta)$

The following theorems are used to reduce trigonometric functions of angles in the third quadrant. Angles in the third quadrant are written as $(180° + \theta)$.

a) $\sin(180° + \theta) = -\sin\theta;\ 0° \leq \theta < 90°$
b) $\cos(180° + \theta) = -\cos\theta;\ 0° \leq \theta < 90°$
c) $\tan(180° + \theta) = \tan\theta;\ 0° \leq \theta < 90°$

Worked example 8.14

Rewrite $\tan 210°$ as functions of acute angles. Leave the answer in the simplest form where possible, without the use of a calculator. $0° \leq \theta < 90°$.

Solution

$$\begin{aligned}\tan 210° &= \tan(180° + 30°)\\ &= \tan 30°\\ &= \frac{1}{\sqrt{3}}\end{aligned}$$

Worked example 8.15

Rewrite $\sin 192°$ as functions of acute angles. Leave the answer in the simplest form where possible, without the use of a calculator. $0° \leq \theta < 90°$.

Solution

$$\begin{aligned}\sin 192° &= \sin(180° + 12°)\\ &= -\sin 12°\end{aligned}$$

Worked example 8.16

Rewrite $\cos 260°$ as functions of acute angles. Leave the answer in the simplest form where possible, without the use of a calculator. $0° \leq \theta < 90°$.

Solution

$$\begin{aligned}\cos 260° &= \cos(180° + 80°)\\ &= -\cos 80°\end{aligned}$$

Only the tangent ratio is positive in the third quadrant.

EXERCISE 8.20

Use the reduction formulae to write the following as functions of acute angles. Give the answer in simplified form where necessary. Do not use a calculator. $0° \leq \theta < 90°$.

a) $\sin(180° + \theta)$
b) $\tan(180° + 20°)$
c) $\cos 255°$
d) $\tan 198°$
e) $\sin 207°$
f) $\cos 210°$
g) $\sin 210°$
h) $\tan 225°$
i) $\sin 225°$
j) $\cos 225°$
k) $\sin 240°$
l) $\tan 240°$
m) $\cos 240°$

Reduction formula for $(360° - \theta)$

The following theorems are used to reduce trigonometric functions of angles in the fourth quadrant. Angles in the fourth quadrant are written as $(360° - \theta)$.

a) $\sin(360° - \theta) = -\sin\theta$; $0° \leq \theta < 90°$

b) $\cos(360° - \theta) = \cos\theta$; $0° \leq \theta < 90°$

c) $\tan(360° - \theta) = \tan\theta$; $0° \leq \theta < 90°$

Worked example 8.17

Rewrite $\sin 330°$ as a function of acute angles. Leave the answer in the simplest form where possible, without the use of a calculator. $0° \leq \theta < 90°$.

Solution

$$\begin{aligned}\sin 330° &= \sin(360° - 30°)\\ &= -\sin 30°\\ &= -\frac{1}{2}\end{aligned}$$

Worked example 8.18

Rewrite $\cos 350°$ as a function of acute angles. Leave the answer in the simplest form where possible, without the use of a calculator. $0° \leq \theta < 90°$.

Solution

$$\begin{aligned}\cos 350° &= \cos(360° - 10°)\\ &= \cos 10°\end{aligned}$$

Worked example 8.19

Rewrite $\tan 275°$ as a function of acute angles. Leave the answer in the simplest form where possible, without the use of a calculator. $0° \leq \theta < 90°$.

Solution

$$\begin{aligned}\tan 275° &= \tan(360° - 85°)\\ &= -\tan 85°\end{aligned}$$

Only the cosine ratio is positive in the fourth quadrant.

EXERCISE 8.21

Use the reduction formulae to write the following as functions of acute angles:

a) $\cos(360° - 10°)$

b) $\sin(360° - 50°)$

c) $\tan 320°$

d) $\cos 305°$

e) $\sin 282°$

f) $\tan 330°$

g) $\sin 330°$

h) $\cos 330°$

i) $\sin 300°$

j) $\cos 300°$

k) $\tan 300°$

l) $\cos 315°$

m) $\tan 315°$

n) $\sin 315°$

o) $\cos 275°$

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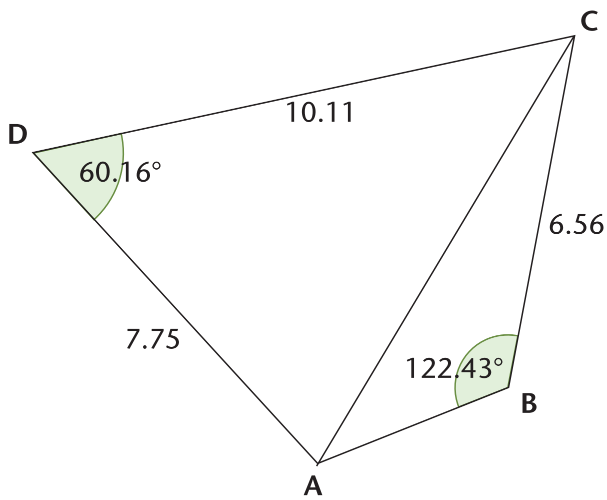
CONSOLIDATION EXERCISE

1. Solve the following triangles.
   a) ΔABC, $\hat{A} = 59°$, $\hat{C} = 75°$ and $a = 49$
   b) ΔPQR, $p = 6$ cm, $q = 4$ cm and $r = 8$ cm
2. Calculate the area of a triangle with sides $a = 17$ cm, $c = 20$ cm and $\hat{B} = 50°$
3. A piece of metal is in the shape of a triangle with sides of lengths 11,4 cm; 7,2 cm and 8,7 cm. Calculate its area.
4. The Mlondo family owns a property whose boundaries appear as shown in the sketch below. What is the total area of the property?



5. Grade 10 learners were given a task to calculate the height of a tall building near their school. They first measured the angle of elevation to the top of the building as 45°. They then walked 15 metres closer to the building and measured the angle of elevation to be 70°.
   a) Draw a sketch to represent the above situation.
   b) Calculate the height of the building.
   c) If each floor of the building is 4,7 metres in height, how many floors are in this building?
7. Consider the functions given below.
   a) Draw graphs of each of the given functions on the same set of axes for the indicated values of k.

	Function	$k = -2$	$k = -1$	$k = 1$	$k = 2$
(i)	$y = k \sin x$				
(ii)	$y = \cos kx$				
(iii)	$y = k \tan x$				

   b) Describe the change caused by changing the value of k in each of the case.

Summary

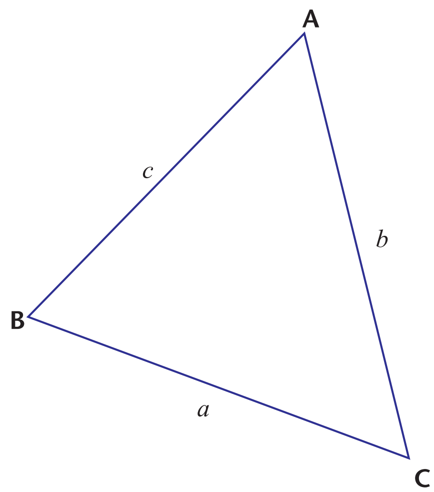
Solving a triangle

To solve a triangle is to calculate the lengths of each of its sides and all its angles.

The sine rule or the law of sine

- The sine rule is useful for solving triangles.
- The sine rule is used when we know the following:
  - Two angles and one side of the triangle
  - Two sides and a non-included angle

Statement of the sine rule



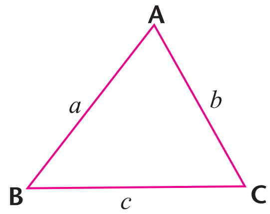
or

- $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

The cosine rule

- The cosine rule is also useful for solving triangles.
- We use the cosine rule when we know the following:
  - Three sides of the triangles.
  - Two sides and the included angle.

Statement of the cosine rule



$a^2 = b^2 + c^2 - 2bc \cos A$

$b^2 = a^2 + c^2 - 2bc \cos B$

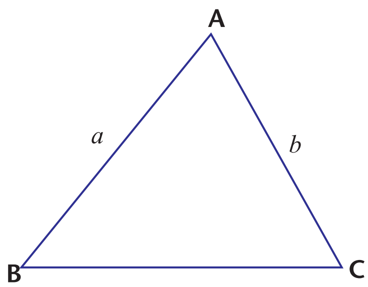
$c^2 = a^2 + b^2 - 2bc \cos C$

The area rule

The area rule is one way to compute the area of a triangle.

We use the area rule when we know two sides and the included angle of the triangle.

Statement of the area rule



Area of $ABC = \frac{1}{2}\, ab \sin C$

Area of $ABC = \frac{1}{2}\, ac \sin B$

Area of $ABC = \frac{1}{2}\, bc \sin A$

Transformation of the basic trigonometric functions

Some transformations of the sine function

Function	Property that is affected	Property that is not affected
$y = k \sin x$	Amplitude, range, minimum and maximum value	x-intercepts, period and domain
$y = \sin kx$	Period, x-intercepts	y-intercept, amplitude, range, domain
$y = \sin (x + p)$	x-intercepts, y-intercept	Amplitude, range, domain, period

Function	Domain	Range	Period	Amplitude	x-intercept(s)	y-intercept	Max value	Min Value
$y = \sin x$	$x \in \mathbb{R}$	[–1; 1]	360°	1	0°; 180°; 360°	0	1	–1
$y = 2 \sin x$	$x \in \mathbb{R}$	[–2; 2]	360°	2	0°; 180°; 360°	0	2	–2
$y = \sin 2x$	$x \in \mathbb{R}$	[–1; 1]	$\frac{360°}{2} = 180°$	1	0°; 90°; 180°; 270°; 360°	0	1	–1
$y = \sin(x + 30°)$	$x \in \mathbb{R}$	[–1; 1]	360°	1	30°; 150°; 330°	$\frac{1}{2}$	1	–1

Some transformations of the cosine function

Function	Property that is affected	Property that is not affected
$y = k \cos x$	Amplitude, range, minimum and maximum value	x-intercepts, period and domain
$y = \cos kx$	Period, x-intercepts	y-intercept, amplitude, range, domain
$y = \cos (x + p)$	x-intercepts, y-intercept	Amplitude, range, domain, period

Function	Domain	Range	Period	Amplitude	x-intercept(s)	y-intercept	Max Value	Min Value
$y = \cos x$	$x \in \mathbb{R}$	[–1; 1]	360°	1	90°; 270°	1	1	–1
$y = 3 \cos x$	$x \in \mathbb{R}$	[–3; 3]	360°	3	90°; 270°	3	3	–3
$y = \cos 3x$	$x \in \mathbb{R}$	[–1; 1]	$\frac{360°}{3} = 120°$	1	30°; 90°; 150°; 210°	1	1	–1
$y = \cos(x + 45°)$	$x \in \mathbb{R}$	[–1; 1]	360°	1	130°; 315°		1	–1

Solving trigonometric equations

To solve trigonometric equations we use algebraic techniques such as collecting like terms and factoring.

Trigonometric identities

- $\tan \theta = \frac{\sin\theta}{\cos\theta}$
- $\sin^2 \theta + \cos^2 \theta = 1$
- $1 + \tan^2 \theta = \sec^2 \theta$

Reduction formulae

Quadrant II	Quadrant III	Quadrant IV
$(180° - \theta)$	$(180° + \theta)$	$(360° - \theta)$
$\sin(180° - \theta) = \sin \theta$	$\sin(180° + \theta) = \sin \theta$	$\sin(360° - \theta) = \sin \theta$
$\cos(180° - \theta) = \cos \theta$	$\cos(180° + \theta) = \cos \theta$	$\cos(360° - \theta) = \cos \theta$
$\tan(180° - \theta) = \tan \theta$	$\tan(180° + \theta) = \tan \theta$	$\tan(360° - \theta) = \tan \theta$

9 Finance, growth and decay

Objectives

In this chapter you will:

- revise simple and compound interest
- calculate straight-line and reducing balance depreciation
- use simple and compound decay formulae to solve problems
- determine the effects of nominal and effective interest rates
- use timelines to calculate interest

Finances can grow or decay exponentially.

9.1 Revision: Simple and Compound Interest

Simple interest formula

In Grade 10 you learned that interest is the cost of borrowing money when you make a loan, or the amount of money paid out when a sum of money is invested or saved in a financial institution.

Generally, interest is calculated in two different ways, that is, simple interest and compound interest. In simple interest, interest is calculated on the original amount, that is, interest remains constant over a period of time. In other words, simple interest is earned only on the amount of money borrowed or invested and not on a balance. The hire purchase agreement is a type of loan used as a short-term loan to buy furniture, cars and so on. With a hire purchase agreement, interest charged is calculated as simple interest on the full amount of the loan over the repayment period.

The following formula is used to calculate simple interest $A = P(1 + \text{in})$ and interest earned $I = A - P$, where:

A = accumulated amount, it includes the original amount plus interest

P = Principal amount, it includes the original amounts plus interest

i = interest rate

n = time period in years

Worked example 9.1

Thobeka works part-time at the supermarket. Out of her total earnings, she decides to invest R3 650 for 5 years. Trust Bank offers her a savings account which pays simple interest at a rate of 12,5% per annum. Calculate:

a) Thobeka's investment at the end of 5 years

b) the interest amount she earned.

Solution

a) $A = ?$

P = R3 650

i = 12,5%

n = 5 years

$I = ?$

$A = P(1 + \text{in})$

$= 3650(1 + 0{,}125 \times 5)$

= R5 931,25

b) $SI = A - P$

= R5 931,25 – R3 650

= R2 281,25

The problem can also be solved in the following way:

$SI = P \times I \times n$ = R3 650 $\times$ 0,125 $\times$ 5 = R 2 281,25

EXERCISE 9.1

1. The inflation rate over the past two years was 5,3% and 5,9%, respectively.
   What are the present prices of these articles if they cost the following amounts two years ago:
   a) Music Center: R2 400
   b) DVD player: R820
   c) CD: R119
2. The inflation rate is about 5,7% each year. Suppose that 5 years ago a loaf of bread cost R8. How much would you expect a loaf of bread to cost today?
3. The inflation rate is about 7,5% each year. Suppose that 10 years ago a movie ticket cost R25. How much would you expect it to cost today?

Compound interest formula

Compound interest is calculated both on the total amount borrowed or invested and also on the accumulated interest. In other words, interest is earned on interest.

The following formula is used to calculate compound interest: $A = P(1 + i)^n$, where:

A = accumulated amount, it includes the original amount plus interest

P = Principal amount, it includes the original amounts plus interest

i = interest rate

n = time period in years

Worked Example 9.2

Problem revisited: Thobeka works part-time at the supermarket. Out of her total earnings, she decides to invest R3 650 for 5 years. United Bank offers her a savings account which pays compound interest at a rate of 12,5% per annum.

Calculate:
a) Thobeka's investment at the end of 5 years
b) the interest amount she earned.

Solution

a) $A = ?$
$P = \text{R}3\ 650$
$i = 12{,}5\%$
$n = 5$ years
$I = ?$
$A = P(1 + i)^n$
$= 3\ 650(1 + 0{,}125)^5$
$= \text{R}6\ 577{,}42$

b) $CI = A - P$
$= \text{R}6\ 577{,}42 - \text{R}3\ 650$
$= \text{R}2\ 927{,}42$

Note:
Compare these amounts to the amounts in Example 9.1.
It is advisable to do calculations on the calculator only at the end.
Calculations in this chapter are rounded off to two decimal places which correspond to the nearest cent.

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Worked Example 9.3

Siphiwe invests R38 500 in a savings investment account for a period of 9 years. The total growth of the funds amounts to R90 781 after 9 years.

a) Calculate the interest rate per annum compounded annually that would yield the same return.

b) Calculate the simple interest rate per annum that would yield the same return.

c) What compound interest rate must the savings investment account achieve for Siphiwe to triple his money in 9 years?

Solution

a) $A = \text{R}90\ 781$

$P = \text{R}38\ 500$

$i = ?$

$n = 9$ years

$A = P(1 + i)^n$

Make i the subject of the formula and calculate: $i = \sqrt[n]{\frac{A}{P}} - 1$

$\therefore = \sqrt[9]{\frac{90\ 781}{38\ 500}} - 1$

$= 0,10$

$= 10\%$ p.a.

b) $A = \text{R}90\ 781$

$P = \text{R}38\ 500$

$i = ?$

$n = 9$ years

$A = P(1 + in)$

Make i the subject of the formula and calculate: $i = \frac{1}{n}\left(\frac{A}{P} - 1\right)$

$\therefore = \frac{1}{9}\left(\frac{90\ 781}{38\ 500} - 1\right)$

$= 0,15$

$= 15\%\ p.a.$

c) $A = \text{R}115\ 500$

$P = \text{R}38\ 500$

$i = ?$

$n = 9$ years

$A = P(1 + i)^n$

Make i the subject of the formula and calculate: $i = \sqrt[n]{\frac{A}{P}} - 1$

$= \sqrt[9]{\frac{115\ 500}{38\ 500}} - 1$

$= 0,13$

$= 13\%\ p.a.$

EXERCISE 9.2

1. Determine the value of an investment of R8 000 at the end of 5 years if it is invested at:
   a) 8% p.a. simple interest
   b) 8% p.a. compound interest.
2. Nazim invested R70 000 for 7 years at 11% p.a. compounded annually.
   Calculate the total amount of money that will be available at the end of 7 years.
3. Calculate the total amount of money that will be available after 3 years when Tshilidzi invests R120 000 at 15% p.a. simple interest.
4. Nkateko deposits R4 500 into a fixed deposit account and does not touch the money for 9 years.
   Calculate how much money she will have in the account at the end of 9 years if the interest is calculated at:
   a) 13,5% p.a. simple interest
   b) 13,5% p.a. compound interest compounded annually.
   c) Which type of interest is better? Why?
5. Calculate how much interest Sharon will earn if she invests R3 000 for 4 years when interest is calculated at:
   a) 6,7% p.a. simple interest
   b) 5,4% p.a. compound interest
6. Mrs Moreki buys a new laptop, colour printer and software at a cost of R25 000 on a hire purchase agreement. She pays 16,4% p.a. simple interest on the full amount for 3 years.
   a) Calculate the total amount that she must repay over 3 years.
   b) Calculate Mrs Moreki's equal monthly instalment for 3 years.
7. Shoni deposits a lump sum into an account giving 15% p.a. compound interest. The balance in the account is R30 591 after 8 years. What was the initial amount?
8. Shake's investment grows from R4 400 to R7 700 with 8 years' simple interest.
   Determine the simple interest rate at which it was invested.
9. Mr Shekinah invests R150 000 into a pension fund. The fund grows to R450 000 after 10 years. Calculate the annual interest rate if the interest was compounded annually.
10. Masello won the lottery and she decided to invest her winnings in a savings account. At the beginning of her investment, the compound interest rate is 8,25% p.a. Three years later, the interest rate rose to 12,5% p.a. Twelve years after the initial deposit, Masello's investment is worth R850 000. How much did she win on the lottery?
11. Tshivhase invests an amount, P and earns an annual compound interest. After 5 years, the amount is R3 500 and after a total of ten years the amount grows to R12 500. Calculate P and the annual interest rate.

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9.2 Depreciation

It does happen in some instances that individuals or companies make costly investments, where an investment is depreciating in value instead of appreciating. Most large companies have fixed assets such as motor vehicles, computers, machinery and so on. The value of these assets decreases as time passes. Each year they get older and more out of date and the price that can be obtained if they are sold will not be the same as the price paid for them. We say that their value depreciates. For example, a brand new car loses its value immediately when it leaves the dealership. We say the car has depreciated in value because now it is a used or second-hand vehicle.

One reason for calculating depreciation is to establish the value of a company's asset. The other reason is financial, that is, depreciation determines how much tax a company must pay by reducing taxable income.

> ***Depreciation*** *is said to be the loss in value of an asset over a period in time.*
>
> ***Decay*** *is also a term used to describe a reduction or decline in value.*

Two different kinds of depreciation are calculated, simple depreciation (decay) also known as straight-line depreciation (decay) and compound depreciation which is also known as reducing balance depreciation (decay).

Simple depreciation

In simple or straight-line depreciation, the value of an asset is reduced to zero over a period of time. Depreciation in this instance is the same every year and it is a percentage of the original value of the asset.

Worked Example 9.4

Novena bought a brand new Alfa Romeo for R250 000. In five years' time she would like to replace this vehicle with a new one. However, Novena wants to know the value of the vehicle at the end of each year over five years. She decides to work on a depreciation rate of 20% per annum on the straight-line basis. Calculate the value of the vehicle at the end of the 1st, 2nd, 3rd, 4th and 5th year respectively.

Solution

Step 1: Calculate depreciation amount

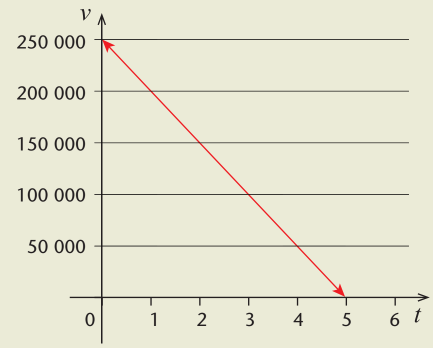
Depreciation $= 250\ 000 \times \frac{20}{100} = 50\ 000$

Therefore, the vehicle depreciates by R50 000 every year.

Step 2: Complete a table of values

	Value at the beginning of the year	Depreciation amount	Value at the end of the year
1	R 250 000	R 50 000	R 200 000
2	R 200 000	R 50 000	R 150 000
3	R 150 000	R 50 000	R 100 000
4	R 100 000	R 50 000	R 50 000
5	R 50 000	R 50 000	R 0

The diagram on the right shows a graph of the depreciation over the 5 years.
The graph represents a straight line, and illustrates why this method of depreciation is called straight-line depreciation.



In the worked example above, you will notice that the vehicle reduces in value by the same amount of R50 0000 in each successive year.

The depreciation is always a percentage of the original value of the asset in straight-line depreciation. In the worked example, the depreciation at the end of each year is 20% of R250 000.

The value of the vehicle is reduced to a zero balance after 5 years.

The depreciated value of the asset is also called the book value of the asset.

EXERCISE 9.3

1. Three years after the purchase, Fresh's BMW motorbike was worth R75 000. Four years later it was worth only R15 000. Given that the value of the motorbike depreciated on a straight-line basis, calculate:
   a) the annual rate of depreciation
   b) the original cost of the motorbike.
2. Moshe Civil & Construction company has office furniture and drawing equipment valued at R320 000. The value of the furniture and drawing equipment depreciates at a rate of 15% p.a. on a straight-line basis. Calculate the value of the furniture and drawing equipment at the end of 4 years.
3. Nokuthula's parents bought her a second hand car costing R120 000 when she started working after graduating at university. Nokuthula expects the car to have a lifetime of at least another 8 years, then she will sell the car and use the money as a deposit on a new car. If the depreciation rate is 5% p.a. on a straight-line basis, how much will be available as a deposit on the new car 8 years from now?
4. A company buys computers at a cost of R400 000. They expect to replace the computers at the end of 4 years.
   a) What annual rate of depreciation will be used if the computers are discarded at the end of 4 years? You may assume that the computers have no resale value.
   b) Write down the depreciated value of the computers year by year for the 4 years.

Compound depreciation (reducing balance)

Book value is the value of an asset after taking into account depreciation for a given period or number of years.

In compound depreciation, the value of the asset decreases at a certain annual rate, but the initial value of the asset in the current year, is the book value of the asset at end of last year. In other words, the depreciation is based on the previous year's value. In reducing balance depreciation, the asset has some value at the end of the period as the asset devalues.

The significant difference between simple depreciation and compound depreciation is that simple depreciation is based on the original amounts only, whereas compound depreciation is based on the reducing balance basis.

Worked Example 9.5

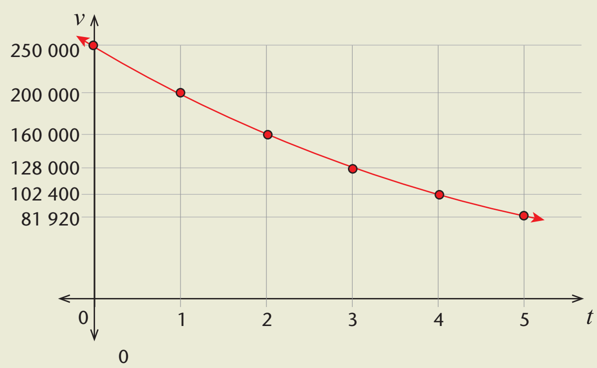
Novena bought a brand new Alfa Romeo at R250 000. In five years' time she would like to replace this vehicle with a new one. However, Novena wants to know the value of the vehicle at the end of each year over five years. She decides to work on a depreciation rate of 20% per annum on the reducing balance basis. Calculate the value of the vehicle at the end of the 1st, 2nd, 3rd, 4th and 5th year respectively.

Solution

Note: $20\% = \frac{20}{100} = 0{,}2$

Year	Book Value (R)	Calculation of De-preciation	Depreciation (R)	Value at the end of year (R)
1	250 000	250 000 × 0,2	50000	200 000
2	200 000	200 000 × 0,2	40000	160 000
3	160 000	160 000 × 0,2	32000	128 000
4	128 000	128 000 × 0,2	25600	102 400
5	102 400	102 400 × 0,2	20480	81 920

A graph to show the depreciation of the vehicle over 5 years:



In worked example 9.5, you will notice that the depreciation each year is 20% of the preceding year's value. Again, note that as the value of Novena's vehicle decreases, the depreciation each year is less than that of the previous year. This means that Novena's vehicle will always have some value and the vehicle's value will never be reduced to zero.

Selection of the method to calculate depreciation depends largely on the type of business. But in the business world, it is standard that for the relatively inexpensive assets, or assets that have negligible resale value after a few years, depreciation be calculated on a straight-line basis.

Assets like vehicles, machinery, equipment and expensive durable assets have a significant resale value. Thus, in most cases reducing balance basis is used to calculate depreciation.

EXERCISE 9.4

1. A small business enterprise wins an Eskom tender to transport coal. The company buys a used horse-'n-trailer truck for R840 000. The company anticipates that the life span of the truck will be five years, and it will depreciate at the rate of 15% p.a.
   a) Calculate the book value of the truck at the end of 1st, 2nd, 3rd, 4th and 5th year respectively if depreciation is calculated on a (i) straight-line basis, and (ii) reducing balance basis.
   b) Draw two graphs on the same system of axes to illustrate the difference in the value of the truck, year by year, when depreciation is calculated on a straight-line basis and on a reducing balance basis.

Simple and compound depreciation formula

The formulae for depreciation are similar to the formulae for simple and compound interest. In simple and compound interest, the principal amount (P) accumulates (appreciate) based on the interest rate given, but in the simple and compound depreciation the principal amount depreciates. The following illustrates the similarities and differences:

	Increase	Decrease
Simple	$A = P(1 + in)$	$A = P(1 - in)$
Compound	$A = P(1 + i)^n$	$A = P(1 - i)^n$

P is the original value of the asset, and A is the value of the asset after depreciation.

Worked example 9.6

A printing company buys a big photocopy-printer machine that has multiple functions valued at R72 784. Calculate the value of the photocopy-printer machine at the end of 6 years if depreciation is calculated at 14% p.a. on:

a) a straight-line basis
b) a reducing balance basis.
c) Which is the better option?

Solution

a) $A = ?$

$P = \text{R}72\ 784$

$i = 0{,}14$

$n = 6$ years

$A = P(1 + in)$

$= 72\ 784(1 - (0{,}14 \times 6))$

$= \text{R}11\ 645{,}44$

b) $A = ?$

$P = \text{R}72\ 784$

$i = 0{,}14$

$n = 6$ years

$A = P(1 + i)^n$

$= 72\ 784(1 - 0{,}14)^6$

$= \text{R}29\ 446{,}02$

c) After a period of 6 years, the value of the photocopy-printer machine calculated on the straight-line basis is less than the value of the photocopy-printer calculated on the reducing balance basis. This means that the value of the photocopy-printer depreciated less on the reducing balance basis because the amount of depreciation is calculated on a smaller amount every year, whereas the straight-line basis is calculated on the full value of the photocopy-printer every year.

EXERCISE 9.5

1. Style's Mercedes Benz, which is worth R225 000, depreciates by 6,6% p.a. based on a reducing balance. Determine the value of the car after 4 years.
2. Determine the original value of a company's assets if it depreciated to R149 500, after 3 years, at 3,5% p.a. compounded.
3. A used car is worth R145 000 today. It depreciates at a rate of 14% on a straight-line depreciation. What will the book value be in 4 years' time?
4. Tshepo bought a computer worth R22 000 to use it as an administrator in the school's computer lab. The computer will depreciate after five years to R10 000 on a reducing-balance method. At what rate will this take place?
5. Rose invested R15 400 for a period of 3 years which depreciated to an amount of R9 832.
   a) Determine the flat rate at which the money depreciated.
   b) Suppose the money depreciated on the reducing balance. Determine the reducing balance rate.
6. Vosburg is a small town in the upper Karoo in the Northern Cape Province. According to 2011 census figures, Vosburg had a population of 1 259. A drought caused people to move closer to the cities. This resulted in a loss of 8% per annum in the population. How many people are in this town in 2016 if measured on:
   a) compound depreciation
   b) straight-line depreciation.

9.3 Nominal and effective interest rates

The examples examined so far all have compounding periods of one year. Compounding can, and often do, occur more frequently. In other words, interest can be added (compounded) at any frequency or regular period of time during the year. The standard practice, when quoting interest rates, is to quote them as annual interest rates, irrespective of the compounding period.

When working with problems involving interest, we use the term payment period as follows:

Annually	Once a year
Semi-annually	Twice a year
Quarterly	4 times a year
Monthly	12 times a year
Daily	365 times a year

In the financial world, interest rates are distinguished as effective interest rate and nominal interest rate respectively. In nominal interest rate the stated period and the compounding period are not the same, for example 12% per annum (stated period) compounded quarterly (compounding period). Whereas in effective interest rate the stated period and the compounding period are the same, for example, 3% per quarter compounded quarterly.

To illustrate what we mean, consider 12% p.a. compounded quarterly on R1, the information is summarised in the table below:

	Percentage	Stated Period	Compounding Period
Nominal rate	12	Per annum	Quarterly
Effective quarterly rate	3	Per quarter	Quarterly
Effective annual rate	12.55	Per annum (correct to 2 decimal places)	

Usually the symbol i is used for an effective annual interest rate, and $i^{(m)}$ is used for the nominal interest rate compounded (m) times per annum. When working with nominal interest rates, the compound interest rate formula is altered to accommodate the different time intervals at which interest is calculated:

$A = P\left(1 + \frac{i^{(m)}}{m}\right)^{n \times m}$, where:

A = Accumulated amount, it includes the original amounts plus interest.

P = Principal amount, it is the original amount that is invested or borrowed.

$i^{(m)}$ = nominal interest rate

n = time period in years

m = frequency at which interest is calculated

Worked example 9.7

R9 500 is deposited into a savings account for two years at an interest rate of 14,5% p.a. compounded quarterly.

a) What is the nominal interest rate?
b) What is the quarterly interest rate?
c) Calculate the total amount of money accumulated in the savings account at the end of two years.
d) Calculate the effective annual interest rate.

Solution

Nominal interest rate is $i(4) = 14{,}5\%$ p.a.

Quarterly interest rate is $\frac{0{,}145}{4}$

Total accumulated amount: $A = P\left(1 + \frac{i^{(m)}}{m}\right)^{n \times m} = \text{R}9\ 500\left(1 + \frac{0{,}145}{4}\right)^{2 \times 4} = \text{R}12\ 631{,}06$

Effective annual interest rate: $A = P(1 + i)^n$

Make i the subject of the formula and calculate: $i = \sqrt[n]{\frac{A}{P}} - 1$

$= \left(\sqrt[2]{\frac{12\ 631{,}06}{9\ 500}} - 1\right) \times 100$

$= 15{,}31\,\%$ p.a. (correct to two decimal places)

The example above suggests that we can establish a conversion formula to convert nominal interest rate p.a. to effective annual interest rate and effective annual interest to nominal interest rate:

$$A = P\left(1 + \frac{i^{(m)}}{m}\right)^{n \times m}$$

$$A = P(1 + i)^n$$

$$\therefore\ P(1 + i)^n = A = P\left(1 + \frac{i^{(m)}}{m}\right)^{n \times m}$$

$$\text{hence } 1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

$$i = \left(1 + \frac{i^{(m)}}{m}\right)^m - i$$

Worked example 9.8

a) Convert 17% p.a. compounded quarterly to the effective annual interest rate.
b) Convert an effective annual interest rate of 15.31% to a nominal interest rate compounded monthly.

Solution

a) $i = 17\% = \frac{17}{100} = 0{,}17;\ m = 4$

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

$$1 + i = \left(1 + \frac{i^{(4)}}{4}\right)^4$$

$$i = \left(1 + \frac{0{,}17}{4}\right)^4 - 1$$

$$i = \left(\left(1 + \frac{0{,}17}{4}\right)^4 - 1\right) \times 100$$

Therefore, the effective annual interest rate is 18,11% p.a. correct to two decimal places.

b) $i = 15{,}31\% = \frac{15,31}{100} = 0{,}1531$; $m = 12$, because we are adding interest monthly (that is, 12 times per year)

$$1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m$$

$$1 + 0,1531 = \left(1 + \frac{i^{(12)}}{12}\right)^{12}$$

$$\frac{i(12)}{12} = \sqrt[12]{1,1531} - 1$$

$$\text{rate } i\,(12) = \left(\sqrt[12]{1,1531} - 1\right) \times 12 \times 100$$

Therefore, the nominal interest rate is 14,33% p.a. compounded monthly.

EXERCISE 9.6

1. Convert the following nominal interest rates to effective annual interest rates:
   a) 9% p.a. compounded quarterly
   b) 11% p.a. compounded semi-annually
   c) 9,5% p.a. compounded monthly
   d) 12,3% p.a. compounded daily
   e) 11% p.a. compounded yearly
2. a) Find i if $i^{(12)} = 0{,}086$.
   b) Find i if $i^{(2)} = 0{,}086$
   c) Convert an effective interest rate of 7,5% p.a. to a nominal interest rate per annum compounding daily.

Worked example 9.9

Change a nominal rate of 14% per annum compounded weekly to an equivalent effective monthly rate.

Solution

$$\left(1 + \frac{i^{(12)}}{12}\right)^{12} = \left(1 + \frac{i^{(52)}}{52}\right)^{52}$$

$$\therefore \frac{i^{(12)}}{12} = \left(1 + \frac{0,14}{52}\right)^{\frac{52}{12}} - 1$$

$$\therefore \text{rate} = \left(\left(1 + \frac{0,14}{52}\right)^{\frac{52}{12}} - 1\right) \times 100$$

Therefore, the effective rate is 1,7% per month compounded monthly.

Worked example 9.10

Change a nominal monthly rate of 16% p.a. compounded monthly to anequivalent effective semi-annual rate.

Solution

$$\left(1 + \frac{i^{(2)}}{2}\right)^2 = \left(1 + \frac{i^{(12)}}{12}\right)^{12}$$

$$\therefore \ \frac{i^{(2)}}{2} = \left(1 + \frac{0,16}{12}\right)^{\frac{12}{2}} - 1$$

$$\therefore \ \text{rate} = \left(\left(1 + \frac{0,16}{12}\right)^6 - 1\right) \times 100$$

Therefore, the effective rate is 8,27% per month compounded semi-annually.

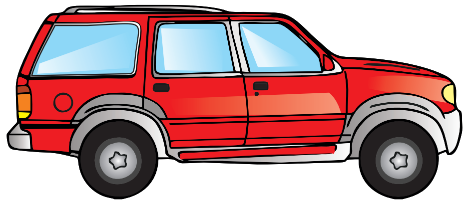
EXERCISE 9.7

1. Thobile has investment opportunities listed below.
   Advise her which interest rate is more favourable over a period of one year.
   a) 10,3% p.a. compounded monthly
   b) 10,4% p.a. compounded quarterly
   c) 10% p.a. compounded daily
   d) 10,2% p.a. compounded weekly
   e) 10,7% p.a. compounded annually
2. Pule invests R12 000 in a savings account. The interest paid is 14,5% p.a. compounded monthly.
   Calculate the value of Pule's investment after:
   a) 5 years
   b) 10 years
   c) 15 years
   d) What do you notice about the relative increase in the investment every five years?
   e) Which investment will be the best, 13% simple interest for two years or 12% p.a. compounded monthly for two years?
3. R4 300 is deposited into a savings account. Calculate how much money is in the savings account at the end of 18 months if the interest is 8,75% p.a. compounded half yearly.
4. For any savings account, which is the better option: 7% p.a. compounded monthly or 7,5% p.a. compounded semi-annually?
5. R1 000 is invested at a rate of 14% p.a. compounded quarterly for three years.
   Determine its value after three years.
6. It takes 12 years for R4 500 to accumulate to R25 073.
   Find the effective annual rate.
7. Pearl deposits a lump sum into an account giving 1,25% compound interest per month.
   If she can withdraw R15 565 after 7 years, what was the original amount?

8. R3 500 is invested at 14,4% p.a. compounded quarterly. After 6 months, R1 000 is added to the investment, and the amount is reinvested at 16% p.a. compounded monthly.

   Find the accumulated amount after five years.

9. Tintswalo buys a car for R289 900. She also invests R50 000 in a savings account that will yield interest at 13,4% per annum compounded monthly. The rate of depreciation on Tintswalo's car is 10% per annum compounded and the rate of inflation is 12% per annum. Given that Tintswalo trades in her car after 5 years and buys a new one, also using her savings, how much extra cash will she need?



9.4 Timelines

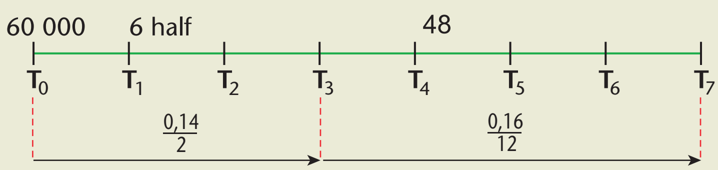
Timelines are used to visualise multiple deposits or withdrawals and also multiple changes in interest rates. In other words, timelines are suitable to use when dealing with more complicated problems that involves interest rate changes, or when several deposits are made into a savings account, or withdrawals are made from the account.

Worked example 9.11

Mohau deposited R60 000 into a savings account. The financial institution offers interest rate of 14% per annum compounded semi-annually for the first three years. Thereafter, the interest rate changes to 16% per annum compounded monthly. Mohau decides to leave the money in the savings account for four more years.

Calculate how much investment money Mohau will have at the end of the seven-year savings period.

Solution



Note:

In this case the time line has been done in years. T4 for example, represents the end of four years. The time line could be drawn up in months, days or quarter-years.

The way to calculate the value of the investment at T7 is to grow R60 000 through two interest rates. First we accumulate the money for the first three years at 14% interest rate, and then we accumulate the amount realised at the end of the third year for the next four years at 16% interest rate.

$$\text{At } T_3\text{: } A_3 = 60\ 000\left(1 + \frac{0,14}{2}\right)^{3 \times 2}$$

$$\text{At } T_7\text{: } A_7 = A_1\left(1 + \frac{0,16}{12}\right)^{4 \times 12}$$

Consider the investment over the seven years:

$$A = 60\ 000\left(1 + \frac{0,14}{2}\right)^{6}\left(1 + \frac{0,16}{12}\right)^{48}$$

$$\therefore A = \text{R}170\ 045,72$$

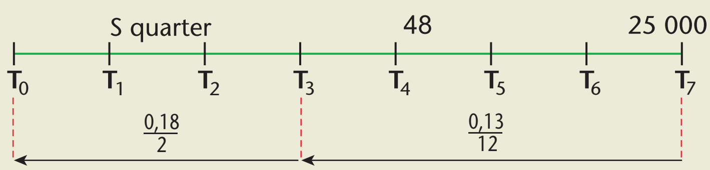
Therefore, the value of the investment after 7 years is R170 045,72.

Note:
For a more accurate answer, it is highly recommended that you do all calculations in a single step on your calculator. In other words, enter your computations on your calculator and press the equal sign once to obtain an answer.

Worked example 9.12

Pheello invests a certain amount of money for six years at 18% per annum compounded quarterly for the first three years and 13% per annum compounded monthly for the remaining term. The future value of the investment at the end of the 6-year period is R25 000. How much did Pheello originally invest?

Solution



$$25\ 000 = P_3\left(1 + \frac{0,13}{12}\right)^{4 \times 12}$$

$$P_3 = \frac{25\ 000}{\left(1 + \frac{0,13}{12}\right)^{48}} = 25\ 000\left(1 + \frac{0,13}{12}\right)^{-48}$$

$$P_3 = P_0\left(1 + \frac{0,18}{4}\right)^{2 \times 4}$$

$$P_0 = \frac{P3}{\left(1 + \frac{0,18}{4}\right)^{8}} = P_3\left(1 + \frac{0,18}{4}\right)^{-8}$$

$$P = 25\ 000\left(1 + \frac{0,13}{12}\right)^{-48}\left(1 + \frac{0,18}{4}\right)^{-8}$$

$$\therefore P = 10\ 480,72$$

Therefore, the amount that was originally invested is R10 480,72.

EXERCISE 9.8

1. Rhirhanzu invest R28 000 for 10 years. The interest she receives is calculated at 9,3% p.a. compounded monthly for the first four years. After four years the interest rate is increased to 11,8% p.a. compounded quarterly. Calculate the value of Rhirhanzu's investment at the end of 10 years.

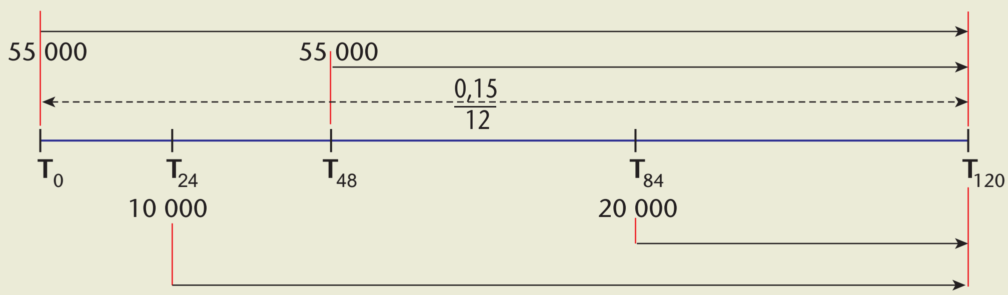
Note:
For a better accurate answer, it is highly recommended that you do all calculations in a single step on your calculator. In other words, enter your computations on your calculator and press the equal sign once to obtain an answer.

2. Thokozile invested a certain amount in the bond market for a period of 20 years. The value of the investment at the end of the 20-year period is R1 927 713,61. Calculate the original amount that Thokozile invested, if the interest was calculated at 9% p.a. for the first 8 years, and 11% p.a. compounded monthly for the remaining 12 years.
3. Keisha borrows R16 000. For the first 3 years, the interest is calculated at 5,5% p.a. compounded monthly. Thereafter, for the next 4 years, interest is calculated at 6,2% p.a. compounded daily. Determine how much Keisha owes after 7 years.
4. Jabulane invests R3 500 in a savings account. The interest rate for the first 4 years is 8% p.a. compounded monthly, thereafter the interest rate is changed to 9% p.a. compounded semi-annually for the next 5 years. Determine the amount of money that Jabulane had in his savings account at the end of this period.
5. A man named Solomon invests R10 800 into an account at U-save Bank at an interest rate of 7,5% p.a. compounded monthly. After 5 years, the bank changes the interest rate to x% p.a. compounded quarterly. Solomon have R21 546,67 in his account 9 years after the original deposit. Find x.

Worked Example 9.13

Zukiswa invests R55 000 now and a further R50 000 in 4 years' time. However, after two years she withdraws R10 000 to cover for sibling's school fees. Seven years later, she withdraws a further R20 000 to pay off her car loan. Interest is constant at 15% p.a. compounded monthly. How much can Zukiswa withdraw after 10 years?

Solution



Note:
The times given on the time line are in terms of months since interest is calculated monthly.

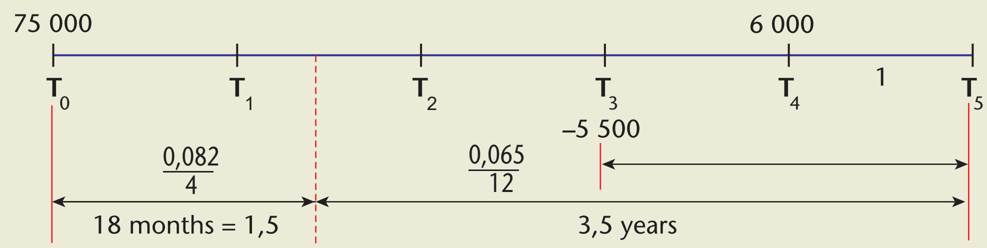
In this case it is suitable to bring all amounts to T120, hence the total amount that Zukiswa withdraws after 10 years is:

$$A = 55\,000\left(1+\frac{0,015}{12}\right)^{120} + 50\,000\left(1+\frac{0,15}{12}\right)^{72} - 10\,000\left(1+\frac{0,15}{12}\right)^{96} - 20\,000\left(1+\frac{0,15}{12}\right)^{3}$$
$$= \text{R}302\,273,73$$

Worked Example 9.14

R75 000 is invested in an account that offers interest at 8,2% p.a. compounded quarterly for the first 18 months. Thereafter the interest rate changes to 6,5% p.a. compounded monthly. Three years after the initial investment, R5 500 is withdrawn from the account. A year later after the withdrawal, R6 000 is deposited into the account. How much will be in the account at the end of 5 years?

Solution



At T 1,5: $75\ 000\left(1+\frac{0,082}{4}\right)^{1,5\times 4}$

At T 5: $75\ 000\left(1+\frac{0,065}{12}\right)^{3,5\times 12}$ $\quad 6\ 000\left(1+\frac{0,065}{12}\right)^{1\times 12} - 5\ 500\left(1+\frac{0,065}{12}\right)^{2\times 12}$

The total calculation:

$$A = 75\ 000\left(1+\frac{0,082}{4}\right)^{6}\left(1+\frac{0,065}{12}\right)^{42} - 5\ 000\left(1+\frac{0,065}{12}\right)^{6} + 6\ 000\left(1+\frac{0,065}{12}\right)^{12}$$

$$= \text{R}106\ 995,35$$

EXERCISE 9.9

1. Tobias deposits R120 000 into a savings account and, two years later, Tobias deposits R6 000 into the same savings account. Calculate the amount of money that Tobias will have in the savings account at the end of 6 years if the interest rate is 8% p.a. compounded monthly.
2. Mazola wants to buy a motorcycle. The cost of the motorcycle is R65 000. In 2003 Mazola opened an account at Perm Society Bank with R18 000. Then in 2007 he added R2 500 more into the account. In 2011 Mazola made another change: she took R3 000 from the account. If the account pays 6% p.a. compounded half-yearly, will Mazola have enough money in the account at the end of 2017 to buy the motorcycle?
3. Thembelihle's lump sum investment matures to an amount of R415 550 after a 25-year period. How much did she invest if her money earned interest at a rate of 14,65% p.a. compounded half yearly for the first 10 years, 9,4% p.a. compounded quarterly for the next six years and 8,6% p.a. compounded monthly for the remaining period?
4. Mr. Sobopha pays a deposit of R35 000 for a new car. He repays the remaining loan by paying a further R24 000 in two years' time and R120 000 six years thereafter. Interest is 18,6% p.a. compounded monthly during the first three years and 31,5% p.a. compounded annually for the remaining five years. What was Mr. Sobopha's original loan amount?
5. R3 500 is invested at x% p.a. compounded quarterly. After 6 months, R1 000 is added to the investment, and the amount is reinvested at 16% p.a. compounded monthly. The accumulated amount after five years is R9 725,60. Find x.

CONSOLIDATION EXERCISE

1. Nomfundo take out a loan of R12 800 and the bank charges her 15% compound interest per year. If Nomfundo does not pay off any of the loan in 5 years, how much would she owe the bank?
2. Sthembiso deposited R8 900 into a bank account paying 7% simple interest per year. How much interest would Sthembiso earn after 5 years?
3. If R60 000 is deposited in a bank paying 12,75% simple interest per annum. How much interest will have been paid after 11 years?
4. R16 900 is deposited in a bank paying 7,75% compound interest per annum. What is the balance after 5 years?
5. A bank account pays 7,3% interest on the first R3 000 and 5,2% on anything above this amount. Keneilwe deposits R12 600. How much interest will Keneilwe earn in a year if interest is calculated on simple interest and on compound interest?
6. Somagaxa makes an investment that has annual compound interest. After 4 years, Somagaxa's amount is R4 000 and after twelve years in total this amount grows to R20 000. Calculate Somagaxa's original investment amount and the annual interest rate.
7. Sechaba buys a Polo Vivo worth R185 000 in 2009. What will the value of Polo Vivo be at the end of 2016 if the car:
   a) depreciates at 6% p.a. straight-line depreciation.
   b) depreciates at 6% p.a. reducing-balance depreciation.
8. Mopadi Projects & Consulting has office equipment valued at R950 000. The value of the equipment depreciates to an amount of R142 500 after 5 years. Calculate the depreciation interest rate if depreciation is calculated on a straight-line basis.
9. Suppose Bloemhof dam contains 75 billion litres of water. Each day, 5% of the volume of water at the start of the day is released.
   a) Will the dam ever become empty if 5% of water is continually released? Explain
   b) Calculate the volume of the water in the dam after 15 days.
   c) The dam project manager decides to change the rate of release of water after 15 days, such that in the following 30 days, half the remaining volume of the water is lost. What percentage of the water is being released each day?
10. Patrice invests R 45 000 in the bank for a period of 30 months. Calculate how much money he will have at the end of the period and the effective annual interest rate if the nominal interest of 9% is compounded:

	Calculation	Accumulated Amount	Effective annual interest rate
Daily			
Weekly			
Monthly			
Quarterly			
Semi-annually			
Yearly			

11. Botle-Buhle Senior Citizen Society has fundraised R40 000. The society decides to invest the money for a period of not more than one year. The business consultant at Trust Bank informs them that the following investment opportunities are available:
    a) 9% p.a. compounded monthly
    b) 9,2% p.a. compounded quarterly
    c) 9,3% p.a. compounded annually
    d) Advise Botle-Buhle Senior Citizen Society which of the three options would provide the best return on their investment over a period of one year.
12. Mrs. Mokoena, a retired professional, invests R145 000 in an account which offers interest at 9% p.a. compounded half-yearly for the first 2 years. Then the interest rate changes to 4%p.a. compounded quarterly. Four years after the initial investment, Mrs. Mokoena withdraws R20 000 to supplement her savings whilst touring Africa. 6 years after the initial investment, Mrs. Mokoena's other investment policy matures, she then decided to take R15 000 from the other policy's maturity value and deposit it into her current investment account. Determine the balance of Mrs. Mokoena's account at the end of 8 years.
13. Mrs. Erasmus invests R6 000 now, R10 000 after 2 years and R50 000 after a further 4 years. However, Mr. Erasmus withdraws R15 000 after three years and the remaining amount after 7 years. If Mr. Erasmus' final withdrawal is R58 602,87, calculate the per annum interest rate compounded quarterly.

Summary

Simple interest:

Interest is calculated on the original amount

Interest remains constant over the period of time

$A = P(1 + in)$

and interest earned

$I = A - P$

A = Accumulated amount, it includes the original amounts plus interest.

P = Principal amount, it is the original amount that is invested or borrowed.

i = interest rate

n = time period in years

Compound interest:

Interest is calculated on the original amount plus the interest added at each time interval

Effective interest rate is calculated per annum

$A = \mathrm{P}(1 + i)^n$

A = Accumulated amount, it includes the original amounts plus interest.

P = Principal amount, it is the original amount that is invested or borrowed.

i = interest rate

n = time period in years

Simple depreciation:

Depreciation or decay occurs based on the simple interest formula

It is also known as "Straight-line" depreciation

$A = P(1 - in)$

Compound depreciation:

Depreciation or decay occurs based on the compound interest formula

Also known as "Reducing-balance" depreciation

$A = P(1 - i)n$

Nominal and effective interest rates:

Compound interest is calculated on the original amount plus the interest added at each time interval

Nominal interest rate is calculated at different time intervals, for example, monthly, daily, or half-yearly – compound interest is calculated per time period

Effective interest rate, compound interest is calculated per annum

The formula to calculate effective interest is as follows:

$i = \left(1 + \frac{i^{(m)}}{m}\right) - 1$

Timelines:

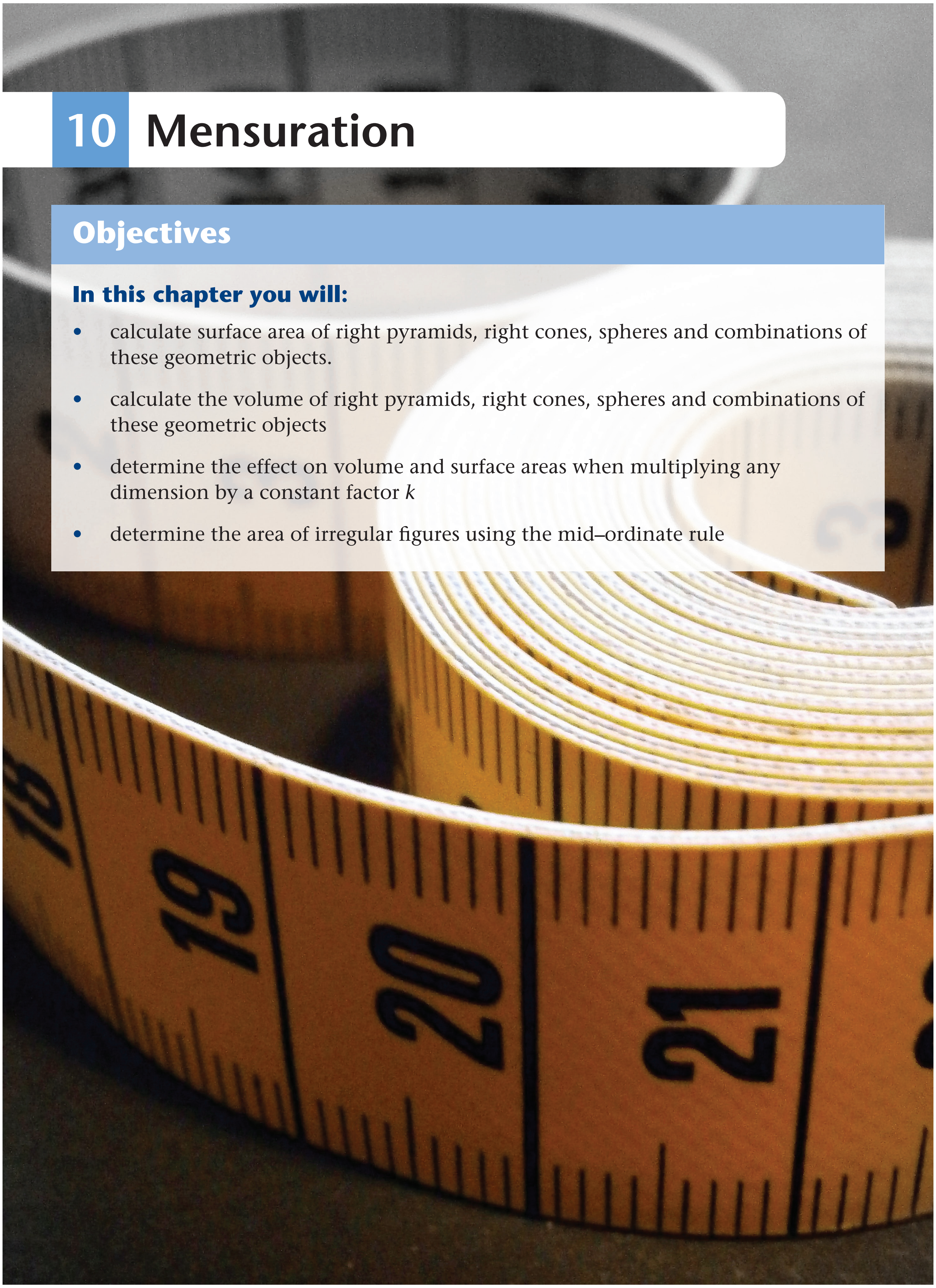
Timelines are used to visualise multiple deposits/withdrawals and also multiple changes in interest rates

10 Mensuration

Objectives

In this chapter you will:

- calculate surface area of right pyramids, right cones, spheres and combinations of these geometric objects.
- calculate the volume of right pyramids, right cones, spheres and combinations of these geometric objects
- determine the effect on volume and surface areas when multiplying any dimension by a constant factor *k*
- determine the area of irregular figures using the mid–ordinate rule



Measurement is the assignment of a number to a characteristic of an object and in which a measuring tool is used.

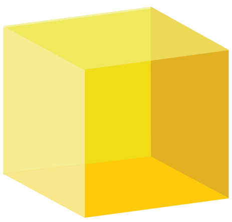
10.1 What is surface area?

The area of a 2-D shape is the amount of surface covered by the shape. A 3-D object is made up of a number of surfaces, thus when we refer to the area of a 3-D object, we refer to the surface area.

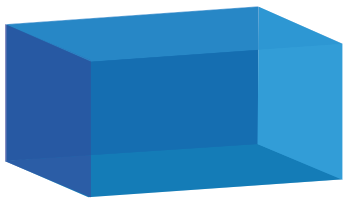
To calculate the surface area of a 3-D object, we need to calculate the total of the areas of all the surfaces of the object. The unit of measure for surface area is a square unit, for example, m^2.

10.2 Surface area of prisms

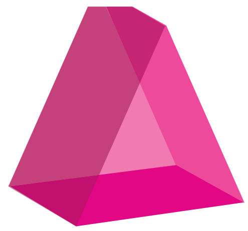
A prism is a type of three-dimensional (3-D) object that has flat sides. It has two ends that are the same shape and size (and look like a 2-D shape). Here are some examples of prisms:



Cube

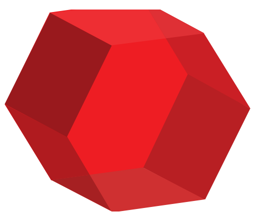


Cuboid (rectangular prism)

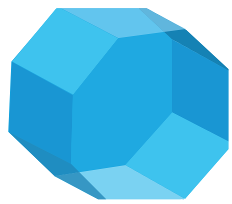


Triangular prism

Pentagonal prism



Hexagonal prism



Octagonal prism

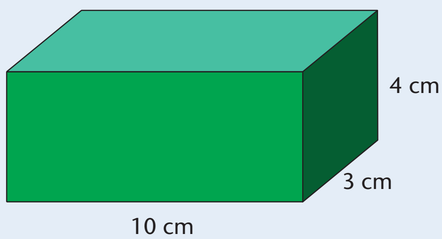
Each prism has the same cross-section all along the shape from end to end. This means that if we cut through the prism, we would see the same 2-D shape on either end. That is why we use a special formula to calculate surface area.

To find the surface area of a prism, use the formula:

Surface area of a prism $= 2 \times$ base area $+$ base perimeter $\times$ length/height of prism

Worked example 10.1

Sandra wants to make a small tissue box. She designed a cuboid as shown below: Determine the amount of material she will need before she opens the top part.

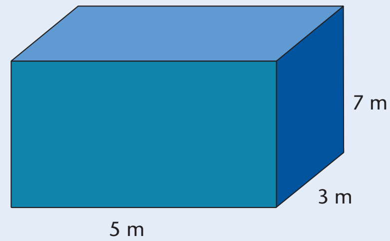


Solution

$$\begin{aligned}\text{Surface area of the prism} &= 2 \times \text{area of the base} + \text{base perimeter} \times \text{height}\\ &= 2 \times (10 \times 3) + (10 + 3 + 10 + 3) \times 4\\ &= 60 + 104\\ &= 164\ \text{cm}^2\end{aligned}$$

Worked example 10.2

Sandra made another tissue box with dimensions 5 m by 3 m by 7 m, and she removed the top face. Determine the surface area of the box.

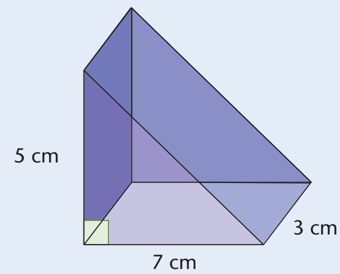


Solution

$$\begin{aligned}\text{Surface area of the prism} &= 2 \times \text{area of the base} + \text{base perimeter} \times \text{height}\\ &= 2 \times (5 \times 3) + (5 + 3 + 5 + 3) \times 7\\ &= 30 + 112\\ &= 127 \text{ cm}^2\end{aligned}$$

Worked example 10.3

Determine the surface area of this prism with a right-angled triangle base. Give your answer correct to the nearest cm^2.



Solution

$$\begin{aligned}\text{Hypotenuse of the triangle} &= \sqrt{7^2 + 5^2}\\ &= \sqrt{74} \text{ cm}\end{aligned}$$

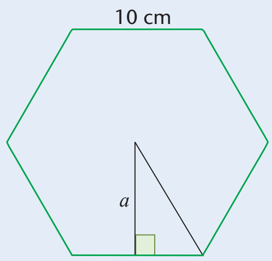
$$\begin{aligned}\text{The surface area of the prism} &= 2 \times \text{area of the triangular face} + \text{base perimeter} \times \text{length of the prism}\\ &= 2 \times \left(\tfrac{1}{2} \times 7 \times 5\right) + (7 + 5 + \sqrt{74}) \times 3\\ &= 35 + 36 + 3\sqrt{74}\\ &= 71 + 3\sqrt{74}\\ &= 96{,}81 \text{ cm}^2\\ &\approx 97 \text{ cm}^2\end{aligned}$$

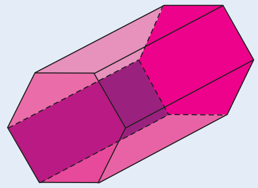
Worked example 10.4

Maggie bought chocolates packaged in a hexagonal container. If each side of the hexagonal face is 10 cm and the length of the container is 12 cm.

a) Calculate the amount of material used to make the container.

b) If 1 m^2 of the material cost R15,50, how much will it cost to make 20 containers?





Base of the prism

Solution

Each interior angle of a hexagon $= 120^\circ$.

To find the length of a, the height of the isosceles triangle, we have: tangent $60^\circ = \frac{a}{5}$;

$a = 5\tan 60^\circ$

$a = 5\sqrt{3}$ cm

Area of base of prism $=$ area of the isosceles triangle $\times$ 6

$= 6\left(\frac{1}{2} \times 10 \div 5\sqrt{3}\right)$

$= 150\sqrt{3}$ cm^2

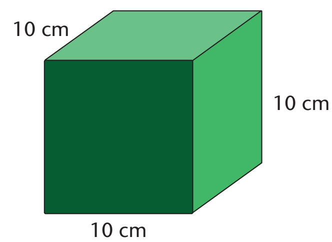
Surface area of the prism $= 2 \times$ area of the base $+$ base perimeter $\times$ height

$= 2 \times (150\sqrt{3}) + (10 \times 6) \times 12$

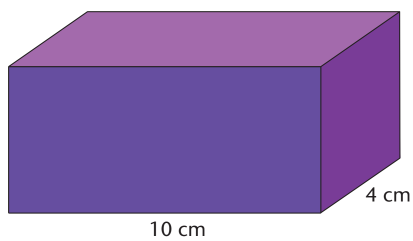
$= 300 + 720$

$= 1\ 020$ cm^2

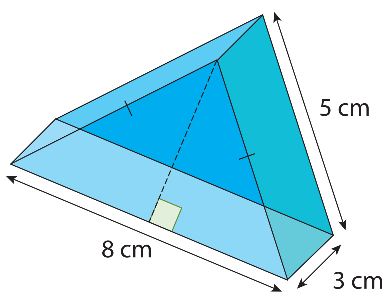
EXERCISE 10.1



1. A money box looks like a cube as shown in the diagram alongside. Determine the amount of metal needed to make the money box.
2. The surface area of this cuboid is 332 cm^2. What is its height?



3. Maria wants to make sandwich containers as shown below:



   a) Determine the amount of material needed to make sandwich container.

4. What is the surface area of a prism where the base area is 25 m^2, the base perimeter is 24 m, and the length is 12 m.

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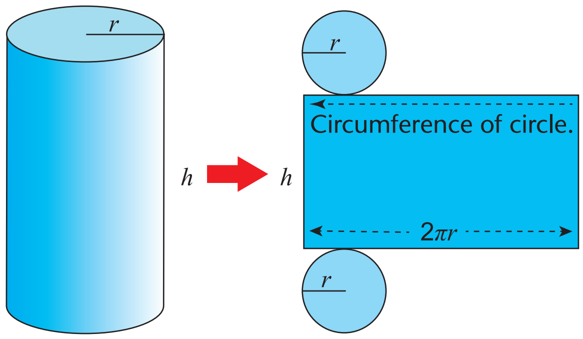
10.3 Surface area of cylinders

A cylinder is made up of two congruent circles and a rectangle which forms the 'wall' of the cylinder.

One of the circles is taken to be the base of the cylinder.



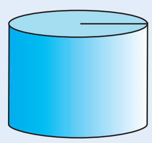
Note:
The surface area of a cylinder = (2 × area of a circle) + (circumference × height)

$= 2 \times \pi r^2 + 2\pi r \times h$

$= 2\pi r\,(r + h)$

Worked example 10.5

Calculate the surface area of the cylinder alongside.
Give your answer in cm².



Solution

Convert mm to cm: 10 mm = 1 cm and 20 mm = 2 cm

Surface area of a cylinder $= 2 \times \pi r^2 + 2\pi r \times h$

$= 2\pi(1\text{ cm})^2 + 2\pi \times 1\text{ cm} \times 2\text{ cm}$

$= 2\pi.1\text{ cm}^2 + 2\pi \times 2\text{ cm}^2$

$= 2\pi + 4\pi\text{ cm}^2$

$= 6\pi\text{ cm}^2$

$= 18{,}84955592\text{ cm}^2$

$\approx 18{,}85\text{ cm}^2$

OR

Surface area of a cylinder

$= 2\pi r(r + h)$

$= 2\pi(1\text{ cm})\,(1\text{ cm} + 2\text{ cm})$

$= 2\pi(1\text{cm})(3\text{cm})$

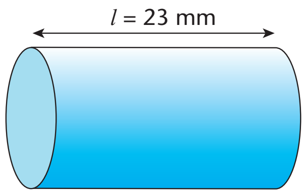
$= 6\pi$

$= 18{,}84955592\text{ cm}^2$

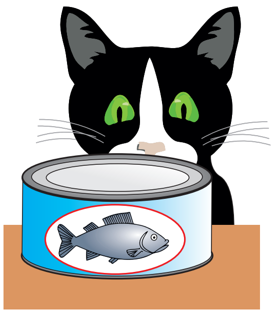
$\approx 18{,}85\text{ cm}^2$

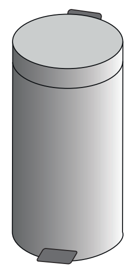
EXERCISE 10.2

1. Calculate the surface area of a sewerage pipe with a length of 23 m and a radius of 50 cm.



2. The diameter of a cool drink can is 6 cm and its height is 120 mm. Calculate the surface area of the can. Give your answer in cm^2.
3. A tin of tuna chunks is 8,5 cm in diameter and is 35 mm high. Calculate the approximate area of the label around the tin in cm^2.





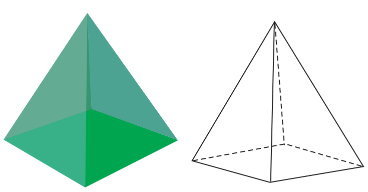
4. A cylindrical dustbin with a lid has a total surface area of 9 161 cm^2.
   The diameter of the dustbin is 360 mm. Calculate the height of the dustbin.

10.4 Surface area of pyramids

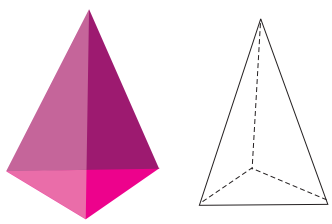
A pyramid is another type of three-dimensional (3-D) object.

It has a polygon base and flat (triangular) faces that join at a common point, called the apex.

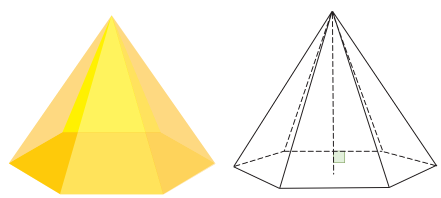
Here are some examples of pyramids:



Square – based pyramid



Triangular – based pyramid

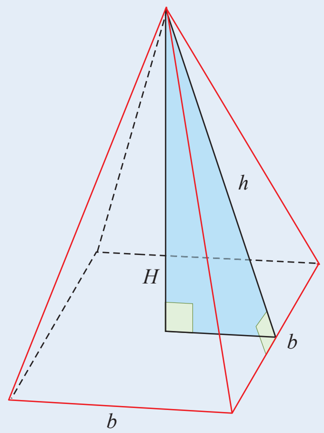


Hexagonal – based pyramid

Note:
The surface area of a pyramid = area of base + area of all the triangular faces

Worked example 10.6

A square-based pyramid has a base with edges of 10 mm and sloping edges of 13 cm. Calculate the surface area of the pyramid.



h = height of triangular face (the slant height of the pyramid)
H = height of pyramid
b = length of base

Area of a triangle $= \frac{1}{2}bh$

Area of a square $= b^2$

Solution

Area of a square $= 10^2 = 100 \text{ cm}^2$

Area of a triangle $= \frac{1}{2} \times 10 \times h$; $h = \sqrt{13^2 - 5^2} = 12$ cm

$= \frac{1}{2} \times 10 \times 12$

$= 60 \text{ cm}^2$

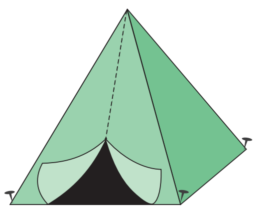
Therefore:

Area of the pyramid = area of base + 4 × area of a triangle

$= 100 + 4 \times 60$

$= 340 \text{ cm}^2$

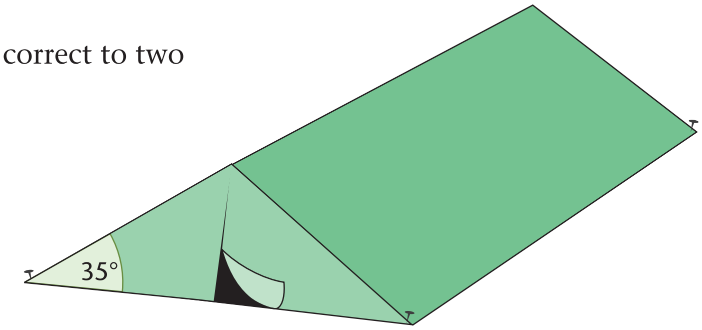
EXERCISE 10.3



1. Find the surface area of a tent in the shape of a square-based pyramid with each edge 3 m long and isosceles triangle faces of side length 4 m.

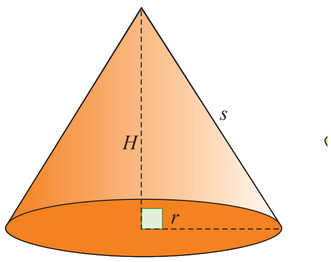
2. The metal roof is made of two rectangular sheets of metal of length 8 m and breadth of 5 m. The roof has a pitch of 35˚ as shown.

   Calculate the total surface area of the roof correct to two decimal places.

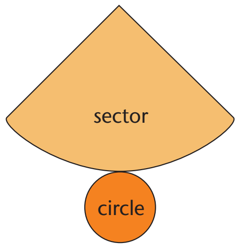


10.5 Surface area of a right cone

Here is an example of a right cone.

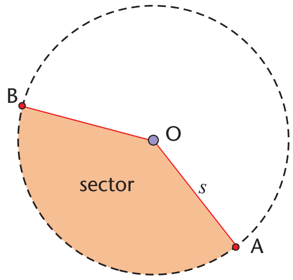


The net of a cone consists of two parts: a circle that gives the base; and a sector that gives the curved surface.



The area of the sector is a fraction of the area of the circle with radius *s*.

To find this fraction we make use of equivalence:



$$\frac{\text{area sector BOA}}{\text{area circle centre O}} = \frac{\text{length arc BA}}{\text{circumference circle centre O}}$$

$$= \frac{2\pi r}{2\pi s}$$

$$= \frac{r}{s}$$

Thus, the area of sector BOA $= \text{area of the circle} \times \frac{r}{s}$

$= \pi s\ 2 \times \frac{r}{s}$

$= \pi rs$

The area of the base of the cone $= \pi r^2$

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Note:

The surface area of a right cone = area of the sector + area of the base

$$= \pi rs + \pi r^2$$
$$= \pi rs + \pi r^2$$
$$= \pi r(s + r)$$

Worked example 10.7

A model of a pine tree will be made in the shape of a right cone and be put onto a cylindrical base.

a) Calculate the surface area of the tree (correct to the nearest square metre) if it has a radius of 1 m and a slant height of 2,4 m. Round your answer off to the nearest whole number.

b) If a 5-litre paint covers 0,1 m^2, about how much paint will be needed to paint the tree?

Solution

a) surface area of the tree $= \pi r(s + r)$

$= 3{,}14(1 \text{ m})(1 + 2{,}4)\text{m}$

$= 10{,}676 \text{ m}^2$

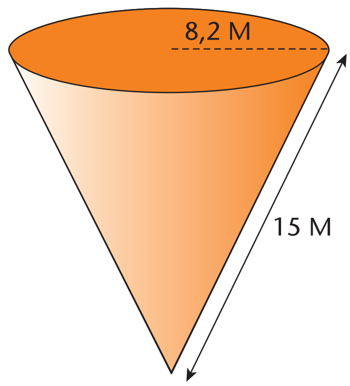
$\approx 11 \text{ m}^2$

b) If 5 litre paint covers 0,1 m^2, for 11 m^2 we will need:

$\frac{11 \times 5}{0{,}1} =$ 550 litre

EXERCISE 10.4

1. Calculate the surface area of a right cone with radius = 8,2 m and slant height = 15 m. Use π on your calculator.

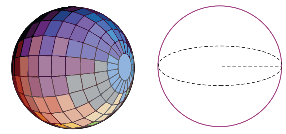


2. Mapule wants to make party hats in the shape of a right cone. The hat has a diameter of 18 cm and a height of 26 cm.
   a) Calculate the amount of cardboard needed to make one party hat.
   b) Determine the area of 200 party hats.
   c) If the cardboard cost R109,90 per m^2, how much will it cost Mapule to buy material for 200 hats?

10.6 Surface area of a sphere

To calculate the surface area of a sphere, we use the following equation:

Surface area of a sphere $= 4\pi r^2$



Worked example 10.8

Calculate the surface area of a hemisphere with diameter 3,5 m (correct to one decimal place). Use $\pi = 3{,}14$.

Solution

The surface area of a hemisphere $= \frac{1}{2} \times 4\pi r^2$

$= 2 \times 3{,}14 \times \left(\frac{3{,}5}{2}\right)^2$

$= 19{,}2325$

$\approx 19{,}2\ \text{cm}^2$

Note:
A hemisphere is half a sphere.

EXERCISE 10.5

The radius of the Earth is 6 378,1 km.

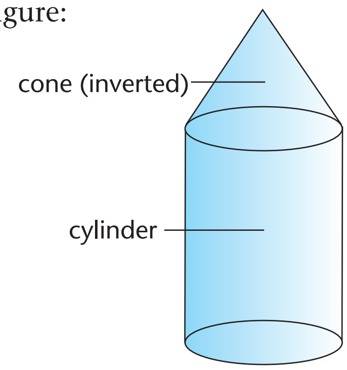
a) What is the surface area of the southern hemisphere? (Assume that the Earth is spherical.)

b) If 71% of the Earth is covered by water, what is the total area of land on the Earth. Give your answer correct to the nearest million square kilometers?

10.7 Surface area of composite figures

Some objects are made up of a combination of different objects such as prisms, pyramids, cylinders, spheres and cones. We call these objects composite figures.

Here is an example of a composite figure:



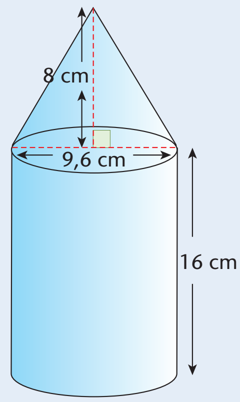
This composite figure is made up of a right cone placed above a right cylinder.

To calculate the surface area of this composite figure, we will need to calculate the surface area of each object separately, before adding the results.

Note:
Remember to exclude the base of the cone and one base of the cylinder when adding your results!

Worked example 10.9

The slant height of the cone in this composite figure is 8 cm.
The height of the cylinder is 16 cm and the diameter of the cylinder is 9,6 cm.
Calculate the surface area of the composite figure.



Solution

Surface area of composite figure
= surface area cylinder with one base + surface area cone with no base

$= \pi r^2 + 2\pi r H + \pi r s$

$= \pi r(r + 2H + s)$

$= \pi(4{,}8\text{ cm})\left[(4{,}8\text{ cm}) + 2(16\text{ cm}) + \left(\frac{4\sqrt{34}}{5}\text{cm}\right)\right]$

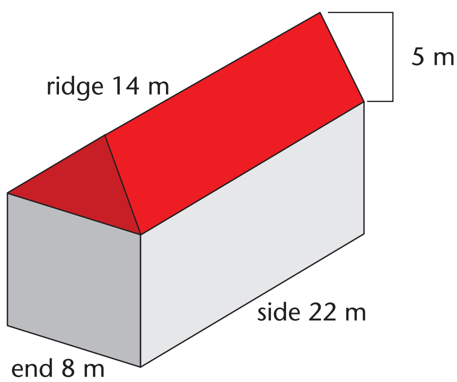
$s = \sqrt{4.8^2 + 8^2}$

$= \frac{4\sqrt{34}}{5}$

$= 695{,}616\ldots\text{cm}^2$

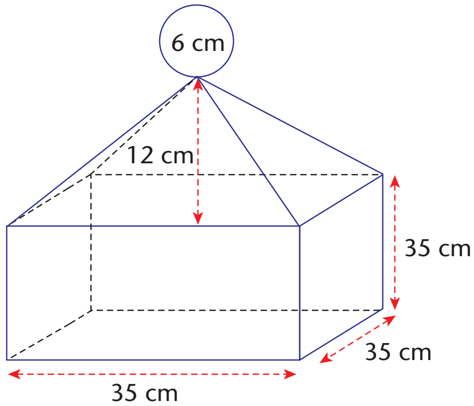
$\approx 695{,}62\text{ cm}^2$

EXERCISE 10.6



1. A house with a rectangular floor-plan measuring 22 m by 8 m, has a roof that has four faces. Each of the two larger faces has the shape of an isosceles trapezium, and they meet in a 14 m long ridge at the top. On the ends of the house there are two faces – each with the shape of an isosceles triangle.
   a) Write down the full calculations to show that the surface area of the roof is 220 m^2.
   b) The roof will be covered in concrete tiles that measure 25 cm by 35 cm.
      Estimate the number of tiles (correct to the nearest 10).
   c) Give two reasons that explain why the answer to b) will be an underestimate.
2. The frame of a lid that will be placed over food to keep it warm is shown below.

The lid and its handle will be made from stainless steel.



a) What three shapes is this lid made up of?

b) Work out the amount of stainless steel needed to cover this frame. Give your answer correct to the nearest square metre.

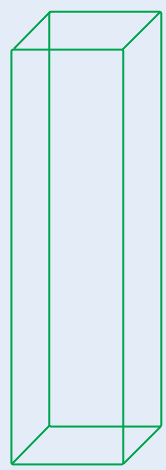
10.8 The effect that changing dimensions will have surface area

When changing the dimensions of an object, there is no simple numeric relationship between the original surface area and the new surface area. However, if all the dimensions are increased by the same factor (k), the new surface area becomes: $k^2 \times$ original area.

Worked example 10.10

The prism alongside has dimensions: $l = 2$ cm, $b = 1$ cm and $h = 6$ cm.
What effect will a change in one of the dimensions have on the surface area of the prism?

Solution



Using the original dimensions:

Surface area of a cuboid $= 2 \times$ area of base + base perimeter × height

$= 2 \times (2 \times 1) + (2 + 1 + 2 + 1) \times 6$

$= 4 + 36$

$= 40 \text{ cm}^2$

If the dimensions are each multiplied by a constant factor of 5:

Length becomes: $5 \times 2 = 10$ cm

Surface area becomes: $= 2 \times (10 \times 1) + (10 + 1 + 10 + 1) \times 6$

$= 20 + 132$

$= 152 \text{ cm}^2$

Original surface area $= 40 \text{ cm}^2$

New surface area $= 152 \text{ cm}^2$

Breadth becomes: $5 \times 1 = 5$ cm

Surface area becomes: $= 2 \times (2 \times 5) + (2 + 5 + 2 + 5) \times 6$

$= 20 + 84$

$= 104\ \text{cm}^2$

Original surface area $= 40\ \text{cm}^2$

New surface area $= 104\ \text{cm}^2$

Height becomes: $5 \times 6 = 30$ cm

Surface area becomes: $= 2 \times (2 \times 1) + (2 + 1 + 2 + 1) \times 30$

$= 4 + 180$

$= 184\ \text{cm}^2$

Original surface area $= 40\ \text{cm}^2$

New surface area $= 184\ \text{cm}^2$

The new surface area is different each time.

EXERCISE 10.7

1. Investigate the effect on the surface area of the following objects if you multiply any one of the given dimensions by the given constant factor.
   a) Tetrahedron: side = 6 mm. Let the constant factor $k = 2$ then let $k = 3$
   b) Right cylinder: $r = 6{,}3$ m, $h = 2{,}7$ m. Let the constant factor $k = 8$
2. Investigate the effect on the surface area of the following objects if you multiply all the given dimensions by the given constant factor.
   a) Right cone: $r = 4$ m, $s = 5$ m. Let the constant factor be $k = 3$
   b) Sphere: $r = 1{,}5$ m. Let the constant factor be $k = 6$
3. Investigate the overall effect on surface area of a cuboid if you multiply the length by 5, the breadth by 2 and the height by 6.
4. Investigate the effect on surface area of any cuboid by multiplying all the dimension by a constant factor k.

10.9 Volume of right prisms

If you picture a right prism as of a lot of 2-D shapes piled one on top of the other, the volume of a right prism is: (area of base) $\times$ h

Note:

Volume of any right prism = (area of base) $\times$ height

The formula for the area of the base depends on the prism.

Worked example 10.11

A water tank is in the shape of a rectangular prism with length 8 m, breadth 5 m and height 4 m. What length of cylindrical irrigation pipe can be filled from the tank if the pipe has a diameter of 10 cm? Use $\pi = 3{,}14$ and round your answer to one decimal place.

Solution

$$\begin{aligned}\text{Volume of the tank} &= \text{area of base} \times \text{height}\\ &= 8 \times 5 \times 4\\ &= 160 \text{ m}^3\end{aligned}$$

$$\begin{aligned}\text{Volume of pipe} &= \text{area of circle} \times \text{length of pipe}\\ &= \pi r^2 h\\ &= 3{,}14(0{,}05)^2 h\end{aligned}$$

The volume of the pipe is the same as the volume of the tank. Therefore:

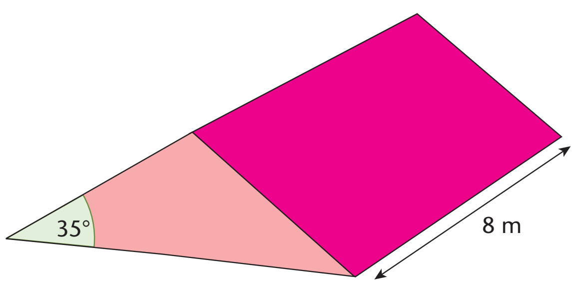
$160 \text{ m}^3 = 3{,}14(0{,}05)^2 h$

$$\begin{aligned}h &= \frac{160}{0{,}00785}\\ &= 20328 \text{ m}\\ &= 20{,}4 \text{ km}\end{aligned}$$

EXERCISE 10.8

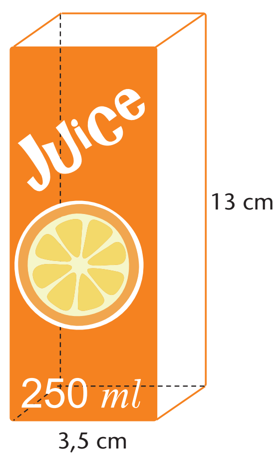
1. The metal roof shown in the diagram is made from two rectangular sheets of metal, of length 8 m and breadth 5 m. The roof has a pitch of 35˚ as shown.

   Calculate the volume of the roof, correct to two decimal places.

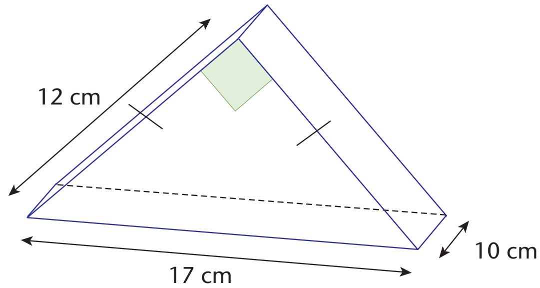


2. A 250 mℓ juice box is 13 cm high and 3,5 cm deep.

   Find its width correct to the nearest cm.



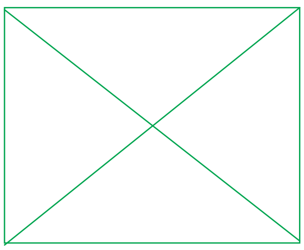
3. A sandwich container is 10 cm deep, 17 cm long and has a slant height of 12 cm.



a) Find the height of the triangle. (Leave your answer in surd form.)

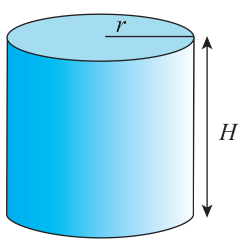
b) How much space is taken up by the container? Give your answer correct to the nearest whole number.

c) The sandwich containers are packed into a box so that four containers form a layer as shown below. Three layers of containers fit perfectly into the box. What are the dimensions of the box?



10.10 Volume of cylinders

To calculate the volume of a cylinder, we need to multiply the area of the base by the height of the cylinder. The base of a cylinder is a circle, thus, to calculate the area of a circle, we use the following equation, Area of circle = πr^2.



Area of base = πr^2

Height of cylinder = H

Volume of a cylinder = $\pi r^2 \times H$

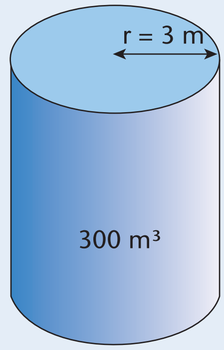
Note:

Volume of a cylinder = (area of base) × height

When using Pi, use the π key on your calculator and round your answers off to two decimal places (unless stated otherwise).

Worked example 10.12

Find the height of a cylindrical tank if its volume is 300 m^3 and its radius is 3 m. Round your answer off to two decimal places.



Solution

Volume of cylinder $= \pi r^2 \times H$

$$300\ \text{m}^3 = \pi(3\ \text{cm})^2 \times H$$

$$\frac{300\ \text{m}^3}{\pi 3^2 \text{m}^2} = H$$

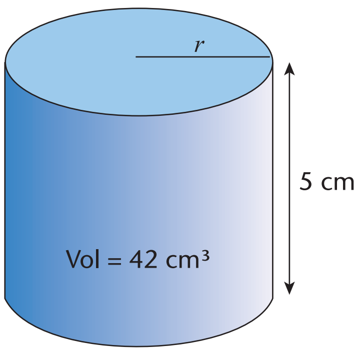
$$H = 10{,}61032954\ \text{m}$$

$$H \approx 10{,}61\ \text{m}$$

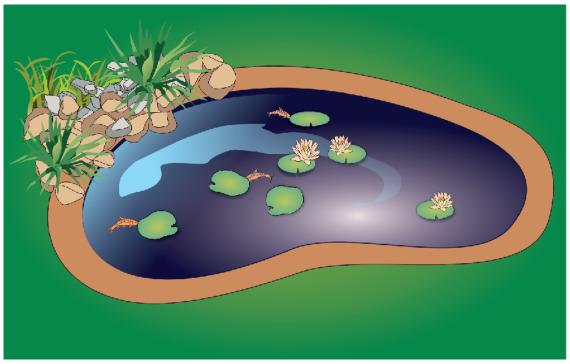
Height of the tank is 10,61 m

EXERCISE 10.9

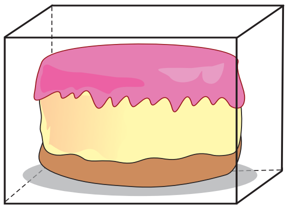
1. Find the values of the missing measurement (correct to 2 decimal places).



2. The surface area of the water in a circular garden pond is 6,2 m^2. It is 0,45 m deep at all points.
   a) What is the volume of the pond?
   b) What is the capacity of the pond?
   c) If the pond is filled at a rate of 30 litres per second, how long will it take to fill the pond?



3. A round chocolate cake has a radius of 8 cm and a height of 9 cm.
   It will be placed in a cardboard cake box so that there is a gap of 2 cm on each side and below the top of the box.
   a) Add suitable measurements to the diagram to show the dimensions of the box.
   b) Calculate the volume of the box.



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10.11 Volume of pyramids

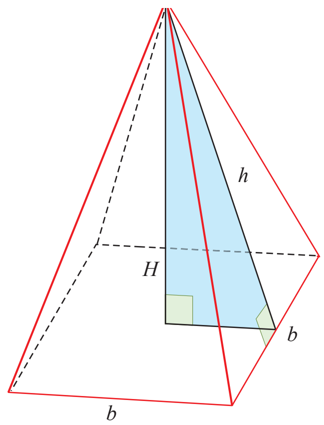
To calculate the volume of a pyramid, we need to multiply $\frac{1}{3}$ of the area of the base by the height of the pyramid. Use the equations below to calculate the volume for:

- a square-based pyramid
- regular tetrahedron.

Note:
Volume of a pyramid $= \frac{1}{3}$ (area of base) $\times$ height

Square-based pyramid

Here is an example of a square-based pyramid.



Note:
Height prism $= H$

Height triangular side $= h$

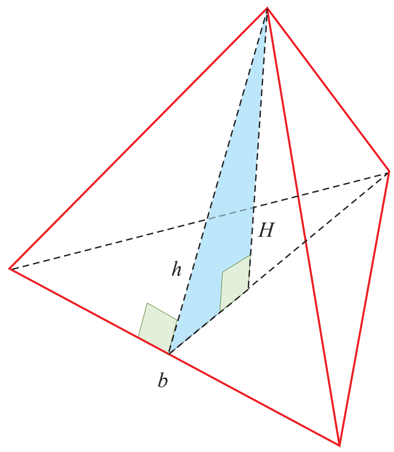
To calculate the volume, first find the area of the base. Thus:

Area of the base $= b^2$

Then calculate the volume using the equation, $\frac{1}{3}(b^2 \times H)$

Regular tetrahedron

Here is an example of a regular tetrahedron.



Note:
In both cases the slant height of the prism = height of triangle $= h$

To calculate the volume, first find the area of the base. Thus:

Area base $= \frac{1}{2} \times b \times h$

Then calculate the volume using the equation, $\frac{1}{3} \times \left(\frac{1}{2} \times b \times h\right) \times H$

$= \frac{1}{6} \times b \times h \times H$

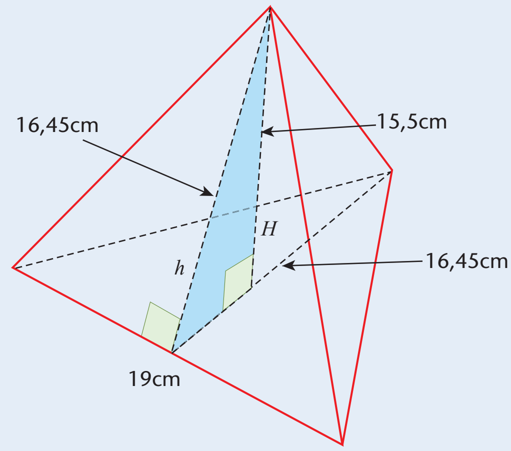
Note:
Height prism $= H = \sqrt{\frac{2}{3}}\,b$

Height triangular side $= h = \frac{\sqrt{3}}{2}b$

Thus $\frac{1}{6} \times b \times h \times H = \frac{\sqrt{2}}{12}b^3$

Worked example 10.13

Find the volume of this regular tetrahedron, where $H = 15{,}5$ cm, $h = 16{,}45$ cm and $b = 19$ cm. Give your answer to the nearest cubic metre.



Solution

Volume regular tetrahedron $= \frac{1}{3} \times \text{area of base} \times H$

$= \frac{1}{3} \times \left(\frac{1}{2} \times 19\text{ cm} \times 16,45\text{ cm}\right) \times H$

$= \frac{1}{3} \times \left(\frac{1}{2} \times 19\text{ cm} \times 16,45\text{ cm}\right) \times 15,5\text{ cm}$

$= 807,420\ldots\text{ cm}^3$

$= 0,8074\ldots\text{ m}^3$

$\approx\ 1\text{ m}^3$

OR

Volume regular tetrahedron

$= \frac{\sqrt{2}}{12} b^3$

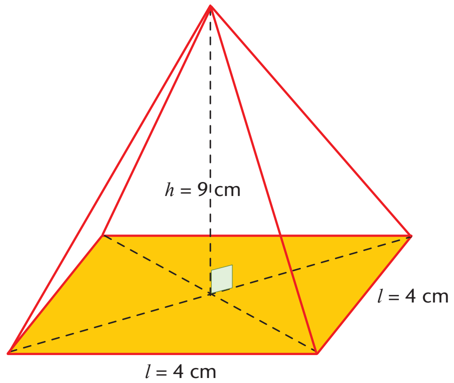
$= \frac{\sqrt{2}}{12}(19)^3$

$= 808,340\text{ cm}^3$

$\approx\ 1\text{ m}^3$

EXERCISE 10.10

1. A pyramid has a square base with sides of 4 cm and a height of 9 cm. Find its volume.



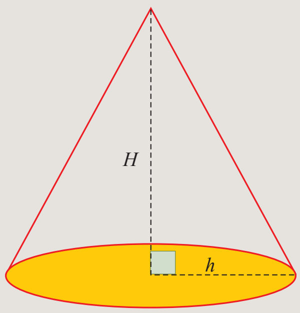
2. Calculate the height of a triangular pyramid that has a volume of 350 m^3 if its base is a right-angled triangle whose hypotenuse is 12 m and another side of the right-angled triangle equals 7 m.

   Give your answer correct to two decimal places.

10.12 Volume of cones

A cone is like a circular pyramid, because it has a circular base. To calculate the volume of a cone, we need to multiply $\frac{1}{3}$ of the area of the base by the height of the cone. Use the equations below to calculate the volume of a cone: Volume of a cone $= \frac{1}{3}$ (area of base) $\times$ height, where the area of the base $= \pi r^2$, and the height of the cone $= H$.

Note:



Volume of a cone $= \frac{1}{3}$ (area of base) $\times$ height

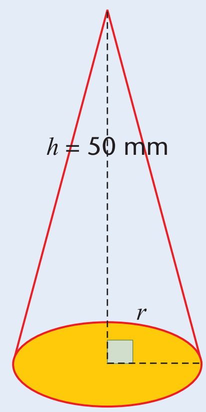
Area of the base $= \pi r^2$

Height of a cone $= H$

Volume of cone $= \frac{1}{3}(\pi r^2 \times H)$

Worked example 10.14

The volume of this cone is 7,54 cm^3 and its height is 50 mm. Find the radius of the cone. Give your answer in cm and correct to one decimal place.



Solution

Volume cone $= \frac{1}{3} \times$ area of base $\times H$

$= \frac{1}{3} \times \pi r^2 \times H$

$7{,}54 \text{ cm}^3 = \frac{1}{3} \times \pi \times r^2 \times 5 \text{ cm}$

$r^2 = \dfrac{3 \times 7{,}54 \text{ cm}^3}{\pi \times 5 \text{ cm}}$

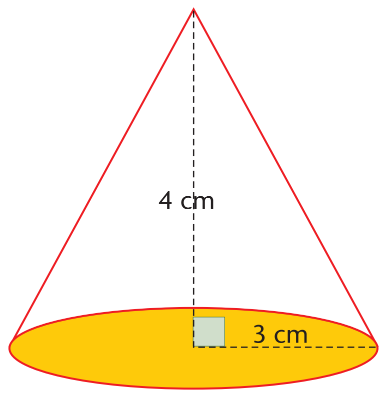
$r = \pm\sqrt{\dfrac{3 \times 7{,}54 \text{ cm}^2}{\pi \times 5}}$

$r = +1{,}200$ cm

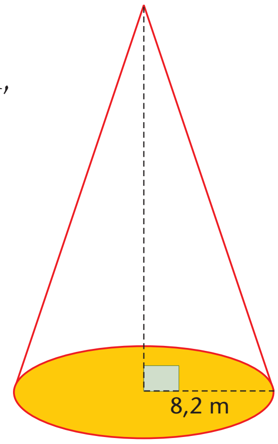
$r \approx 1{,}2$ cm

EXERCISE 10.11

1. Find the volume of the cone alongside:



2. If the capacity of this cone is 1 056,20 kℓ and its radius is 8,2 m, find the height of the cone correct to the nearest metre.



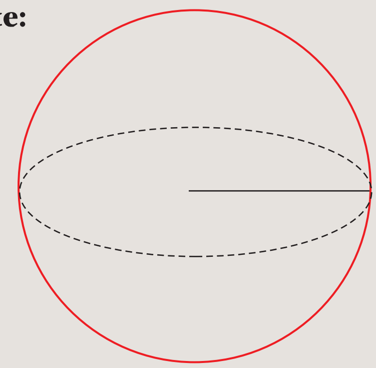
3. A cone of height 15 cm has a volume of 500 cm^3.

   Calculate the radius of the cone, correct to one decimal place.

10.13 Volume of a sphere

A sphere is like a circular object. To calculate the volume of a sphere, we need use the equation: Volume of a sphere $= \frac{4}{3}\pi r^3$.

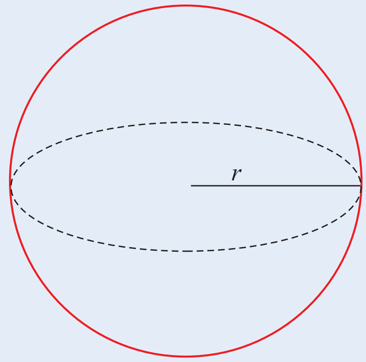
Note:



Volume of a sphere $= \frac{4}{3}\pi r^3$

Worked example 10.15

The volume of this sphere is 524 m^3. Find the radius of the sphere. Give your answer correct to the nearest metre.



Solution

Volume sphere $= \frac{4}{3} \times \pi \times r^3$

$524 \text{ m}^3 = \frac{4}{3} \times \pi \times r^3$

$r\,3 = \frac{3 \times 524 \text{ m}^3}{4 \times \pi}$

$r = \sqrt[3]{\frac{3 \times 524 \text{ m}^3}{4 \times \pi}}$

$r = 5\ 0012\ldots$ mm

$r \approx 5$ m

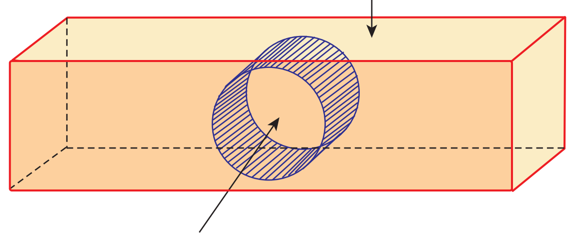
EXERCISE 10.12

1. The volume of a sphere is 4 575 m^3, find its radius.
2. Calculate the volume of a hemisphere with diameter 3,5 m.
3. How many spherical pellets 5 mm in diameter can be moulded from a sphere of radius 100 mm, assuming no material is wasted?
4. A metal sphere of radius 8 mm is to be silver plated to a thickness of 1 mm. What volume of silver is required?

10.14 Volume of composite figures

Some figures are made up of a combination of different objects such as prisms, pyramids, cylinders, spheres and cones. These are called composite figures.

Here is an example of a composite figure made up of a rectangular prism with a cylindrical hollow.



To calculate the volume of this shape you would work with the prism and the cylinder separately and then subtract.

Worked example 10.16

Find the volume of cement that would be needed to make a rectangular block with dimensions of 5 m × 5 m × 0,25 m, with a cylindrical hollow that has a radius of 5 m in the middle. Give your answer correct to the nearest cubic metre.

Solution

Volume rectangular prism $= l \times b \times H$

$= 5 \text{ m} \times 5 \text{ m} \times 25 \text{ m}$

$= 625 \text{ m}^3$

Volume cylinder = area of base × length

$= \pi \times r^2 \times l$

$= \pi \times (5)^2 \times l$

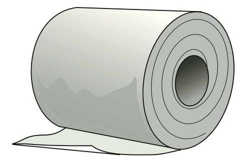
$= \pi (5)^2 \times 5$

$= 125\pi \text{ m}^3$

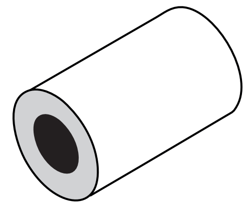
Volume of cement needed to make a rectangular block with a cylindrical hollow

$= 625 - 125\pi = 233{,}009 \approx 232 \text{ m}^3$

EXERCISE 10.13

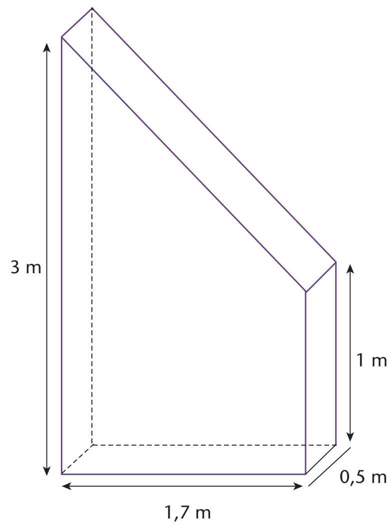


1. An unused roll of toilet paper measures 10 cm high and 10 cm across. The hole in the middle measures 2 cm across.
   a) What shape(s) would make a suitable model to use when calculating the volume of paper on the roll?
   b) Calculate the volume of paper in the roll.



2. A cylindrical concrete pipe has an inner diameter of 26 cm and an outer diameter of 30 cm. Calculate the length of the pipe (correct to two decimal places), if the volume of the concrete is 20 600 cm^3.
3. Maria constructed a room for her son's toys in the shape given below.

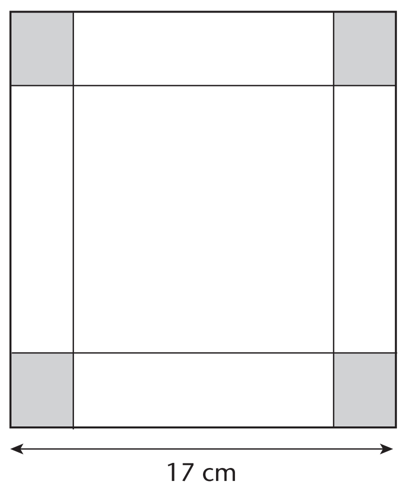
   Using the sketch and dimensions given, calculate the volume of the prism.



4. A square sheet of cardboard has sides of length 17 cm.

   A square of side 4 cm is cut out of each corner.

   The flaps remaining are turned up to form an open box.
   a) Give the depth of the box.
   b) Find the volume of the box.

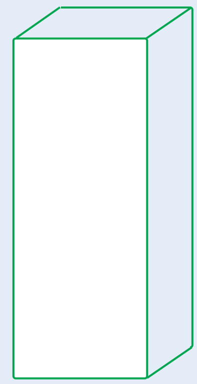


10.15 The effect that changing dimensions will have volume

When increasing any one of the dimensions of a right rectangular prism by a constant factor k, the volume is increased by a factor of k each time.

Let's look at an example to help us understand what this means.

Worked example 10.17



The prism below has dimensions:
$l = 2$ cm, $b = 1$ cm and $H = 6$ cm.

a) Calculate the volume of the prism.
b) Increase each of the following by a factor of 5.
   (i) the length
   (ii) the breadth
   (iii) the height
c) Compare the original volume to each new volume and make a conclusion.

Solution

a) Volume rectangular prism
$= l \times b \times H$
$= 2 \times 1 \times 6$
$= 12 \text{ cm}^3$

b) (i) Increase length by a factor of 5.
$V = l \times b \times H$
$= 10 \times 1 \times 6$
$= 60 \text{ cm}^3$

(ii) Increase breadth by a factor of 5.
$V = l \times b \times H$
$= 2 \times 5 \times 6$
$= 60 \text{ cm}^3$

(iii) Increase height by a factor of 5.
$V = l \times b \times H$
$= 2 \times 1 \times 30$
$= 60 \text{ cm}^3$

c) Original volume $= 12 \text{ cm}^3$
Each new volume $= 60 \text{ cm}^3$
We know $12 \times 5 = 60 \text{ cm}^3$

So, if you multiply any one of the dimensions of this rectangular prism by 5, the volume will also be increased 5 times.

A generalised form of this effect for a right rectangular prism, with dimensions length = l, breadth = b and height = H; states the following:

Suppose you multiply each of the dimensions, one at a time, by a constant factor, k:

- if height = H, then $V = l \times b \times H$, thus, if height = Hk, then
  $V = l \times b \times Hk = k(l \times b \times H)$
- if breadth = b, then $V = l \times b \times H$, thus, if breadth = bk, then
  $V = l \times bk \times H = k(l \times b \times H)$
- if length = l, then $V = l \times b \times H$, thus if length = lk, then
  $V = lk \times b \times H = k(l \times b \times H)$

Note:
In general, if one of the dimensions of a right rectangular prism is increased by a constant factor k, the volume is increased by a factor of k each time.

EXERCISE 10.14

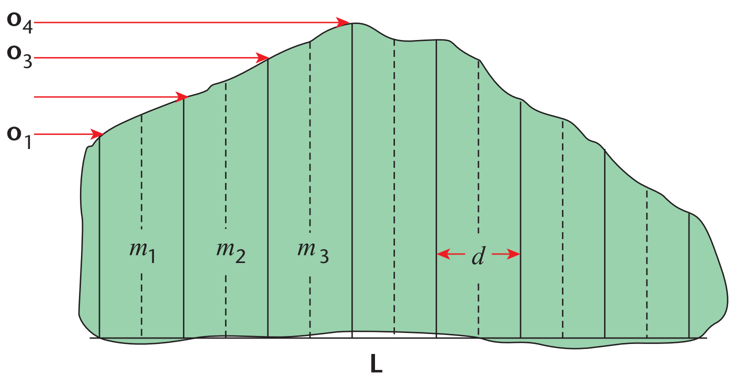
1. Investigate the effect on the volume of the following objects if you multiply any one of the given dimensions by the given constant factor.
   a) Tetrahedron: side = 5 mm. Let the constant factor $k = 2$
   b) Right cylinder: $r = 6{,}4$ m, $H = 2{,}8$ m. Let the constant factor $k = 7$
   c) Right cone: $r = 4$ m, $s = 5$ m. Let the constant factor $k = 3$
   d) Sphere: $r = 1{,}5$ m. Let the constant factor $k = 6$ then let $k = 3$
2. Investigate the overall effect on the volume of a cuboid if you multiply the length by 5, the breadth by 5 and the height by 5.
3. Investigate the overall effect on the volume of a right cylinder if you multiply the radius by 3 and the height by 5.

10.16 Mid-ordinate rule

To calculate areas of irregular shapes, the approximate area is given by the sum of all the ordinates taken at midpoints of each division multiplied by the length of the base line having the ordinates.

In mathematical terms we say: Area of an irregular figure $= a(m^1 + m_2 + m_3 + \ldots m_n)$; $m = \frac{O_1 + O_2}{2}$

So, assume we want to find the area of this shape using the mid-ordinate method. The area has a width of L.



Divide the shape into n equal rectangles. The width of each rectangle will equal $\frac{L}{n} = a$, where L is the length of the baseline and n is the number of rectangles.

Determine the height (m1 to mn) of each rectangle. $m = \frac{O_1 + O_2}{2}$

Find the approximate area using the formula: Area $= a(m_1 + m_2 + m_3 \ldots m_n)$

Worked example 10.18

A surveyor wants to determine the area of a land. He takes perpendicular offsets at 10m interval from a survey line to an irregular boundary line. The ordinates measured at midpoint of the division are 10, 13, 17, 16, 19, 21, 20 and 18 m. Calculate the area enclosed using the mid-ordinate rule.

Solution

Given: Ordinates: $m_1 = 10$ m; $m_2 = 13$ m; $m_3 = 17$ m; $m_4 = 16$ m; $m_5 = 19$ m; $m_6 = 21$ m; $m_7 = 20$ m; $m_8 = 18$ m

Common distance, $a = 10$ m

Area $= 10$ m$(10 + 13 + 17 + 16 + 19 + 21 + 20 + 18)$ m

$= 10$ m $\times$ 134 m

$= 1340$ m^2

Worked example 10.19

Sindi has a small garden in her compound. She wants to find the area of her garden so that she knows how much vegetables she can grow in her space. She put pegs around her garden as shown below: If the pegs mark a side of a rectangle, use the mid-ordinate rule to find the approximate area of Sindi's garden.

$l = 3\ 000$ m; $\quad n = 6$. Therefore: $a = \frac{30 \times 100}{6} = 500$ m;

$m_1 = \frac{1+0}{2} = 0,5$; $m_2 = \frac{2,8+1}{2} = 1,9$; $m_3 = \frac{5+2,8}{2} = 3,9$

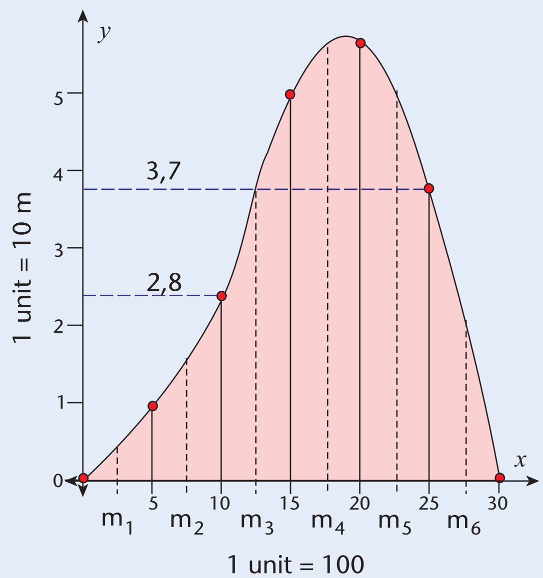
$m_4 = \frac{6+5}{2} = 5,5$; $m_5 = \frac{3,7+6}{2} = 4,85$; $m_6 = \frac{0+3,7}{2} = 1,85$

Approximate Area $= a(m_1 + m_2 + m_3 \ldots m_n)$

$= 500(0,5 + 1,9 + 3,9 + 5,5 + 4,85 + 1,85) \times 10$

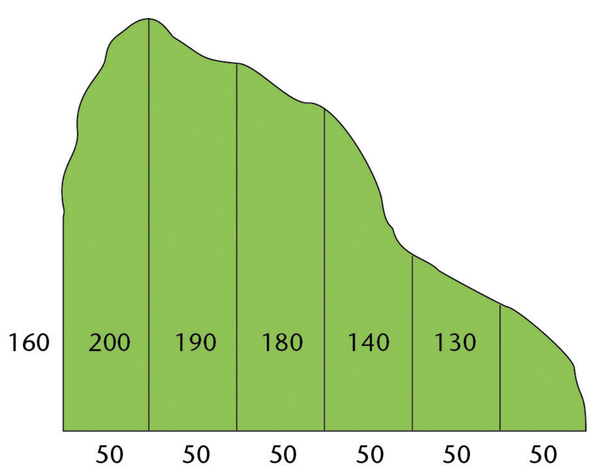
$= 500(185)$

$= 92\ 500\ m^2$

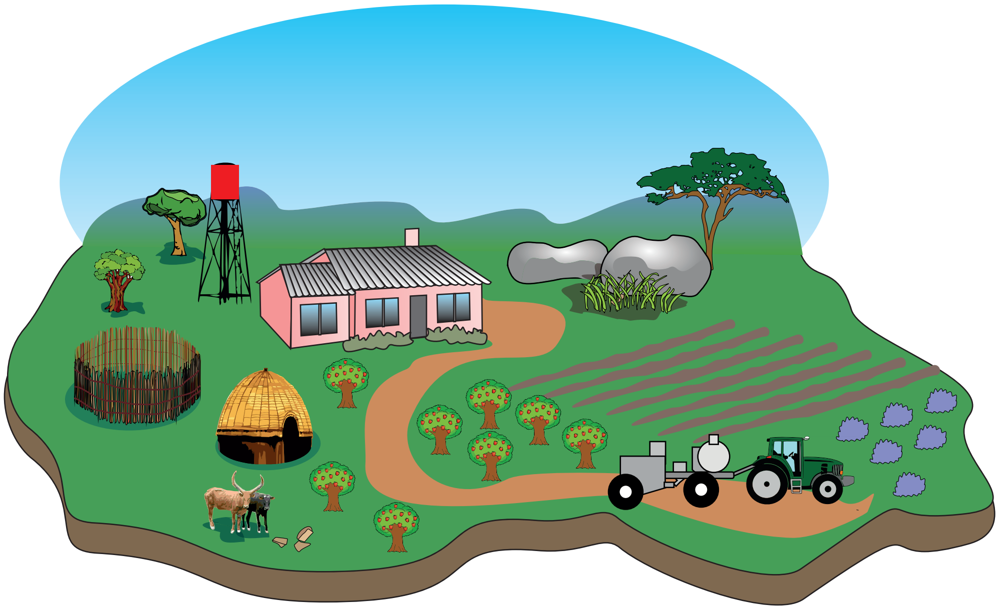


EXERCISE 10.15

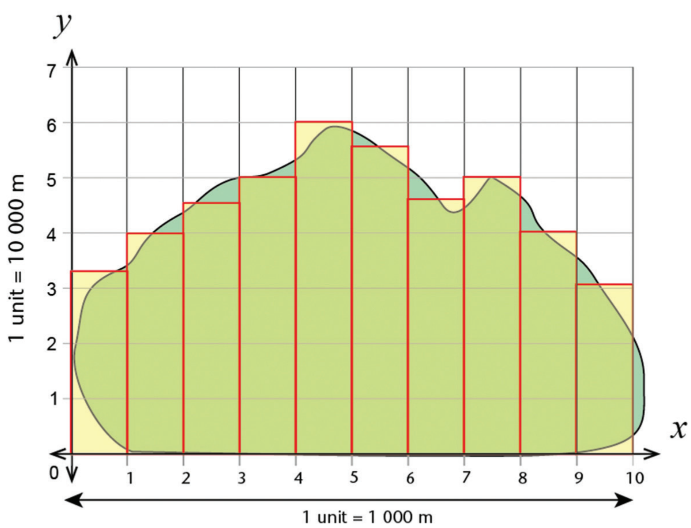
1. The diagram below shows a portion of land to be developed. The surveyor partitions the area by drawing perpendicular lines at an interval of 50 m from the straight boundary line to the irregular boundary line as shown below. Use the mid-ordinate rule to estimate the area of the land in hectares (1 $ha = 10^4\ m^2$).



2. Mr Simelane wants to sell his property (shown in the diagram below). He consulted a land surveyor to help him evaluate his land and buildings.



Use the graph to help Mr Simelane determine the approximate area of his land.



a) Determine the value of a, the distance from one ordinate to another.

b) Find the value of m, the height of each rectangle.

c) Determine the approximate area of mr Simelane's land.

d) If 1 $ha = 10^4$ m^2 is priced at R55 900, how much will Mr Simelane sell his property?

SELECTED ANSWERS

CHAPTER 1 Exponents and surds

Exercise 1.1

a) 2×2^5
$= 2^{1+5}$
$= 2^6$
$= 64$

b) $2^3 \times 2 \times 2^3 = 2^{3+1+3}$
$= 2^7$
$= 128$

c) $2^4 \times 3^4 = 16 + 81$
$= 97$

d) $2^5 + 2^3 = 32 + 8$
$= 40$

e) $\frac{2^{11}}{2^4} = 2^{11-4} = 2^7 = 128$

f) $\frac{2^7}{2^{11}} = 2^{7-11} = 2^{-4} = \frac{1}{2^4} = \frac{1}{16}$

g) $2 \div 2^2 = 2^{1-2} = \frac{1}{2}$

h) $3^{-2}.2.5^{-1} = \frac{2}{9.5} = \frac{2}{45}$

i) $3^{-2}(3^2.2.5) = \frac{9.2.5}{9}$
$= 10$

j) $6 - 2^3 = 6 - 8 = -2$

k) $\frac{10^8}{20^2.25^3.8} = \frac{5^8 \times 2^8}{5^2 \times 2^2 \times 5^5 \times 5^5 \times 8}$
$= 2$

l) $3^{-2}(2.5)^{-1} = \frac{1}{3^2.2.5}$
$= \frac{1}{90}$

m) $(1000^{100} \times 6013^{-1})^0 = 1$

n) $\frac{9 \times 12^5}{4^3 \times 6^4} = \frac{9 \times 6^5 \times 2^5}{4^3 \times 6^4}$
$= 3^3$

o) $3^3 \div 4^{-3} = \frac{3^3}{4^{-3}}$
$= 3^3 \times 4^3$

p) $2^3 \times 2^{-3} = 2^{3-3}$
$= 1$

q) $5 + 5^{-1} = 5 + \frac{1}{5}$
$= 5\frac{1}{5}$

r) $3^3 - 2^2 = 27 - 4$
$= 23$

s) $\frac{4^4 \times 18^3}{9^2 \times 6^7 \times 2^{-1}} = \frac{2^4.2^4 \times 9^3.2^3 \times 2}{9^2 \times 3^7.2^7}$
$= \frac{2^5}{3^5} = \frac{32}{243}$

t) $\frac{45^2.12}{36000} \times \frac{20^4}{30^4}$
$= \frac{9^2 \times 5^2 \times 5^4 \times 4^4}{5^3 \times 2^3 \times 6^2 \times 5^4 \times 2^4 \times 3^4}$
$= \frac{2}{15}$

u) $6^{-2}.36 + \frac{3}{4} \times 2^3$
$= \frac{1}{6^2} \times \frac{6^2}{1} + \frac{3}{2^2} \times \frac{2^3}{1}$
$= 1 + 6$
$= 7$

Exercise 1.2

a) $x^2y \times x^4y^5 = x^{2+4}y^{1+5}$
$= x^6y^6$

b) $x^2y^{-3} \times x^2y^3 = x^{2+2}y^{-3+3}$
$= x^4$

c) $\frac{2}{3}a^{-2} - \left(\frac{3}{a^{-2}}\right)^{-1} = \frac{2}{3a^2} - \frac{1}{3a^2}$
$= \frac{1}{3a^2}$

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d) $2^{3x} \times 8^{y-2x}$

$= 2^{3x} \times 8^{y-2x}$

$= 2^{3x} \times 2^{3y} \times 2^{-6x}$

$= 2^{3y-3x}$

e) $(3m^3)^2 + 3(2m^2)^3$

$= 3^2m^6 + 3.2^3m^6$

$= 9m^6 + 24m^6$

$= 33m^6$

f) $\frac{1}{2}\left(\frac{2^{2h+10}}{16.4^h}\right) = \frac{1.2^{2n}.2^9}{2.16.2^{2n}}$

$= 2^5$

g) $\frac{3^{5x-1}.81^{2x+1}}{3^{12x+3}}$

$= \frac{3^{5x}.3^{8x}.3^4}{3.3^{12x}.3^3}$

$= 3^x$

h) $\frac{16^{2y+2} \times 4^{1-y}}{64^{y+2}}$

$= \frac{16^{2y}.16^2.4}{16^y.16^2.4^{2y}.4^2}$

$= \frac{16^{2y}}{4^{2y}.4}$

$= \frac{1}{4}$

i) $\frac{a^{2x-1}.a^{-6x+1}}{a^{-4x}}$

$= a^{2x-1}.a^{-6x+1}.a^{4x}$

$= a^{2x-6x+4x-1+1}$

$= a^0$

$= 1$

j) $\frac{49a^5b^2c}{14(a^2bc)^2}$

$= \frac{7a^{5-4}b^{2-2}c^{1-2}}{2}$

$= \frac{7a}{2c}$

k) $2^x.10^{2x-1}.25^{1-x}.4^{3x}.8^{-3x}$

$= 2^x2^{2x-1}5^{2x-1}.5^{2-2x}.2^{6x}.2^{-9x}$

$= 2^{9x-9x-1}5^{2x-2x-1+2}$

$= \frac{5}{2}$

l) $\frac{ab \times (2b^2)^3 \times 6a^9b^7}{(2b^6)^2 \times 8(a^5b^{-2})^2}$

$= \frac{ab \times 2^3b^6 \times 6a^9b^7}{4b^{12} \times 8a^{10}b^{-4}}$

$= a^{1+9-10}.b^{1+6+7-8} \times \frac{6}{4 \times 8}$

$= \frac{3}{2}b^6$

Exercise 1.3

a) $2^x = 128$

$2^x = 2^7$

$x = 7$

b) $2^{n-1} = 8$

$2^{n-1} = 2^3$

$n - 1 = 3$

$n = 4$

c) $7^{x-1} = 1$

$7^{x-1} = 7^0$

$x - 1 = 0$

$x = 1$

d) $(b^k)^2 = b^2$

$b^{2k} = b^2$

$2k = 2$

$k = 1$

e) $8\left(\frac{7}{2}\right)^r = 343$

$\frac{2^3.7^r}{2^r} = 7^3$

$\frac{2^3}{7^3} = \frac{2^r}{7^r}$

$\left(\frac{2}{7}\right)^r = \left(\frac{2}{7}\right)^3$

$r = 3$

f) $2.3^x = 18$

$3^x = 9$

$3^x = 3^2$

$x = 2$

g) $2^{n+1} = 64$

$2^{n+1} = 2^6$

$n + 1 = 6$

$n = 5$

h) $\frac{5^{2k-1}}{5} = 25$

$5^{2k-1} = 5^3$

$2k - 1 = 3$

$2k = 4$

$k = 2$

i) $b^{k+2} = b^4$

$k + 2 = 4$

$k = 2$

j) $\frac{64^{2m-1}}{16^{m+1}} = 4$

$\frac{4^{6m-3}}{4^{2m+2}} = 4$

$4^{6m-3-2m-2} = 4$

$4^{4m-5} = 4$

$4m - 5 = 1$

$m = \frac{3}{2}$

k) $2^{\frac{a}{3}} = 8$

$2^{\frac{a}{3}} = 2^3$

$\frac{a}{3} = 3$

$a = 9$

l) $(3^n)^3 = 27$

$3^{3n} = 3^3$

$3n = 3$

$n = 1$

m) $6^x = \frac{1}{216}$

$6^x = 6^{-3}$

$x = -3$

n) $b^k.b^2 = b^4$

$b^{2+k} = 4$

$2 + k = 4$

$k = 2$

o) $2^n.4^{n+1} = 16^n$

$2^n.2^{2n+2} = 2^{4n}$

$.2^{3n+2} = 2^{4n}$

$2^2 = 2^{4n-3n}$

$n = 2$

Exercise 1.4

a) $\sqrt{16+9} = \sqrt{25}$

$= 5$

b) $\sqrt{100-36} = \sqrt{64}$

$= 8$

c) $\sqrt{16} + \sqrt{25} = 4 + 5$

$= 9$

d) $\sqrt{100} - \sqrt{36} = 10 - 6$

$= 4$

Exercise 1.5

1. a) $\sqrt{225} = 225^{\frac{1}{2}} = 15$

b) $\sqrt{289} = 289^{\frac{1}{2}} = 17$

c) $\sqrt{625} = 625^{\frac{1}{2}} = 25$

d) $\sqrt{676} - \sqrt{144}$

$= 26 - 12$

$= 14$

e) $\sqrt{169} + \sqrt{400}$

$= 13 + 20$

$= 33$

2. a) $9^{\frac{1}{2}} = 3^{2\times\frac{1}{2}}$

$= 3$

b) $-16^{\frac{1}{2}} = -(4)^{2\times\frac{1}{2}}$

$= -4$

c) $(-16)^{\frac{1}{2}} = (4i)^{2\times\frac{1}{2}}$

$= 4i$ (imaginary solution)

d) $\left(\frac{1}{9}\right)^{\frac{1}{2}} = \left(\frac{1}{3}\right)^{2\times\frac{1}{2}}$

$= \frac{1}{3}$

e) $\left(\frac{9}{16}\right)^{\frac{1}{2}} = \left(\frac{3}{4}\right)^{2\times\frac{1}{2}}$

$= \frac{3}{4}$

f) $(0.01)^{\frac{1}{2}} = (0,1)^{2\times\frac{1}{2}}$

$= 0,1$

g) $-(25)^{\frac{1}{2}} = -(5^2)^{\frac{1}{2}}$
$= -5$

h) $(0{,}25)^{\frac{1}{2}} = 0{,}5^{2\times\frac{1}{2}}$
$= 0{,}5$

3. a) $9^{\frac{1}{2}} = \sqrt{9}$
$= 3$

b) $-16^{\frac{1}{2}} = -\sqrt{16}$
$= -4$

c) $(-16)^{\frac{1}{2}} = \sqrt{-16}$
$= \sqrt{16i^2}$
$= 4i$ Imaginary solution

d) $\left(\frac{1}{9}\right)^{\frac{1}{2}} = \frac{\sqrt{1}}{\sqrt{9}}$
$= \frac{1}{3}$

e) $\left(\frac{9}{16}\right)^{\frac{1}{2}} = \sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}}$
$= \frac{3}{4}$

f) $(0{,}01)^{\frac{1}{2}} = \left(\frac{1}{100}\right)^{\frac{1}{2}}$
$= \frac{\sqrt{1}}{\sqrt{100}}$
$= \frac{1}{10}$

g) $-(25^{\frac{1}{2}}) = -\sqrt{25}$
$= -5$
$(0{,}25)^{\frac{1}{2}} = \left(\frac{25}{100}\right)^{\frac{1}{2}}$
$= \frac{\sqrt{25}}{100}$
$= \frac{1}{2}$

Exercise 1.6

1. a) $125^{\frac{1}{3}} = 5^{3\times\frac{1}{3}}$
$= 5$

b) $-125^{\frac{1}{3}} = -5^{3\times\frac{1}{3}}$
$= -5$

c) $216^{\frac{1}{3}} = 6^{3\times\frac{1}{3}}$
$= 6$

d) $-216^{\frac{1}{3}} = -6^{3\times\frac{1}{3}}$
$= -6$

e) $729^{\frac{1}{3}} = 9^{3\times\frac{1}{3}}$
$= 9$

f) $-729^{\frac{1}{3}} = -9^{3\times\frac{1}{3}}$
$= -9$

g) $(0{,}001)^{\frac{1}{3}} = 0{,}1^{3\times\frac{1}{3}}$
$= 0{,}1$

h) $(-0{,}001)^{\frac{1}{3}} = -0{,}1^{3\times\frac{1}{3}}$
$= -0{,}1$

i) $\left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{2^{3\times\frac{1}{3}}}{3^{3\times\frac{1}{3}}}$
$= \frac{2}{3}$

2. a) $-27^{\frac{1}{3}} = -3^{3\times\frac{1}{3}}$
$= -3$

b) $\left(\frac{64}{27}\right)^{\frac{1}{3}} = \frac{4^{3\times\frac{1}{3}}}{3^{3\times\frac{1}{3}}}$
$= \frac{4}{3}$

c) $\left(\frac{125}{216}\right)^{\frac{1}{3}} = \frac{5^{3\times\frac{1}{3}}}{6^{3\times\frac{1}{3}}}$
$= \frac{5}{6}$

d) $4096^{\frac{1}{3}} = 16^{3\times\frac{1}{3}}$
$= 16$

e) $\left(\frac{8\times x^3}{343}\right)^{\frac{1}{3}} = \frac{2^{3\times\frac{1}{3}}\times x^{3\times\frac{1}{3}}}{7^{3\times\frac{1}{3}}}$
$= \frac{2x}{7}$

f) $(-512)^{\frac{1}{3}} = -8^{3\times\frac{1}{3}}$
$= -8$

Exercise 1.7

a) $32^{\frac{1}{5}} = (2)^{5\times\frac{1}{5}}$
$= 2$

b) $-32^{\frac{1}{5}} = (-2)^{5\times\frac{1}{5}}$
$= -2$

c) $128^{\frac{1}{7}} = (2)^{7 \times \frac{1}{7}}$
$= 2$

d) $-128^{\frac{1}{7}} = (-2)^{7 \times \frac{1}{7}}$
$= -2$

e) $64^{\frac{1}{2}} = (8)^{2 \times \frac{1}{2}}$
$= 2$

f) $-64^{\frac{1}{2}} = -(8)^{2 \times \frac{1}{2}}$
$= 8i$

g) $16^{\frac{1}{4}} = (2)^{4 \times \frac{1}{4}}$
$= 2$

h) $64^{\frac{1}{6}} = (2)^{6 \times \frac{1}{6}}$
$= 2$

i) $256^{\frac{1}{8}} = (2)^{8 \times \frac{1}{8}}$
$= 2$

Exercise 1.8

1. a) $8^{\frac{2}{3}} = \left(\sqrt[3]{2^3}\right)^2$
$= 2^2$
$= 4$

b) $27^{\frac{2}{3}} = \left(\sqrt[3]{3^3}\right)^2$
$= 3^2$
$= 9$

c) $-64^{\frac{2}{3}} = \left(\sqrt[3]{-4^3}\right)^2$
$= -4^2$
$= 16$

d) $125^{\frac{2}{3}} = \left(\sqrt[3]{5^3}\right)^2$
$= 5^2$
$= 25$

e) $216^{\frac{2}{3}} = \left(\sqrt[3]{6^3}\right)^2$
$= 6^2$
$= 36$

f) $343^{\frac{1}{3}} = \sqrt[3]{7^3}$
$= 7$

g) $512^{\frac{2}{3}} = \left(\sqrt[3]{8^3}\right)^2$
$= 8^2$
$= 64$

h) $1000^{\frac{2}{3}} = \left(\sqrt[3]{10^3}\right)^2$
$= 10^2$
$= 100$

i) $16^{\frac{5}{4}} = \left(\sqrt[4]{2^4}\right)^5$
$= 2^5$
$= 32$

j) $9^{\frac{3}{2}} = \left(\sqrt{3^2}\right)^3$
$= 3^3$
$= 27$

k) $\left(\frac{8}{27}\right)^{\frac{2}{3}} = \left(\sqrt[3]{\frac{2^3}{3^3}}\right)^2 = \left(\frac{2}{3}\right)^2$
$= \frac{4}{9}$

l) $\left(\frac{27}{64}\right)^{\frac{1}{3}} = \sqrt[3]{\frac{3^3}{4^3}}$
$= \frac{3}{4}$

m) $\left(\frac{27}{64}\right)^{-\frac{1}{3}} = \left(\frac{64}{27}\right)^{\frac{1}{3}}$
$= \sqrt[3]{\frac{4^3}{3^3}}$
$= \frac{4}{3}$

n) $\left(\frac{64}{125}\right)^{-\frac{2}{3}} = \left(\frac{125}{64}\right)^{\frac{2}{3}} = \left(\sqrt[3]{\frac{5^3}{4^3}}\right)^2$
$= \left(\frac{5}{4}\right)^2$
$= \frac{25}{16}$

o) $8^{-\frac{2}{3}} = \left(\frac{1}{8}\right)^{\frac{2}{3}}$
$= \left(\sqrt[3]{\frac{1}{2^3}}\right)^2$
$= \left(\frac{1}{2}\right)^2$
$= \frac{1}{4}$

p) $64^{-\frac{2}{3}} = \left(\frac{1}{64}\right)^{\frac{2}{3}}$

$= \left(\sqrt[3]{\frac{1}{4^3}}\right)^2$

$= \left(\frac{1}{4}\right)^2$

$= \frac{1}{16}$

2. a) $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}}$

$= 2^{3 \times \frac{2}{3}}$

$= 2^2$

$= 4$

b) $27^{\frac{2}{3}} = (3^3)^{\frac{2}{3}}$

$= 3^{3 \times \frac{2}{3}}$

$= 3^2$

$= 9$

c) $-64^{\frac{2}{3}} = (-4^3)^{\frac{2}{3}}$

$= -4^{3 \times \frac{2}{3}}$

$= -4^2$

$= 16$

d) $125^{\frac{2}{3}} = (5^3)^{\frac{2}{3}}$

$= (5)^{3 \times \frac{2}{3}}$

$= 5^2$

$= 25$

e) $216^{\frac{2}{3}} = (6^3)^{\frac{2}{3}}$

$= 6^{3 \times \frac{2}{3}}$

$= 6^2$

$= 36$

f) $343^{\frac{1}{3}} = (7^3)^{\frac{1}{3}}$

$= 7$

g) $512^{\frac{2}{3}} = (8^3)^{\frac{2}{3}}$

$= 8^{3 \times \frac{2}{3}}$

$= 8^2$

$= 64$

h) $1000^{\frac{2}{3}} = (10^3)^{\frac{2}{3}}$

$= 10^{3 \times \frac{2}{3}}$

$= 10^2$

$= 100$

i) $16^{\frac{5}{4}} = (2^4)^{\frac{5}{4}}$

$= 2^{4 \times \frac{5}{4}}$

$= 2^5$

$= 32$

j) $9^{\frac{3}{2}} = (3^2)^{\frac{3}{2}}$

$= 3^{2 \times \frac{3}{2}}$

$= 3^3$

$= 27$

k) $\left(\frac{8}{27}\right)^{\frac{2}{3}} = \left(\frac{2}{3}\right)^{3 \times \frac{2}{3}}$

$= \left(\frac{2}{3}\right)^2$

$= \frac{4}{9}$

l) $\left(\frac{27}{64}\right)^{\frac{1}{3}} = \left(\frac{3}{4}\right)^{3 \times \frac{1}{3}}$

$= \frac{2}{3}$

m) $\left(\frac{27}{64}\right)^{-\frac{1}{3}}$

$= \left(\frac{4}{3}\right)^{3 \times \frac{1}{3}}$

$= \frac{4}{3}$

n) $\left(\frac{64}{125}\right)^{-\frac{2}{3}}$

$= \left(\frac{5}{4}\right)^{3 \times \frac{2}{3}}$

$= \left(\frac{5}{3}\right)^2$

$= \frac{25}{16}$

o) $8^{-\frac{2}{3}} = \left(\frac{1}{8}\right)^{\frac{2}{3}}$

$= \left(\frac{1}{2}\right)^{3 \times \frac{2}{3}}$

$= \frac{1}{4}$

p) $64^{-\frac{2}{3}} = \left(\frac{1}{64}\right)^{\frac{2}{3}}$

$= \left(\frac{1}{4}\right)^{3 \times \frac{2}{3}}$

$= \frac{1}{16}$

Exercise 1.9

a) $\sqrt{16x^2} = (16x^2)^{\frac{1}{2}}$
$= (4x)^{2 \times \frac{1}{2}}$
$= 4x$

b) $\sqrt[4]{64x^4y^8z^{12}} = (4^4x^4y^{4\times2}y^{4\times3})^{\frac{1}{4}}$
$= (4xy^2z^3)^{4\times\frac{1}{4}}$
$= 4xy^2z^3$

c) $\sqrt[6]{729m^{12}n^{18}} = (3^6m^{6\times2}n^{6\times3})^{\frac{1}{6}}$
$= (3m^2n^3)^{6\times\frac{1}{6}}$
$= 3m^2n^3$

d) $\sqrt[7]{128a^{14}} = (128a^{14})^{\frac{1}{7}}$
$= (2^7a^{14})^{\frac{1}{7}}$
$= 2a^2$

e) $\sqrt[3]{8x^6} = (8x^6)^{\frac{1}{3}}$
$= (2^3x^6)^{\frac{1}{3}}$
$= 2x^2$

f) $\sqrt[3]{27y^9} = (27y^9)^{\frac{1}{3}}$
$= (3^3y^9)^{\frac{1}{3}}$
$= 3y^3$

Exercise 1.10

1. a) $4x^{\frac{5}{2}} = 128$
$x^{\frac{5}{2}} = 32$
$x^{\frac{5}{2}\times\frac{2}{5}} = 32^{\frac{2}{5}} = 2^2$
$x = 4$

b) $25^{2x} = 5^{x-3}$
$5^{2\times2x} = 5^{x-3}$
$4x = x - 3$
$3x = -3$
$x = -1$

c) $x^{\frac{1}{3}} = 27$
$x^{\frac{1}{3}\times\frac{3}{1}} = 3^{3\times\frac{3}{1}}$
$x = 3^9$
$x = 19683$

d) $125^{x+1} = 5$
$5^{3(x+1)} = 5^1$
$3x + 3 = 1$
$x = -\frac{2}{3}$

e) $3x^{\frac{3}{7}} = 81$
$x^{\frac{3}{7}} = \frac{81}{3}$
$x^{\frac{3}{7}\times\frac{7}{3}} = 3^{3\times\frac{7}{3}}$
$x = 3^7$
$x = 2187$

f) $6^{x-1} = 36^{-x}$
$6^{x-1} = 6^{-2x}$
$x - 1 = -2x$
$3x = 1$
$x = \frac{1}{3}$

g) $3x^{\frac{3}{4}} = 375$
$x^{\frac{3}{4}} = 125$
$x^{\frac{3}{4}\times\frac{4}{3}} = 5^{3\times\frac{4}{3}}$
$x = 5^4$
$x = 625$

h) $7x^{\frac{2}{3}} = 343$
$x^{\frac{2}{3}} = 49$
$x^{\frac{2}{3}\times\frac{3}{2}} = 7^{2\times\frac{3}{2}}$
$x = 7^3$
$x = 343$

i) $4^{3(x-1)} = 4^{2x}$
$3x - 3 = 2x$
$x = 3$

2. $6^5 \times 6^3 = 6^8 = 1679616\ kg = 1679,67\ tonnes$
3. Number of days $= 10^{12} \div 10^5 = 10^{12-5}$
$= 10^7\ days$
4. $2,2 \times 10^{-1} \times 1,2 \times 10^3 = 2,64 \times 10^2 = 264kg$

Exercise 1.11

1.

Number	1	2	3	4	5	6	7	8	9	10
Perfect square of the number	1	4	9	16	25	36	49	64	81	100

2. a) $\sqrt{9} = 3$

   b) $\sqrt{900} = 30$

   c) $\sqrt{16} = 4$

   d) $\sqrt{1600} = 40$

3.

$\times$	2	3	4	5	6	7	8	9	10	100
4	8	12	16	20	24	28	32	36	40	400
16	32	48	64	80	96	112	128	144	160	176
25	50	75	100	125	150	175	200	225	250	275
36	72	108	144	180	216	252	288	324	360	396
49	98	147	196	245	294	343	392	441	490	539
64	128	192	256	320	384	448	512	576	640	704
81	162	243	324	405	486	567	648	729	810	891
100	200	300	400	500	600	700	800	900	1000	10000

4. a) True

   b) True

   c) True

   d) True

   e) True

   f) True

Exercise 11.12

1. a) $\sqrt{392} = \sqrt{196 \times 2}$
   $= 14\sqrt{2}$

   b) $\sqrt{600} = \sqrt{100 \times 6}$
   $= 10\sqrt{6}$

   c) $\sqrt{343} = \sqrt{49 \times 7}$
   $= 7\sqrt{7}$

   d) $\sqrt{810} = \sqrt{81 \times 10}$
   $= 9\sqrt{10}$

   e) $\sqrt{128} = \sqrt{64 \times 2}$
   $= 8\sqrt{2}$

   f) $\sqrt{12} = \sqrt{4 \times 3}$
   $= 2\sqrt{3}$

   g) $\sqrt{98} = \sqrt{49 \times 2}$
   $= 7\sqrt{2}$

   h) $\sqrt{175} = \sqrt{25 \times 7}$
   $= 5\sqrt{7}$

   i) $\sqrt{392} \times \sqrt{8}$
   $= \sqrt{196 \times 2} \times \sqrt{4 \times 2}$
   $= 14\sqrt{2} \times 2\sqrt{2}$
   $= 28 \times 2$
   $= 56$

   j) $3 \times \sqrt{12} = 3 \times \sqrt{4 \times 3}$
   $= 3 \times 2\sqrt{3}$
   $= 6\sqrt{3}$

   k) $\sqrt{3} \times \sqrt{12} = \sqrt{3} \times 2\sqrt{3}$
   $= 2 \times 3$
   $= 6$

   l) $\sqrt{192} = \sqrt{64 \times 3}$
   $= 8\sqrt{3}$

2. a) $\frac{\sqrt{75}}{5} = \frac{\sqrt{25 \times 3}}{5}$

$= \frac{5\sqrt{3}}{5}$

$= \sqrt{3}$

b) $\frac{\sqrt{125}}{\sqrt{5}} = \frac{\sqrt{25 \times 5}}{\sqrt{5}}$

$= \frac{5\sqrt{5}}{\sqrt{5}}$

$= 5$

c) $\frac{\sqrt{192}}{\sqrt{108}} = \frac{\sqrt{64 \times 3}}{\sqrt{36 \times 3}}$

$= \frac{8\sqrt{3}}{6\sqrt{3}}$

$= \frac{4}{3}$

d) $\frac{\sqrt{5}}{\sqrt{125}} = \frac{\sqrt{5}}{\sqrt{25 \times 5}}$

$= \frac{\sqrt{5}}{5\sqrt{5}}$

$= \frac{1}{5}$

e) $\frac{1}{\sqrt{3}} \times \frac{5}{\sqrt{3}} = \frac{1 \times 5}{\sqrt{3} \times \sqrt{3}}$

$= \frac{5}{3}$

f) $\frac{2}{\sqrt{7}} \times \frac{\sqrt{7}}{2} = \frac{2\sqrt{7}}{2\sqrt{7}}$

$= 1$

g) $\frac{2}{\sqrt{8}} \times \frac{\sqrt{8}}{\sqrt{8}}$

$= \frac{2}{\sqrt{2 \times 4}}$

$= \frac{2}{2\sqrt{2}}$

$= \frac{1}{\sqrt{2}}$

h) $\frac{\sqrt{25}}{\sqrt{75}} = \frac{\sqrt{25}}{\sqrt{25 \times 3}}$

$= \frac{5}{5\sqrt{3}}$

$= \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

Exercise 1.13

a) $7\sqrt{5} + 3\sqrt{2} + 8\sqrt{5} - 2\sqrt{2} = 15\sqrt{5} + \sqrt{2}$

b) $3\sqrt{3} - \sqrt{2} - 4\sqrt{3} = -\sqrt{3} - \sqrt{2}$

c) $2\sqrt{3} + 2\sqrt{3} + 3\sqrt{3} + 4\sqrt{3} + 5\sqrt{3} + 6\sqrt{3}$

$= 22\sqrt{3}$

d) $10\sqrt{7} - 3\sqrt{7} - 7\sqrt{7} = 0$

Exercise 1.14

a) $\sqrt{48} - \sqrt{50} + \sqrt{5} - 4\sqrt{3}$

$= \sqrt{16 \times 3} - \sqrt{25 \times 2} + \sqrt{5} - 4\sqrt{3}$

$= 4\sqrt{3} - 5\sqrt{2} + \sqrt{5} - 4\sqrt{3}$

$= -5\sqrt{2} + \sqrt{5}$

b) $\sqrt{18} + \sqrt{12} - 2\sqrt{2} + 7\sqrt{3}$

$= \sqrt{9 \times 2} + \sqrt{4 \times 3} - 2\sqrt{2} + 7\sqrt{3}$

$= 3\sqrt{2} + 2\sqrt{3} - 2\sqrt{2} + 7\sqrt{3}$

$= 9\sqrt{3} + \sqrt{2}$

c) $-5\sqrt{20} + \sqrt{28} - \sqrt{5}$

$= -5\sqrt{4 \times 5} + \sqrt{4 \times 7} - \sqrt{5}$

$= -10\sqrt{5} + 2\sqrt{7} - \sqrt{5}$

$= -11\sqrt{5} + 2\sqrt{7}$

d) $\sqrt{243} + \sqrt{300} + \sqrt{108}$

$= \sqrt{81 \times 3} + \sqrt{100 \times 3} + \sqrt{36 \times 3}$

$= 9\sqrt{3} + 10\sqrt{3} + 6\sqrt{3}$

$= 25\sqrt{3}$

e) $\sqrt{700} + \sqrt{112} - \sqrt{175}$

$= \sqrt{100 \times 7} + \sqrt{16 \times 7} - \sqrt{25 \times 7}$

$= 10\sqrt{7} + 4\sqrt{7} - 5\sqrt{7}$

$= 9\sqrt{7}$

f) $\sqrt{54} + 6\sqrt{216} + \sqrt{6} - 2\sqrt{3}$

$= \sqrt{9 \times 6} + 6\sqrt{36 \times 6} + \sqrt{6} - 2\sqrt{3}$

$= 3\sqrt{6} + 36\sqrt{6} + \sqrt{6} - 2\sqrt{3}$

$= 40\sqrt{6} - 2\sqrt{3}$

g) $5\sqrt{180} + \sqrt{63} - 2\sqrt{112}$

$= 5\sqrt{36 \times 5} + \sqrt{9 \times 7} - 2\sqrt{16 \times 7}$

$= 5 \times 6\sqrt{5} + 3\sqrt{7} - 2 \times 4\sqrt{7}$

$= 30\sqrt{5} - 5\sqrt{7}$

h) $-\sqrt{18} + \sqrt{27} + \sqrt{147} - \sqrt{162}$

$= -\sqrt{9 \times 2} + \sqrt{9 \times 3} + \sqrt{49 \times 3} - \sqrt{81 \times 2}$

$= -3\sqrt{2} + 3\sqrt{3} + 7\sqrt{3} - 9\sqrt{2}$

$= -12\sqrt{2} + 10\sqrt{3}$

i) $\sqrt{96} + \sqrt{180} - \sqrt{24}$

$= \sqrt{16 \times 6} + \sqrt{36 \times 5} - \sqrt{4 \times 6}$

$= 4\sqrt{6} + 6\sqrt{5} - 2\sqrt{6}$

$= 2\sqrt{6} + 6\sqrt{5}$

j) $\sqrt{245} + \sqrt{147} + \sqrt{96}$

$= \sqrt{49 \times 5} + \sqrt{49 \times 3} + \sqrt{16 \times 6}$

$= 7\sqrt{5} + 7\sqrt{3} + 4\sqrt{6}$

Exercise 1.15

a) $(\sqrt{3} + 3)(\sqrt{3} - 3)$

$= 3 - 3\sqrt{3} + 3\sqrt{3} - 9$

$= -6$

b) $(3 + \sqrt{3})(3 - \sqrt{3})$

$= 9 - 3\sqrt{3} + 3\sqrt{3} - 3$

$= 6$

c) $(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})$

$= 3 + \sqrt{15} - \sqrt{15} - 5$

$= -2$

d) $(a - \sqrt{b})(a + \sqrt{b})$

$= a^2 + a\sqrt{b} - a\sqrt{b} - b$

$= a^2 - b$

e) $(\sqrt{7} - x)(\sqrt{7} + x)$

$= 7 + x\sqrt{7} - x\sqrt{7} - x^2$

$= 7 - x^2$

f) $(1 + \sqrt{2})(1 - \sqrt{2})$

$= 1 - \sqrt{2} + \sqrt{2} - 2$

$= -1$

Exercise 1.16

a) $(1 - \sqrt{3})^2$

$= (1 - \sqrt{3})(1 - \sqrt{3})$

$= 1 - 2\sqrt{3} + 3$

$= 4 - 2\sqrt{3}$

b) $(\sqrt{3} - 1)^2$

$= (\sqrt{3} - 1)(\sqrt{3} - 1)$

$= 3 - 2\sqrt{3} + 1$

$= 4 - 2\sqrt{3}$

c) $(2\sqrt{3} + 1)^2$

$= (2\sqrt{3} + 1)(2\sqrt{3} + 1)$

$= 4.3 + 2\sqrt{3} + 2\sqrt{3} + 1$

$= 13 + 4\sqrt{3}$

d) $(\sqrt{3} + \sqrt{2})^2$

$= 3 + 2\sqrt{2}\sqrt{3} + 2$

$= 5 + 2\sqrt{6}$

e) $(-1 - \sqrt{5})^2$

$= 1 + 2\sqrt{5} + 5$

$= 6 + 2\sqrt{5}$

f) $(3 - 3\sqrt{3})^2$

$= 9 - 18\sqrt{3} + 9.3$

$= 36 - 18\sqrt{7}$

Exercise 1.17

a) $\sqrt{3}(5 + \sqrt{3})$

$= 5\sqrt{5} + 3$

b) $\sqrt{2}(\sqrt{2} - 7\sqrt{3})$

$= 2 - 7\sqrt{6}$

c) $-5(1 - \sqrt{125})$

$= -5 + 5.5\sqrt{5}$

$= -5 + 25\sqrt{5}$

d) $\sqrt{6}(\sqrt{2} + 1)$

$= \sqrt{12} + \sqrt{6}$

$= 2\sqrt{3} + \sqrt{6}$

e) $2\sqrt{7}(1-2\sqrt{7})$

$= 2\sqrt{7} - 4.7$

$= 2\sqrt{7} - 28$

f) $9(\sqrt{27}+1)$

$= 9.3\sqrt{3} + 9$

$= 27\sqrt{3} + 9$

Exercise 1.18

a) $\frac{3\sqrt{2}}{\sqrt{6}} = \frac{3\sqrt{2}}{\sqrt{2}\sqrt{3}}$

$= \frac{3}{\sqrt{3}}$

$= \sqrt{3}$

b) $\frac{\sqrt{15}+\sqrt{3}}{\sqrt{3}}$

c) $\frac{\sqrt{27}}{3} = \frac{3\sqrt{3}}{3}$

$= \sqrt{3}$

d) $\frac{(\sqrt{3}-1)(\sqrt{3}+1)}{\sqrt{3}}$

$= \frac{2}{\sqrt{3}}$

$= \frac{2\sqrt{3}}{3}$

e) $\frac{\sqrt{3}}{\sqrt{2}} \div \sqrt{15}$

$= \frac{\sqrt{3}}{\sqrt{2}\sqrt{5}\sqrt{3}}$

$= \frac{1}{\sqrt{10}} = \frac{\sqrt{10}}{10}$

f) $6\frac{\sqrt{7}}{\sqrt{14}}$

$= \frac{6\sqrt{7}}{\sqrt{7}\sqrt{2}}$

$= \frac{6}{\sqrt{2}} = \frac{6\sqrt{2}}{2}$

$= 3\sqrt{2}$

Consolidation exercise

1. a) $81^{\frac{1}{4}} = (3^4)^{\frac{1}{4}}$

$= 3$

b) $81^{\frac{1}{2}} = (9^2)^{\frac{1}{2}}$

$= 9$

c) $81^{\frac{3}{4}} = (3^4)^{\frac{3}{4}}$

$= 3^3$

$= 27$

d) $\left(\frac{5}{6}\right)^3 = \frac{125}{216}$

e) $\left(\frac{125}{216}\right)^{\frac{1}{3}} = \left(\frac{5^3}{6^3}\right)^{\frac{1}{3}}$

$= \frac{5}{6}$

f) $\left(\frac{16}{81}\right)^{\frac{3}{4}}$

$\left(\frac{16}{81}\right)^{\frac{3}{4}} = \left(\frac{2^4}{3^4}\right)^{\frac{3}{4}}$

$= \frac{2^3}{3^3}$

$= \frac{8}{27}$

2. a) $2^{x+1} = 8$

$2^{x+1} = 2^3$

$x + 1 = 3$

$x = 2$

b) $\left(\frac{2}{3}\right)^x = \frac{8}{27}$

$\left(\frac{2}{3}\right)^x = \left(\frac{2}{3}\right)^3$

$x = 3$

c) $3^{3x} = 27$

$3^{3x} = 3^3$

$3x = 3$

$x = 1$

3. a) $\frac{4\sqrt{16}}{8} = \frac{4 \times 4}{8}$

$= 2$

Not a surd

b) $5\sqrt{3}$ a surd

c) $3 + \sqrt[3]{64} = 3 + (4^3)^{\frac{1}{3}}$

$= 3 + 4 = 7$

Not a surd

4. a) $\sqrt{\frac{45}{25}} = \frac{\sqrt{9 \times 5}}{\sqrt{5 \times 5}}$

$= \frac{3\sqrt{5}}{5}$

b) $3\sqrt[4]{16} = 3(2^4)^{\frac{1}{4}}$

$= 3 \times 2$

$= 6$

c) $(2\sqrt{5})^2 = 2^2 \times (5^{\frac{1}{2}})^2$

$= 4 \times 5$

$= 20$

5. a) $45m^2$

Side length $= \sqrt{45m^2}$

$= \sqrt{9 \times 5}\,m$

$= 3\sqrt{5}\,m$

b) $112\ cm^2$

Side length $= \sqrt{112m^2}$

$= \sqrt{16 \times 7}\,m$

$= 4\sqrt{7}\,m$

c) $552\ cm^2$

Side length $= \sqrt{552m^2}$

$= \sqrt{4 \times 138}\,m$

$= 2\sqrt{138}\,m$

6. a) $\sqrt{75}$ and $\sqrt{25}$

$\sqrt{75} = 5\sqrt{3}$ and $\sqrt{25} = 5$

Unlike terms

b) $3\sqrt{72}$ and $\sqrt{378}$

$3\sqrt{72} = 18\sqrt{2}$ and $\sqrt{378} = 3\sqrt{42}$

Unlike terms

c) $-3\sqrt{363}$ and $7\sqrt{847}$

$-3\sqrt{363} = -33\sqrt{3}$

$7\sqrt{847} = 77\sqrt{7}$

Unlike terms

7. a) $\sqrt{7}(1 + 2\sqrt{7})$

$= \sqrt{7} + 2.7$

$= 14 + \sqrt{7}$

b) $\sqrt{15} \times \sqrt{3} = \sqrt{3}\sqrt{5} \times \sqrt{3}$

$= 3\sqrt{5}$

c) $(1 - \sqrt{11})(1 + \sqrt{11})$

$= 1 - 11$

$= -10$

8. a) $\sqrt{2} + 3\sqrt{8} - 4\sqrt{12} + 10\sqrt{3}$

$= \sqrt{2} + 3.2\sqrt{2} - 4.2\sqrt{3} + 10\sqrt{3}$

$= 7\sqrt{2} + 2\sqrt{3}$

b) $1 + \sqrt[3]{125} + \sqrt{363}$

$= 1 + 5 + \sqrt{121 \times 3}$

$= 6 + 11\sqrt{3}$

c) $\sqrt{20} - 2\sqrt{32} + 8\sqrt{18}$

$= \sqrt{4 \times 5} - 2\sqrt{16 \times 2} + 8\sqrt{9 \times 2}$

$= 2\sqrt{5} - 8\sqrt{2} + 24\sqrt{2}$

$= 2\sqrt{5} + 16\sqrt{2}$

9. a) $\dfrac{\sqrt{108}}{3} = \dfrac{\sqrt{36 \times 3}}{3}$

$= \dfrac{6\sqrt{3}}{3}$

$= 2\sqrt{3}$

b) $\dfrac{\sqrt{75}}{5\sqrt{3}} = \dfrac{\sqrt{25 \times 3}}{5\sqrt{3}}$

$= \dfrac{5\sqrt{3}}{5\sqrt{3}}$

$= 1$

c) $\left(\dfrac{64}{81}\right)^{\frac{3}{4}} \times \dfrac{\sqrt{15}}{2\sqrt{5}}$

d) $\left(\dfrac{16}{81}\right)^{\frac{3}{4}} \times \dfrac{\sqrt{15}}{2\sqrt{5}}$

$= \left(\dfrac{2^4}{3^4}\right)^{\frac{3}{4}} \times \dfrac{\sqrt{3}\sqrt{5}}{2\sqrt{5}}$

$= \dfrac{2^3}{3^3} \times \dfrac{\sqrt{3}}{2}$

$= \dfrac{4\sqrt{3}}{27}$

CHAPTER 2 Logarithms

Exercise 2.1

1.

	Number	The exponent/index to which 10 must be raised to give the number	Number written as a power of 10
Example	1	0	10^0
Example	0,1	–1	10^0
a)	0,000001	–6	10^{-6}
b)	0,00001	–5	10^{-5}
c)	0,0001	–4	10^{-4}
d)	0.001	–3	10^{-3}
e)	0,01	–2	10^{-2}
f)	10	1	10^1
g)	100	2	10^2
h)	1 000	3	10^3
i)	10 000	4	10^4
j)	100 000	5	10^5
k)	1 000 000	6	10^6

2. a) $\log_{10} 1000 = 3$
   b) $\log_{10} 1000000 = 6$
   c) $\log_{10} 10 = 1$
   d) $\log_{10} 1 = 0$
   e) $\log_{10} 10000 = 4$
   f) $\log_{10} 0,01 = -2$
   g) $\log_{10} 0,0001 = -4$
   h) $\log_{10} 0,1 = -1$

3. a) 5
   b) 6
   c) 3
   c) 2
   d) 4
   e) 2
   f) $\frac{1}{2}$
   g) 1
   h) 3
   i) 2
   j) 4
   k) 3

Exercise 2.2

Exponential equation	Logarithmic equation
$729 = 9^3$	a) $\log_9 729 = 3$
$512 = 8^3$	b) $\log_8 512 = 3$
c) $1024 = 4^5$	$\log_4 1024 = 5$
d) $216 = 6^3$	$\log_6 216 = 3$
$1 = 9^0$	e) $\log_9 1 = 0$

Exponential equation	Logarithmic equation
f) $1 = 12^0$	$\log_{12} 1 = 0$
$3^5 = 243$	g) $\log_3 243 = 5$
h) $25^2 = 625$	$2 = \log_{25} 625$
$12^2 = 144$	i) $\log_{12} 144 = 2$
j) $17^2 = 289$	$2 = \log_{17} 289$
k) $100^1 = 100$	$1 = \log_{100} 100$

Exercise 2.3

1.

Exponential form	Logarithmic form
$\frac{1}{729} = 9^{-3}$	a) $\log_9 \frac{1}{729} = -3$
$\frac{1}{512} = 8^{-3}$	b) $\log_8 \frac{1}{512} = -3$
c) $\frac{1}{1024} = 4^{-5}$	$\log_4 \frac{1}{1024} = -5$
d) $\frac{1}{216} = 6^{-3}$	$\log_6 \frac{1}{216} = -3$
$\frac{1}{9} = \frac{1}{3^2} = 3^{-2}$	e) $\log_3 \frac{1}{9} = -2$
f) $\frac{1}{12} = 12^{-1}$	$\log_{12} \frac{1}{12} = -1$
$3^{-5} = \frac{1}{243}$	g) log12 1/12 = –1
h) $25^{-2} = \frac{1}{625} = \frac{1}{252}$	$-2 = \log_{25} \frac{1}{625}$
$12^{-2} = \frac{1}{144}$	i) $\log_{12} \frac{1}{144} = -2$
j) $17^{-2} = \frac{1}{289}$	$-2 = \log_{17} \frac{1}{289}$
$0,1 = \frac{1}{10} = 10^{-1}$	k) $\log_{10} 0,001 = -3$
l) $0,001 = \frac{1}{1000} = 10^{-3}$	$\log_{10} 0,001 = -3$
$0,0001 = 10^{-4}$	m) $\log_{10} \frac{1}{1000} = -4$

2. a) $x = 3^{81}$
   b) $128 = 2^x$
   c) $x = 4^4$
   d) $y = 10^2$
   e) $y = 10^2$
   f) $4 = 4^x$

3. a) $x = 3^{81}$
   b) $128 = 2^x$
   $2^7 = 2^x$
   $x = 7$

   c) $x = 4^4$
   $x = 256$

   d) $y = 2^{10}$
   $y = 1024$

   e) $y = 10^2$
   $y = 100$

   f) $4 = 4^x$
   $4^1 = 4^x$
   $x = 1$

4. a) $y = 4^3 = 64$
   b) $625 = 5^x$
   $5^4 = 5^x$
   $x = 4$

   c) $216 = x^3$
   $6^3 = x^3$
   $x = 6$

   d) $a = 13^2 = 169$
   e) $14^x = 196$
   $14^x = 14^2$
   $x = 2$

   f) $1000 = x^3$
   $x = 10$

5. a) $\log_4 y = 3$
   b) $\log_5 625 = x$
   c) $\log_x 216 = 3$
   d) $\log_{13} a = 2$
   e) $\log_{14} 196 = x$
   f) $\log_x 1000 = 3$

6. a) $64 = 4^y$
   $y = 3$

   b) $625 = 5^x$
   $x = 4$

   c) $216 = x^3$
   $x = 6$

   d) $a = 13^2$
   $a = 169$

   e) $196 = 14^x$
   $x = 2$

   f) $1000 = x^3$
   $x = 10$

Exercise 2.4

a) $\log_3(729 \times 9) = \log_3 729 + \log_3 9$
$= \log_3 3^6 + \log_3 3^2$
$= 6\log_3 3 + 2\log_3 3$
$= (6+2)\log_3 3$
$= 8$

b) $\log_5(625 \times 125)$
$= \log_5 625 + \log_5 125$
$= \log_5 5^4 + \log_5 5^3$
$= 4\log_5 5 + 3\log_5 5$
$= (4+3)\log_5 5$
$= 7$

c) $\log_2 8st$

d) $\log_4(64 \times 256)$
$= \log_4 64 + \log_4 256$
$= \log_4 4^3 + \log_4 4^4$
$= 3\log_4 4 + 4\log_4 4$
$= (3+4)\log_4 4$
$= 7$

e) $\log_{10}(6100xyz)$

$= \log_{10} 100 + \log_{10} x + \log_{10} y + \log_{10} z$

$= \log_{10} 10^2 + \log_{10} x + \log_{10} y + \log_{10} z$

$= 2\log_{10} 10 + \log_{10} x + \log_{10} y + \log_{10} z$

$= 2 + \log_{10} x + \log_{10} y + \log_{10} z$

f) $\log_5 25x$

$= \log_5 25 + \log_5 x$

$= \log_5 5^2 + \log_5 x$

$= 2\log_5 5 + \log_5 x$

$= 2 + \log_5 x$

Exercise 2.5

a) $\log_9 \frac{729}{81}$

$= \log_9 729 - \log_9 81$

$= \log_9 9^3 - \log_9 9^2$

$= 3\log_9 9 - 2\log 9$

$= (3-2)\log_9 9$

$= 1$

b) $\log_4 \frac{64}{256}$

$= \log_4 64 - \log_4 256$

$= \log_4 4^3 - \log_4 4^4$

$= 3\log_4 4 - 4\log_4 4$

$= (3-4)\log_4 4$

$= -1$

c) $\log_5 \frac{625}{25}$

$= \log_5 625 - \log_5 25$

$= \log_5 5^4 - \log_5 5^2$

$= 4\log_5 5 - 2\log_5 5$

$= (4-2)\log_5 5$

$= 2$

d) $\log_3 \frac{k}{27}$

$= \log_3 k - \log_3 27$

$= \log_3 k - \log_3 3^3$

$= \log_3 k - 3\log_3 3$

$= \log_3 k - 3$

e) $\log_{10} \frac{1000}{z}$

$= \log_{10} 1000 - \log_{19} z$

$= \log_{10} 1000 - \log_{10} z$

$= \log_{10} 10^3 - \log_{10} z$

$= 3\log_{10} 10 - \log_{10} z$

$= 3 - \log_{10} z$

f) $\log_7 \frac{A}{B}$

$= \log_7 A - \log_7 B$

Exercise 2.6

1. a) $\log_5 125$

$= \log_5 5^4$

$= 4 \times \log_5 5$

$= 4$

b) $\log_3 81$

$= \log_3 3^4$

$= 4$

c) $\log_{10} 10^5$

$= 5$

d) $\log_2 128$

$= \log_2 2^7$

$= 7$

e) $\log_8 512$

$= \log_8 8^3$

$= 3$

f) $\log_9 59049$

$= \log_9 9^5$

$= 5$

2.

x	1	2	3	4	5	6	7	8	9	10
$\log_x$	0	0,301	0,477	0,602	0,699	0.788	0,845	0,903	0,954	1

3. a) $3\log 7 = 3 \times 0,845$
$= 2.535$

b) $6\log 10 = 6 \times 1$
$= 6$

c) $2\log 3 = 2 \times 0.477$
$= 0,954$

d) $3\log 2 = 3 \times 0,301$
$= 0,602$

e) $4\log 5 = 4 \times 0,699$
$= 2,796$

f) $3\log 9 = 3 \times 0,954$
$= 2,862$

4. a) true
b) true
c) false
d) false
e) true
f) true
g) false
h) false

Exercise 2.7

1. a) $\log_5 625^{10}$
$= 10\log_5 5^4$
$= 4 \times 10\log_5 5$
$= 400$

b) $\log_x x^{1000}$
$= 1000\log_x x$
$= 1000$

c) $\log_5 x^{-3}$
$= -3\log_5 x$

d) $\log_x y^z$
$= z\log_x y$

e) $\log_a a^2$
$= 2\log_a a$
$= 2$

f) $\log_4 \left(\frac{1}{4}\right)^k$
$= k\log_4 \frac{1}{4}$
$= k(\log_4 1 - \log_4 4)$
$= k(0 - 1)$
$= -k$

2. a) $\frac{\log 2}{\log 64}$
$= \frac{\log 2}{\log 2^6}$
$= \frac{\log 2}{6\log 2}$
$= \frac{1}{6}$

b) $\frac{\log 32}{\log 128}$
$= \frac{\log 2^5}{\log 2^7}$
$= \frac{5\log 2}{7\log 2}$
$= \frac{5}{7}$

c) $\frac{\log 343}{\log 49}$
$= \frac{\log 7^3}{\log 7^2}$
$= \frac{3\log 7}{2\log 7}$
$= \frac{3}{2}$

d) $\frac{\log 216}{\log 36}$
$= \frac{\log 6^3}{\log 6^2}$
$= \frac{3\log 6}{2\log 6}$
$= \frac{3}{2}$

e) $\frac{\log 625}{\log 25}$

$= \frac{\log 5^4}{\log 5^2}$

$= \frac{4 \log 5}{2 \log 5}$

$= \frac{4}{2}$

$= 2$

f) $\frac{\log 512}{\log 4}$

$= \frac{\log 2^9}{\log 2^2}$

$= \frac{9 \log 2}{2 \log 2}$

$= \frac{9}{2}$

3. a) $\frac{\log a^3}{\log a} = \frac{3 \log a}{\log a}$

$= 3$

b) $\frac{\log a^3}{3a} = \frac{3 \log a}{3a}$

$= \frac{\log a}{a}$

c) $\frac{2 \log xy}{\log x^2 y}$

Is in simplified form

d) $\frac{\log 8abc}{3 \log 2abc}$

e) $\frac{\log 27}{3a} = \frac{\log 3^3}{3a}$

$= \frac{3 \log 3}{3a}$

$= \frac{\log 3}{a}$

f) $\frac{b \log_5 625}{b}$

$= 4$

4. a) $\log a^2 - \log 3 + \log 10$

$= \log \frac{10a^2}{3}$

b) $5 \log x + 3 \log y - \log w + \log 10$

$= \log \frac{10x^5 y^3}{w}$

c) $\frac{1}{2} \log a - \frac{2}{3} \log 27$

$= \log \frac{a^{\frac{1}{2}}}{27^{\frac{2}{3}}}$

$= \log \frac{\sqrt{a}}{9}$

d) $\log 6 + 2 \log 20 - \log 3 - 3 \log 2$

$= \log \frac{(6 \times 20^2)}{(3 \times 2^{3)}}$

$= \log \frac{2400}{24}$

$= \log 100$

$= 2$

d) $\frac{\log 4 + \log 25}{\log 0,001}$

$= \frac{\log(4 \times 25)}{\log \frac{1}{1000}}$

$= \frac{\log 100}{\log \frac{1}{1000}}$

$= \frac{2}{-3} = -\frac{2}{3}$

e) $\log_x 64 + \log_x 4 - \log_x 8$

$= \log_x \frac{(64 \times 4)}{8}$

$= \log_x \frac{256}{8}$

$= \log_x 32$

g) $\frac{\log abc}{2 \log a + \log b + \log c}$

$= \frac{\log abc}{\log a^2 bc}$

h) $\log 2 + \log 3$

$= \log(2 \times 3)$

$= \log 6$

i) $\log a - \log 27$

$= \log \frac{a}{27}$

Exercise 2.8

1. a) $\log_{11} 10 = \frac{\log 10}{\log 11}$

b) $\log_{12} 11 = \frac{\log 11}{\log 12}$

c) $\log_{12} 6 = \frac{\log 6}{\log 12}$

d) $\log_{14} 7 = \dfrac{\log 7}{\log 14}$

e) $\log_{12} 11 = \dfrac{\log 11}{\log 12}$

f) $\log_{12} 6 = \dfrac{\log 6}{\log 12}$

g) $\log_{32} 64 = \dfrac{\log 64}{\log 32}$

h) $\log_9 4 = \dfrac{\log 4}{\log 9}$

$= 0,631$

i) $\log_{25} 125 = \dfrac{\log 125}{\log 25}$

$= 1,5$

2. a) $\log_{11} 10 = \dfrac{\log 10}{\log 11}$

$= 0,960$

b) $\log_{12} 11 = \dfrac{\log 11}{\log 12}$

$= 0,965$

c) $\log_{12} 6 = \dfrac{\log 6}{\log 12}$

$= 0,721$

d) $\log_{14} 7 = \dfrac{\log 7}{\log 14}$

$= 0,737$

e) $\log_{12} 11 = \dfrac{\log 11}{\log 12}$

$= 0,965$

f) $\log_{12} 6 = \dfrac{\log 6}{\log 12}$

$= 0,721$

g) $\log_{32} 64 = \dfrac{\log 64}{\log 32}$

$= 1,2$

h) $\log_9 4 = \dfrac{\log 4}{\log 9}$

$= 0,631$

i) $\log_{25} 125 = \dfrac{\log 125}{\log 25}$

$= 1,5$

Exercise 2,9

a)
$$\begin{aligned} 2^x &= 11 \\ \log 2^x &= \log 11 \\ x\log 2 &= \log 11 \\ x &= \frac{\log 11}{\log 2} \\ x &= 3,46 \end{aligned}$$

b)
$$\begin{aligned} 3^x &= 2 \\ \log 3^x &= \log 2 \\ x\log 3 &= \log 2 \\ x &= \frac{\log 2}{\log 3} \\ x &= 0,63 \end{aligned}$$

c)
$$\begin{aligned} 3^x &= 5 \\ \log 3^x &= \log 5 \\ x\log 3 &= \log 5 \\ x &= \frac{\log 5}{\log 3} \\ x &= 1,46 \end{aligned}$$

d)
$$\begin{aligned} 7^x &= 5 \\ \log 7^x &= \log 5 \\ x\log 7 &= \log 5 \\ x &= \frac{\log 5}{\log 7} \\ x &= 0,83 \end{aligned}$$

e)
$$\begin{aligned} 5^x &= 27 \\ \log 5^x &= \log 27 \\ x\log 5 &= \log 27 \\ x &= \frac{\log 27}{\log 5} \\ x &= 2,05 \end{aligned}$$

f)
$$\begin{aligned} 2^x &= 12 \\ \log 2^x &= \log 12 \\ x\log 2 &= \log 12 \\ x &= \frac{\log 12}{\log 2} \\ x &= 3,58 \end{aligned}$$

g) $10^x = 99$

$\log 10^x = \log 99$

$x \log 10 = \log 99$

$x = \dfrac{\log 99}{\log 10}$

$x = 2$

h) $4^x = 52$

$\log 4^x = \log 52$

$x \log 4 = \log 52$

$x = \dfrac{\log 52}{\log 4}$

$x = 2{,}85$

i) $6^x = 200$

$\log 6^x = \log 200$

$x \log 6 = \log 200$

$x = \dfrac{\log 200}{\log 6}$

$x = 2{,}96$

j) $8^x = 90$

$\log 8^x = \log 90$

$x \log 8 = \log 90$

$x = \dfrac{\log 90}{\log 8}$

$x = 2{,}16$

k) $9^x = 100$

$\log 9^x = \log 100$

$x \log 9 = \log 100$

$x = \dfrac{\log 100}{\log 9}$

$x = 2{,}10$

l) $13^x = 170$

$\log 13^x = \log 170$

$x \log 13 = \log 170$

$x = \dfrac{\log 170}{\log 13}$

$x = 2{,}00$

Exercise 2.10

1. a) $\log_3 27 = x$

$27 = 3^x$

$3^3 = 3^x$

$x = 3$

b) $\log_x 16 = 4$

$16 = x^4$

$2^4 = x^4$

$x = 2$

c) $\log_5 x = 3$

$x = 5^3$

$x = 125$

d) $\log_3 \dfrac{1}{27} = x$

$\dfrac{1}{27} = 3^x$

$3^{-3} = 3^x$

$x = -3$

e) $\log_2 (x+1) = 3$

$x + 1 = 2^3$

$x = -1 + 8$

$x = 7$

f) $\log x = \log(2 - x)$

$x = 2 - x$

$2x = 2$

$x = 1$

g) $2 \log x = 4$

$\log x^2 = 10^4$

$x^2 = (10^2)^2$

$x = 100$

h) $\log_2 3x = \log_2 300$

$3x = 300$

$x = 100$

i) $\log x + 3 = \log 1000$

$\log x + 3 = \log 10^3$

$\log x = 0$

$x = 1$

j) $\frac{1}{2}\log x = \log 10$

$\log x^{\frac{1}{2}} = \log 10$

$x^{\frac{1}{2}} = 10$

$x = 100$

k) $4\log_2 x - 1 = \log_2 8$

$\log_2 x^4 - \log_2 8 = 1$

$\log_2 \frac{x^4}{8} = 1$

$\frac{x^4}{8} = 2$

$x^4 = 16 = 2^4$

$x = 2$

l) $\log_2 \frac{1}{8} = x$

$\frac{1}{8} = 2^x$

$2^{-3} = 2^x$

$x = -3$

2. a) $\log_7 x + \log_7 7$

$= \log_7 14 \log_7 7x = \log_7 14$

$7x = 14$

$x = 2$

b) $\log_3 (x - 8) + \log_3 x = 2 \log_3 x(x - 8) = 2$

$x(x - 8) = 3^2$

$x^2 - 8x - 9 = 0$

$(x + 1)(x - 9) = 0$

$x = -1$ or $x = 8$

No solution because the logarithm of a negative number does not exist. Also the logarithm of 0 does not exist.

c) $\log_3 x - \log_3 3 = 2$

$\log_3 \frac{x}{3} = 2$

$\frac{x}{3} = 3^2$

$x = 27$

d) $\log_2(x-1) = -\log_2 x + 2\log_2(x-1) + \log_2 x = 2$

$\log_2 x(x-2) = 2$

$x(x-1) - 4 = 0$

$x^2 - x - 4 = 0$

$x = 2{,}56..$ or $x = -1{,}56$ The solution is: $x = 2{,}56$

e) $\log x + 1 = 0$

$\log x = -1$

$x = 10^{-1}$

$x = \frac{1}{10}$

f) $\log_3 3 + \log_3 x = 2$

$\log_3 3x = 2$

$3x = 3^2$

$x = 3$

g) $\log_2 2 - 2 = -\log_2 x$

$\log_2 2x = 2$

$2x = 2^2$

$x = \frac{4}{2}$

$x = 2$

h) $\log_4(x) - \log_4(x-1) = \frac{1}{2}$

$\log_4 \frac{x}{x-1} = \frac{1}{2}$

$\frac{x}{x-1} = 4^{\frac{1}{2}} = 2$

$x - 2x = -2$

$x = 2$

i) $\log(x-3)(x-2) = \log(2x+24)$

$\log(x-3) + \log(x-2) = \log(2x+24))$

$(x-3)(x-2) = 2x+24$

$x^2 - 5x + 6 - 2x - 24 = 0$

$x^2 - 7x - 18 = 0$

$(x-9)(x+2) = 0$

$x = 9 ; x = -2$

The solution is $x = 9$

3. a) $\frac{\log_a 25 - \log_a 125}{2[\log_a 5^4 - \log_a 5^6]} = \frac{\log_a \frac{25}{125}}{\log_a \frac{5^8}{5^{12}}} = \frac{\log_a \frac{1}{5}}{\log_a \frac{1}{5^4}}$

$= \frac{\log_a 5^{-1}}{\log_a 5^{-4}} = \frac{-1\log_a 5}{-4\log_a 5} = \frac{1}{4} = \text{RHS}$

b) $\log_9 81 + \log_9 1 + \log_2 16 - \log_{25} 0,04$

$2 + 0 + 4 - 1 \;=\; 5$

$= \text{R.H.S}$

c) $\log_8 \frac{1}{8} + \log_{49} 7^{\frac{1}{2}} - \log_6 \frac{1}{216} - \log_a 1$ (

$= \log_8 8^{-1} + \frac{\log 7^{\frac{1}{2}}}{\log 49} - \log_6 216^{-1} - \log_a a^0$

$= -1\log_8 8 + \frac{\frac{1}{2}\log 7}{2\log 7} + 3\log_6 6 - 0\log_a a$

$= -1 + \frac{1}{4} + 3 - 0$

$= \frac{9}{4}$

$= \text{R.H.S}$

4. a) In 20 minutes $= 1000 \times 2^{0,05t}$

$= 1000 \times 2^{0,05 \times 20}$

$= 2000$ bacteria

b) $7000 \;=\; 1000 \times 2^{0,05t}$

$7 \;=\; 2^{0,05t}$

$\frac{\log 1}{\log 2} \;=\; 0,05t$

$t \;=\; 56,14$ minutes

5. a) After 5 years $= 1000 \times 2^{0,2t}$

$= 1000 \times 2^{0,2\times 5}$

$= R2000$

b) $7000 \;=\; 1000 \times 2^{0,2t}$

$7 \;=\; 2^{0,05t}$

$\frac{\log 1}{\log 2} \;=\; 0,2t$

$t \;=\; 14,1$years

6. $4 \;=\; 10^{0,0087t}$

$4 \;=\; 10^{0,0087t}$

$\frac{\log 4}{\log 10} \;=\; 0,0087t$

$t \;=\; 20,24$ years

Consolidation exercise

1. a) $\log_3 27 + \log_2 64 = \log_3 3^3 + \log_2 2^6$

$= 3\log_3 + 6\log_2 2$

$= 3 + 6$

$= 9$

b) $\frac{\log_3 27}{\log_2 64} = \frac{\log_3 3^3}{\log_2 2^6}$

$= \frac{3\log_3 3}{6\log_2 2}$

$= \frac{3}{6} \;=\; \frac{1}{2}$

c) $\log_3 27 - \log_2 64$

$= \log_3 3^3 - \log_2 2^6$

$= 3\log_3 - 6\log_2 2$

$= 3 - 6$

$= -3$

d) $\log_3 27 \times \log_2 64$

$= \log_3 3^3 \times \log_2 2^6$

$= 3\log_3 3 \times 6\log_2 2$

$= 3 \times 6$

$= 18$

e) $\log_2 32 + \log_4 64 - \log_{10} 100 \times \log_5 125 + \log_2 1$

$= \log_2 2^5 + \log_4 4^3 - \log_{10} 10^2 \times \log_5 5^3 + \log_2 2^0$

$= 5\log_2 2 + 3\log_4 4 - 2\log_{10} 10 \times 3\log_5 5 + 0$

$= 5 + 3 - 2 \times 3$

$= 8 - 6$

$= 2$

2. a) $2^x = 17$

$\log 2^x = \log 17$

$x \log 2 = \log 17$

$x = \dfrac{\log 17}{\log 2}$

$x = 4,09$

b) $7^x = 100$

$\log 7^x = \log 100$

$x \log 7 = \log 100$

$x = \dfrac{\log 100}{\log 7}$

$x = 2,37$

c) $10^x = 300$

$\log 10^x = \log 300$

$x = \dfrac{\log 300}{\log 10}$

$x = 2,48$

3. a) $3^x = 729$

$3^x = 3^6$

$x = 6$

b) $5^{x-1} = 5^3$

$x - 1 = 3$

$x = 4$

c) $4^{x+1} = 256$

$4^{x+1} = 4^4$

$x+1 = 4$

$x = 3$

4. a) $\log_x 32 = 5$

$32 = x^5$

$2^5 = x^5$

$x = 2$

b) $\log_2 x = 5$

$x = 2^5$

$x = 32$

c) $\log_2 32 = x$

$32 = 2^x$

$2^5 = 2^x$

$x = 5$

d) $\log(x+1) = 0$

$x+1 = 10^0$

$x+1 = 1$

$x = 0$

e) $\log(x-1) = 0$

$x-1 = 10^0$

$x-1 = 1$

$x = 2$

f) $\log x + \log(x-21) = \log 100$

$\log x(x-21) = \log 100$

$x^2 - 21x = 100$

$x^2 - 21x - 100 = 0$

$(x-25)(x+4) = 0$

$x = 25 \text{or } x = -4$

Solution: $x = 25$

5. a) $\log_5 6 = \frac{\log 6}{\log 5}$

$= 1{,}11$

b) $\log_3 11 = \frac{\log 11}{\log 3}$

$= 2{,}18$

c) $\log_8 9 = \frac{\log 9}{\log 8}$

$= 1{,}06$

6. a) $\log \frac{xyz^3}{w^2}$

$= \log w + \log y + 3\log z - 2\log w$

b) $\log \frac{\sqrt{ab}}{\sqrt[3]{c}}$

$= \log\sqrt{a} + \log\sqrt{b} - \log\sqrt[3]{c}$

$= \frac{1}{2}\log a + \frac{1}{2}\log b - \frac{1}{3}\log c$

c) $\log 10x$

$= \log 10 + \log x$

$= 1 + \log x$

7. a) $\log(x+1) - 3 + \log(x-1)$

$= \log(x+1)(x-1) - \log 1000$

$= \log\frac{x^2-1}{1000}$

b) $\log x + 1$

$= \log x + \log 10$

$= \log 10x$

c) $2 - \log_5 3x$

$= \log_5 25 - \log_5 3x$

$= \log_5 \frac{25}{3x}$

CHAPTER 3 Equations

Exercise 3.1

1. and 2.

a) $x^2 + 7x + 12$
$= (x + 3)(x + 4)$

b) $x^2 - 7x + 12$
$= (x - 3)(x - 4)$

c) $x^2 - x - 12$
$= (x + 3)(x - 4)$

d) $x^2 + x - 12$
$= (x - 3)(x + 4)$

e) $x^2 + 5x + 6$
$= (x + 3)(x + 2)$

f) $x^2 - 5x + 6$
$= (x - 2)(x - 3)$

g) $x^2 + x - 6$
$= (x + 3)(x - 2)$

h) $x^2 - x - 6$
$= (x + 2)(x - 3)$

Exercise 3.2

a) $x^2 + 14x + 49$
$= (x + 7)(x + 7)$

b) $x^2 - 14x + 49$
$= (x - 7)(x - 7)$

c) $x^2 - 6x + 9$
$= (x - 3)(x - 3)$

d) $x^2 + 6x + 9$
$= (x - 3)(x + 4)$

e) $p^2 + 2pq + q^2$
$= (p + q)(p + q)$

f) $p^2 + 8p + 16$
$= (p + 4)(p + 4)$

g) $p^2 + 10pq + 25q^2$
$= (p + 5q)(p + 5q)$

h) $r^2 - 12r + 36$
$= (r - 6)(r - 6)$

Exercise 3.3

a) $3m^2 - 5m - 12$
$= 3m^2 - 9m + 4m - 12$
$= 3m(m - 3) + 4(m - 3)$
$= (3m + 4)(m - 3)$

b) $6m^2 - 14m - 12$
$= 6m^2 - 18m + 4m - 12$
$= 6m(m - 3) + 4(m - 3)$
$= 2(3m + 2)(m - 3)$

c) $15b^2 - 11b + 2$
$= 15b^2 - 5b - 6b + 2$
$= 5b(3b - 1) - 2(3b - 1)$
$= (5b - 2)(3b - 1)$

d) $5b^2 + 27b + 10$
$= 5b^2 + 25b + 2b + 10$
$= 5b(b + 5) + 2(b + 5)$
$= (5b + 2)(b + 5)$

e) $12s^2 - 13s - 35$
$= 12s^2 - 28s + 15s - 35$
$= 4s(3s - 7) + 5s(3s - 7)$
$= (4s + 5)(3s - 7)$

f) $24s^2 - 46s - 18$
$= 2(12s^2 - 23s - 9)$
$= 2(12s^2 - 27s + 4s - 9)$
$= 2(3s(4s - 9)) + 1(4s - 9)$
$= 2(3s + 1)(4s - 9)$

g) $6r^2 + 23r + 7$
$= 6r^2 + 2r + 21r + 7$
$= 2r(3r + 1) + 7(3r + 1)$
$= (2r + 7)(3r + 1)$

h) $20r^2 + 19r + 3$
$= 20r^2 + 4r + 15r + 3$
$= 4r(5r + 1) + 3(5r + 1)$
$= (4r + 3)(5r + 1)$

i) $-5x^2 + 7x - 2$
$= -5x^2 + 5x + 2x - 2$
$= -5x(x - 1) + 2(x - 1)$
$= (2 - 5x)(x - 1)$

j) $-3x^2 - x + 14$
$= -3x^2 + 6x - 7x + 14$
$= -3x(x - 2) - 7(x - 2)$
$= (-7 - 3x)(x - 2)$

k) $-x^2 - x + 2$
$= -x^2 + x - 2x + 2$
$= -x(x - 1) - 2(x - 1)$
$= (-2 - x)(x - 1)$

l) $-6x^2 + x + 1$
$= -6x^2 + 3x - 2x + 1$
$= -3x(2x - 1) - 1(2x - 1)$
$= (-1 - 3x)(2x - 1)$

Exercise 3.4

a) $x^2 - 1$
$= (x - 1)(x + 1)$

b) $x^2 - 81$
$= (x - 9)(x + 9)$

c) $a^2 - b^2$
$= (a - b)(a + b)$

d) $x^2 - 4y^2$
$= (x - 2y)(x + 2y)$

e) $9x^2 - 1$
$= (3x - 1)(3x + 1)$

f) $9x^2 - 9$
$= (3x - 3)(3x + 3)$

g) $200 - 2m^2$
$= 2(10 - m)(10 + m)$

h) $64m^2 - 25n^2q^2$
$= (8m - 5nq)(25 + 5nq)$

i) $p^2q^2r^2 - 1$
$= (pqr - 1)(pqr + 1)$

j) $9x^{10} - 4y^8$
$= (3x^5 - 2y^4)(3x^5 + 2y^4)$

k) $5a^4 - 20b^2$
$= 5(a^2 - 2b)(a^2 + 2b)$

l) $225b^2 - 169c^2$
$= (15b - 13c)(15b + 13c)$

Exercise 3.5

a) $y^2 - 3y = 10$
$y^2 - 3y - 10 = 0$
$y^2 - 5y + 2y - 10 = 0$
$y(y - 5) + 2(y - 5) = 0$
$(y - 5)(y + 2) = 0$
$y - 5 = 0$ or $y + 2 = 0$
$y = 5$ or $y = -2$

b) $7t^2 + 14t = 0$
$7t(t + 2) = 0$
$7t = 0$ or $t + 2 = 0$
$t = 0$ or $t = -2$

c) $12y^2 + 24y + 12 = 0$
$y^2 + 2y + 1 = 0$
$(y + 1)(y + 1) = 0$
$y = -1$

d) $y^2 + 48y = 100$
$y^2 + 48y - 100 = 0$
$y^2 + 50y - 2y - 100 = 0$
$y(y - 2) + 50(y - 2) = 0$
$(y - 2)(y + 50) = 0$
$y - 2 = 0$ or $y + 50 = 0$
$y = 2$ or $y = 50$

e) $y^2 - 5y + 6 = 0$

$y^2 - 3y - 2y + 6 = 0$

$(y - 3)(y - 2) = 0$

$y = 3$ or $y = 2$

f) $y^2 + 5y - 36 = 0$

$y^2 + 9y - 4y - 36 = 0$

$(y - 4)(y + 9) = 0$

$y - 4 = 0$ or $y + 9 = 0$

$y = 4$ or $y = -9$

g) $-y^2 - 11y - 24 = 0$

$y^2 - 3y - 10 = 0$

$y^2 - 5y + 2y - 10 = 0$

$y(y - 5) + 2(y - 5) = 0$

$(y - 5)(y + 2) = 0$

$y - 5 = 0$ or $y + 2 = 0$

$y = 5$ or $y = -2$

h) $13y - 42 = y^2$

$y^2 - 13y + 42 = 0$

$y^2 - 7y - 6y + 42 = 0$

$(y - 7)(y + 6) = 0$

$y = 7$ or $y = 6$

i) $\frac{y-2}{y+1} = \frac{2y+1}{y-7}$

$(y - 2)(y - 7) = (2y + 1)(y + 1)$

$y^2 - 9y + 14 = 2y^2 + 3y + 1$

$y^2 + 12y - 13 = 0$

$(y + 13)(y - 1) = 0$

$y = -13$ or $y = 1$

Exercise 3.6

a) $x^2 - 8x + 7 = 0$

$(x - 7)(x - 1) = 0$

$x - 7 = 0$ or $x - 1 = 0$

$x = 7$ or $x = 1$

b) $x^2 + 11x + 18 = 0$

$(x + 2)(x + 9) = 0$

$x + 2 = 0$ or $x + 9 = 0$

$x = -2$ or $x = -9$

c) $2x^2 - x - 3 = 0$

$(2x - 3)(x + 1) = 0$

$2x - 3 = 0$ or $x + 1 = 0$

$x = \frac{3}{2}$ or $x = -1$

d) $3x^2 + 13x + 4 = 0$

$(3x + 1)(x + 4) = 0$

$3x + 1 = 0$ or $x + 4 = 0$

$x = -\frac{1}{3}$ or $x = -4$

e) $14x^2 - 19x - 3 = 0$

$(7x + 1)(2x - 3) = 0$

$7x + 1 = 0$ or $2x - 3 = 0$

$x = -\frac{1}{7}$ or $x = \frac{3}{2}$

f) $2x^2 + 5x - 3 = 0$

$(2x - 1)(x + 3) = 0$

$2x - 1 = 0$ or $x + 3 = 0$

$x = \frac{1}{2}$ or $x = -3$

g) $3x^2 + 19x + 20 = 0$

$(3x + 4)(x + 5) = 0$

$3x + 4 = 0$ or $x + 5 = 0$

$x = -\frac{3}{4}$ or $x = -5$

h) $6x^2 - 7x + 2 = 0$

$(3x - 2)(2x - 1) = 0$

$3x - 2 = 0$ or $2x - 1 = 0$

$x = \frac{2}{3}$ or $x = \frac{1}{2}$

i) $6x^2 - 17x - 3 = 0$

$(6x + 1)(x - 3) = 0$

$6x + 1 = 0$ or $x - 3 = 0$

$x = -\frac{1}{6}$ or $x = 3$

Exercise 3.7

a) $(2x+1)^2 = 1$

$\sqrt{(2x+1)^2} = \pm 1$

$(2x+1) = \pm 1$

$x = 0 \text{ or } x = -1$

b) $\left(x - \frac{1}{2}\right)^2 = 9$

$\sqrt{\left(x - \frac{1}{2}\right)^2} = \pm 3$

$\left(x - \frac{1}{2}\right) = \pm 1$

$x = \frac{3}{2} \text{ or } x = -\frac{1}{2}$

c) $(x+3)^2 = \frac{1}{4}$

$\sqrt{(x+3)^2} = \pm\frac{1}{2}$

$(x+3) = \pm\frac{1}{2}$

$x = -\frac{5}{2} \text{ or } x = -\frac{7}{2}$

d) $\left(\frac{3}{4}x - 7\right)^2 = 5$

$\sqrt{\left(\frac{3}{4}x - 7\right)^2} = \pm\sqrt{5}$

$\left(\frac{3}{4}x - 7\right) = \pm\sqrt{5}$

$x = \frac{4(7 \pm \sqrt{5})}{3}$

$x = \frac{28 + 4\sqrt{5}}{3} \text{ or } x = \frac{28 - 4\sqrt{5}}{3}$

e) $(x+7)^2 = 49$

$\sqrt{(x+7)^2} = \pm 7$

$(x+7) = \pm 7$

$x = -14 \text{ or } x = 0$

f) $(x+5)^2 = 25$

$\sqrt{(x+5)^2} = \pm 5$

$(x+5) = \pm 5$

$x = 0 \text{ or } x = -10$

Exercise 3.8

a) $2x^2 - 2x - 1 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ a = 2; b = –2; c = –1

$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$

$x = \frac{2 \pm 2\sqrt{3}}{4}$

$x = \frac{1 + \sqrt{3}}{2} \text{ or } \frac{1 - \sqrt{3}}{2}$

(b) $x^2 - 2x = 2(x-1)$

$x^2 - 4x + 2 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ a = 1; b = –4; c = 2

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(2)}}{2(1)}$

$x = \frac{4 \pm 2\sqrt{2}}{2}$

$x = 2 + \sqrt{2} \text{ or } 2 - \sqrt{2}$

c) $-3x^2 + 3x + 1 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ a = –3; b = 3; c = 1

$x = \frac{-(3) \pm \sqrt{(3)^2 - 4(-3)(1)}}{2(-3)}$

$x = \frac{-3 \pm \sqrt{21}}{-6}$

$x = \frac{3 + \sqrt{21}}{6} \text{ or } \frac{3 - \sqrt{21}}{6}$

d) $6x^2 - 5x - 2 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ a = 6; b = –5; c = –2

$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(6)(-2)}}{2(6)}$

$x = \frac{5 \pm \sqrt{73}}{12}$

$x = \frac{5 + \sqrt{73}}{12} \text{ or } \frac{5 - \sqrt{73}}{12}$

e) $x^2 - 4x = 3$

$x^2 - 4x - 3 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ a = 1; b = –4; c = –3

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-3)}}{2(1)}$

$x = \frac{4 \pm 2\sqrt{7}}{2}$

$x = 2 + \sqrt{7} \text{ or } 2 - \sqrt{7}$

f) $x^2 + 1 = 6x$

$x^2 - 6x + 1 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ a = 1; b = –6; c = 1

$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(1)(1)}}{2(1)}$

$x = \frac{6 \pm 4\sqrt{2}}{2}$

$x = 3 + 2\sqrt{2}$ or $x = 3 - 2\sqrt{2}$

g) $x^2 - 4x - 21 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ a = 1; b = –4; c = –21

$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-21)}}{2(1)}$

$x = \frac{4 \pm 10}{2}$

$x = 7$ or $x = -3$

h) $x^2 - 3x = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ a = 1; b = –3; c = 0

$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(0)}}{2(1)}$

$x = \frac{3 \pm 3}{2}$

$x = 3$ or $x = 0$

i) $x^2 + 6x = 10$

$x^2 + 6x - 10 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ a = 1; b = 6; c = –10

$x = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(-10)}}{2(1)}$

$x = \frac{-6 \pm 2\sqrt{19}}{2}$

$x = \frac{-3 + \sqrt{19}}{2}$ or $x = \frac{-3 - \sqrt{19}}{2}$

Exercise 3.9

		Nature of the number						
	Number	Real	Non-Real	Rational	Irrational	Integer	Whole	Natural
a)	0	X				X	X	
b)	$\sqrt{3}$	X			X			
c)	–0.7	X		X				
d)	1.234				X			
e)	–2	X		X		X		
f)	π	X			X			
g)	$-\frac{\sqrt{3}}{2}$	X		X				
h)	$-\frac{3}{2}$	X		X				
i)	$\sqrt{16}$	X		X		X	X	
j)	$\sqrt{\frac{25}{3}}$	X		X				
k)	$\sqrt{-25}$		X					

Exercise 3.10

	Equation	Value of the discriminant:	Nature of roots
a)	$2x^2 - 4x - 3 = 0$	$b^2 - 4ac = 16 - 4(2)(-3) > 0$	2 real rational, unequal roots
b)	$9x^2 - 12x + 4 = 0$	$b^2 - 4ac = 144 - 4(9)(4) > 0$	2 real equal roots
c)	$x^2 + x + 1 = 0$	$b^2 - 4ac = 1 - 4(1)(1) < 0$	No real roots
d)	$3x^2 + 5x + 3 = 0$	$b^2 - 4ac = 1 - 4(3)(5) < 0$	No real roots
e)	$x^2 - 5x + 3 = 0$	$b^2 - 4ac = 25 - 4(1)(3) > 0$	2 real rational, unequal roots
f)	$-2x^2 + 5x - 3 = 0$	$b^2 - 4ac = 25 - 4(-2)(-3) > 0$	2 real rational unequal roots
g)	$-x^2 + 1 = 0$	$b^2 - 4ac = 0 - 4(-1)(1) > 0$	2 real rational, unequal roots
h)	$2x^2 + 5x + 6 = 0$	$b^2 - 4ac = 25 - 4(2)(5) < 0$	2 non real roots
i)	$x^2 - 4x + 4 = 0$	$b^2 - 4ac = 16 - 4(1)(4) = 0$	2 real equal roots

Exercise 3.11

a) $x^2 - 6x + 9 > 0$

$(x - 3)(x - 3) > 0$

$x < 3;\ x > 3$

b) $x^2 - x > 0$

$x(x - 1) > 0$

$x < 0;\ x > 1$

c) $x^2 - 1 < 0$

$(x - 1)(x + 1) < 0$

$x < 1;\ x > -1$

d) $-3x^2 + 4x + 4 < 0$

$(x - 2)(-3x - 2) < 0$

$x < -\frac{2}{3};\ x > 2$

e) $-x^2 + 2x - 1 < 0$

$(x - 1)(-x + 1) < 0$

$x < 1;\ x > 1$

f) $x^2 + 2x - 3 > 0$

$(x + 3)(x - 1) > 0$

$x < -3;\ x > 1$

g) $2x^2 - 7x + 3 < 0$

$(2x - 1)(x - 3) < 0$

$x < 3;\ x > \frac{1}{2}$

h) $x^2 + \frac{5}{2}x + 1 > 0$

$(x + 2)(2x + 1) > 0$

$x < -2;\ x > -\frac{1}{2}$

$-5x^2 - 9x + 2 < 0$

$(x + 2)(-5x + 1) < 0$

$x < -2;\ x > \frac{1}{5}$

Exercise 3.12

a) $4x^2 + 4x + 1 \geq 0$

$(2x + 1)^2 \geq 0$

$x \geq -\frac{1}{2};\ x \leq -\frac{1}{2}$

b) $(x - 1)^2 < 0$

no solution

c) $-x^2 + 6x - 9 > 0$

no solution

d) $-x^2 + 4x - 4 \geq 0$

no solution

e) $2x^2 - 50 \geq 0$

$2(x - 5)(x + 5) \geq 0$

$x \geq 5;\ x \leq -5$

f) $2x^2 + 5x - 12 \leq 0$

$2x^2 + 8x - 3x - 12 \leq 0$

$(x + 4)(2x - 3) \leq 0$

$-4 \leq x \leq \frac{3}{2}$

g) $-2x^2 + x + 3 > 0$

$(x+1)(2x-3) > 0$

$x > \frac{3}{2};\ x < -1$

h) $7x^2 + x \leq 0$

$x(7x+1) = 0$

$-\frac{1}{7} \leq x \leq 0$

i) $4x^2 - 9 < 0$

$(2x+3)(2x-3)$

$-\frac{3}{2} < x < \frac{3}{2}$

Exercise 3.13

a) $y = x^2 + 4x + 3$

$y = x + 3$

$x^2 + 4x + 3 = x + 3$

$x^2 + 3x = 0$

$x(x+3) = 0$

$x = 0$ or $x + 3 = 0$

$x = -3$

At $x = 0;\ x = -3$

$y = 3;$ at $y = 0$

The solutions are:

(0; 3) and (–3; 0)

b) $y = x^2 + 2x + 5$

$y - x = 5$

$y = x + 5$

$x^2 + 2x + 5 = x + 5$

$x^2 + x = 0$

$x(x+1) = 0$

$x = 0$ or $x + 1 = 0$

$x = -1$

At $x = 0;\ x = -1$

$y = 5;$ at $y = 4$

The solutions are:

(0; 5) and (–1; 4)

c) $y = x^2 - 6x + 8$

$y = -\frac{x}{2} + 2$

$x^2 + 4x + 3 = -\frac{x}{2} + 2$

$x^2 - \frac{11}{2}x + 6 = 0$

$(2x-3)(x-4) = 0$

$x = \frac{3}{2}$ or $x = 4$

At $x = \frac{3}{2}$

$y = \frac{5}{4}$

; at $x = 4$

$y = 0$

The solutions are:

$\left(\frac{3}{2};\frac{5}{4}\right)$ and (4; 0)

d) $y = x^2$

$y - 2 = x$

$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x - 2 = 0$ or $x + 1 = 0$

$x = 2$ or $x = -1$

$y = 4$ or $y = 2$

The solutions are:

(2; 4) or (–1; 2)

e) $y = x^2 - 3x + 5$

$y - x = 2$

$x^2 - 3x + 5 = x + 2$

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x - 1 = 0$ or $x - 3 = 0$

$x = 1$ or $x = 3$

$y = 3$ or $y = 5$

The solutions are:

(1; 3) or (3; 5)

f) $y = x^2 - 1$

$y = x + 1$

$x^2 - 1 = x + 1$

$x^2 - x - 2 = 0$

$x - 2 = 0$ or $x + 1 = 0$

$x = 2$ or $x = -1$

$y = 3$ or $y = 0$

The solutions are:

(2; 3) or (–1; 0)

g) $y = 2x^2 - 5x - 3$

$y = -6x + 3$

$2x^2 - 5x - 3 = -6x + 3$

$2x^2 + x - 6 = 0$

$(2x - 3)(x + 2) = 0$

$x = \frac{3}{2}$ or $x = -2$

$y = -6$ or $y = 15$

The solutions are:

$\left(\frac{3}{2};\ -6\right)$ or $(-2;\ 15)$

h) $y = 6x^2 + 11x - 10$

$y = 4x$

$6x^2 + 11x - 10 = 4x$

$6x^2 + 7x - 10$

$x = \frac{-7 \pm \sqrt{289}}{12}$

$x = \frac{10}{12} = \frac{5}{6}$ or $x = -\frac{41}{12}$

$y = \frac{10}{3}$ or $y = -\frac{41}{3}$

The solutions are:

$\left(\frac{10}{12}; \frac{10}{3}\right)$ or $\left(-\frac{41}{12};\ -\frac{41}{3}\right)$

i) $y = x^2 - 1,5x - 2$

$y = 0,5x$

$x^2 - 1,5x - 2 = 0,5x$

$x^2 - 2x - 2 = 0$

$x = \frac{2 \pm \sqrt{4 + 8}}{2}$

$x = 1 + \sqrt{3}$

$y = \frac{1 + \sqrt{3}}{2}$

or $x = 1 - \sqrt{3}$

$y = \frac{1 - \sqrt{3}}{2}$

The solutions are:

$\left(1 + \sqrt{3}; \frac{1 + \sqrt{3}}{2}\right)$ or $\left(1 - \sqrt{3}; \frac{1 - \sqrt{3}}{2}\right)$

j) $y = x^2 - 1,3x + 0,3$

$y = 0,3$

$x^2 - 1,3x = 0$

$x(x - 1,3) = 0$

$x = 0$ or $x = 1,3$

$y = 0,3$ or $y = 0$

The solutions are:

(0; 0,3) or (1,3; 0,3)

k) $y = x^2 + 5,5x - 3$

$y = 5,5x$

$x^2 + 5,5 - 3 = 5,5x$

$x^2 - 3 = 0$

$x = \sqrt{3}$ or $x = -\sqrt{3}$

$y = 5,5\sqrt{3}$ or $y = 0$

The solutions are:

$(2;\ 5,5\sqrt{3})$ or $(-1;\ -5,5\sqrt{3})$

l) $y = 3x^2 + xy + 1$

$y + 2x = 5$

$y = 3x^2 + xy + 1$

$y = -2x + 5$

$3x^2 + x(-2x + 5) + 1 = -2x + 5$

$3x^2 - 2x^2 + 5x + 1 + 2x - 5 = 0$

$x^2 + 7x - 4 = 0$

$x = \frac{-7 \pm \sqrt{49 + 16}}{2}$

$x = \frac{-7 + \sqrt{65}}{2}$

$y = 12 - \sqrt{65}$

or $x = \frac{-7 - \sqrt{65}}{2}$

$y = 12 + \sqrt{65}$

3.8 Word problems

1. $A = 12,6w + 18,65w - w^2$

$-w^2 + 31,25w - 70,56 = 0$

$w = \frac{-31,25 \pm \sqrt{31,25^2 - 4(-1)(-70,56)}}{-2}$

$w = \frac{144}{5} = 28,8$ or $w = \frac{49}{20} = 2,45$

3. a) $l = (18,3 + w)cm$

$A = 55000cm^2$

$w^2 + 18,3w - 55000 = 0$

$w = \frac{-b \pm \sqrt{18,3^2 - 4(1)(-55000)}}{2}$

$w = 225,5$ or $-243,8$

$\therefore w = 225,5cm$

b) $l = (18,3 + w)cm$

$l = 18,3 + 225,5$

$l = 243,8cm$

4. $I = 7x^2 + 4x - 3$

$7x^2 + 4x - 3 = 0$

$(x + 1)(7x - 3) = 0$

$x = -1$ or $x = \frac{3}{7}$

So: x = 0,43 m indicate units of x in the exercise, then give correct answer

5. a) Perimeter: $2x + 2y = 8y$

$x = 3y$

Area: $xy = 6y + 3$

$x = \frac{6y + 3}{y}$

b) $3y = \frac{6y + 3}{y}$ $3y^2 = 6y + 3$

$3y^2 - 6y - 3 = 0$

$y^2 - 2y - 1 = 0$

completing a square gives $y = 1 + \sqrt{2}$

Perimeter: $= 8y$

$= 8y$

$= 8 + 8\sqrt{2}$

$= 19,3$ metres

Area $= 6y + 3$

$= 6(1 + \sqrt{2}) + 3$

$= 17,5$ m^2

6. a) perimeter

$5x - 1 + y + 14 + 2x = 607x + y + 13 = 60$

$7x + y - 47 = 0$

b) Area

$\frac{1}{2}(5x - 1)(y + 14) = 10y$

$5xy + 70x - y - 14 = 20y$

$5xy + 70x - 21y - 14 = 0$

c) Perimeter equation: $7x + y - 47 = 0$

Area equation: $5xy + 70x - 21y - 14 = 0$

d) $7x + y - 47 = 0$

$y = 47 - 7x$ substitute this in $5xy + 70x - 21y - 14 = 0$

$5x(47 - 7x) + 70x - 21(47 - 7x) - 14 = 0$

$235x - 35x^2 + 70x - 987 + 147x - 14 = 0$

$-35x^2 + 452x - 1001 = 0$

Using the quadratic formula,

$x_1 = 10,08$ or $x_2 = 2,84$substituting in $y = 47 - 7x$, we have

We take $y_1 = -23,56$or $y_2 = 27,12$

Therefore: $x = 2,84$and $y = 27,12$

e) Dimensions of the triangle are:

$5x - 1 = 13,2$

$y + 14 = 41,12$

$2x = 5,68$

f) Perimeter: $5x - 1 + y + 14 + 2x = 60$

$13,2 + 41,12 + 5,68 = 60$

Area: $\frac{1}{2} \times 13,2 \times 41,12 = 271,4 \approx 10(27,12)$

7. a) $2\pi r = 3\pi r^2$

$3\pi r^2 - 2\pi r = 0$

b) $\pi r(3r - 2) = 0$

$\pi r = 0$ or $r = \frac{2}{3} cm$

$\therefore\ r = \frac{20}{3} mm$

c) $C = 2\pi r = \frac{40}{3}\pi\text{mm}$

d) $A = \pi r^2 = \frac{400}{9}\pi cm^2$

8. Total surface area: $\pi r(r + l) = 486,2cm^2$

$\pi r^2 + 15,3\pi r - 468,2 = 0$ using the quadratic formula, we have:

$r = 6,8cm$ or $r = -22,1cm$

Therefore: $r = 6,8cm$

9. NB: change the area of the path to 9,5 m² not cm²

length of path $= 400 + 2x\text{cm}$

width of path $= x\text{cm}$

area of path $=2(400 + 2x)(x) + 2(200)(x) - 95000 = 0$

$800x + 4x^2 + 400x - 95000 = 0$

$4x^2 + 1200x - 95000 = 0$

Using the quadratic formula, we have:

$x = 65$ cm

The width of the path is: 65 cm

Or

Area of path $=(4+2x)(2+2x)-8 = 9,5$

$8+8x+4x+4x^2-8 = 9,5$

$4x^2+12x-95 = 0$

Using the quadratic formula, we have:

$x = 0,6065 \text{ m} = 65 \text{ cm}$

Exercise 3.14

1. a) $P = \frac{V^2}{R}$

   $R = \frac{V^2}{P}$

   b) $P = \frac{V^2}{R}$

   $V = +\sqrt{RP}$

2. $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

   $\frac{1}{R_1} = \frac{R_2 - R}{R_1 R_2}$

   $R_1 = \frac{R R_2}{R_2 - R}$

3. $F = ma$

   $a = \frac{F}{m}$

4. a) $PV^n = C$

   $P = \frac{C}{V^n}$

   b) $PV^n = C$

   $n \log V = \log \frac{C}{P}$

   $n = \frac{\log C - \log P}{\log V}$

5. $V = IR$.

   $R = \frac{V}{I}$

6. $C = 2\pi r$

   $r = \frac{C}{2\pi}$

7. $A = 2\pi rh + 2\pi r^2$ to r

   We have $2\pi r^2 + 2\pi rh - A = 0$
   using the quadratic formula with

   $a = 2\pi$;

   $b = 2\pi h$

   $c = -A$

   $r = \frac{-2\pi h \pm \sqrt{(2\pi h)^2 - 4(2\pi)(-A)}}{2\pi}$

   $r = \frac{-\pi h \pm \sqrt{\pi^2 h^2 + 2\pi A}}{2\pi}$

8. $T = 2\pi\sqrt{\frac{l}{g}}$

   $l = g\left(\frac{T}{2\pi}\right)^2$

9. a) $E = \frac{V(R+r)}{R}$

   b) $R = \frac{Vr}{E-r}$

   c) $r = \frac{R(E-V)}{V}$

10. $v^2 = 2gh$.

    $h = \frac{v^2}{2g}$

Consolidation Exercise

1. **Equation** | **Solved by factorising**

Equation	Solved by factorising
$x^2 + 13x + 22 = 0$	YES
$x^2 - 10x + 25 = 0$	YES
$3x^2 - 6x + 5 = 0$	NO
$x^2 + 12x + 32 = 0$	YES
$-3x^2 + 5x + 1 = 0$	NO
$-3,1x^2 + 2,5x + 3,75 = 0$	NO

2. Find factors of the constant term which combine to give the middle term. If this is possible, then the equation is factorable.

3. a) $x^2 + 13x + 22 = 0$

$(x + 2)(x + 11) = 0$

$x = -2$ or $x = -11$

b) $x^2 - 10x + 25 = 0$

$(x - 5)(x - 1) = 0$

$x = 5$ or $x = 1$

c) $3x^2 - 6x + 5 = 0$

$b^2 - 4ac = 36 - 4(3)(5) < 0$

No real roots

d) $x^2 + 12x + 32 = 0$

$(x + 4)(x + 8) = 0$

$x = -4$ or $x = -8$

e) $-3x^2 + 5x + 1 = 0$

$x = \frac{-5 \pm \sqrt{25 + 12}}{-6}$

$x = \frac{5 + \sqrt{37}}{6}$ or $x = \frac{5 - \sqrt{37}}{6}$

f) $-3,1x^2 + 2,5x + 3,75 = 0$

$x = \frac{-2,5 \pm \sqrt{6,25 + 46,5}}{-6,2}$

$x = \frac{25 + 5\sqrt{211}}{62}$ or $x = \frac{25 - 5\sqrt{211}}{62}$

$(x - 1)(x + 7) < 0$

$x = 1$ or $x = -7$

$-7 < x < 1$

4. b) $3(x-3)(x-3) \geq 0$

$x = 3$ repeated root

$x \leq 3; x \geq 3$

c) $(x-1)(x+2) \geq 0$

$x = 1$ or $x = -2$

$x \leq -2; x \geq 3$

d) $3x^2 + 18x + 15 \leq 0$

$(x+5)(3x+3) \leq 0$

$x = -5$ or $x = -1$

$-5 \leq x \leq -1$

e) $-2x^2 - 19x - 9 < 0$

$(x+9)(-2x-1)$

$x = -5$ or $x = -1$

$x < -9; x > -\frac{1}{2}$

f) $-3x^2 - x + 2 < 0$

$(x+1)(-3x+2)$

$x = -1$ or $x = \frac{2}{3}$

$x < -1; x > \frac{2}{3}$

5. a) $y = (x-4)(x+13)$

$y = x - 4$

$x^2 + 9x - 52 = x - 4$

$x^2 + 8x - 48 = 0$

$(x-4)(x+12) = 0$

$x = 4$ or $x = -12$

$y = 0$ or $y = -16$

b) $y = x^2 - 36$

$y - x + 34 = 0$

$x^2 - 36 = x - 34$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$

$y = -32$ or $y = -35$

Exercise 4.1

1. a) $m = \frac{y_2 - y_1}{x_2 - x_1}$

$= \frac{1-4}{0-5}$

$= \frac{3}{5}$

b) $m = 3$

c) $m = -3$

d) $m = -1$

e) $m = -3$

f) $m = \frac{5}{9}$

2. a) $y = -x + 3$

b) $y = -\frac{5}{2}x + 5$

c) $y = x + 4$

d) $y = -x + 6$

e) $y = \frac{4}{3}x + 1$

f) $y = 4$

3. a) $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$

$y - 1 = \frac{13-1}{5-1}(x-1)$

$y - 1 = \frac{12}{4}(x-1)$

$y - 1 = 3x - 3$

$y = 3x - 2$

$-11 = 3p - 2$

$\therefore p = -3$

b) $m_1 = \frac{2p+6}{p-4}$

$m_2 = \frac{-6+14}{4-6} = \frac{8}{-2} = -4$

$\frac{2p+6}{p-4} = -4$

$2p + 6 = -4(p-4)$

$2p + 6 = -4p + 16$

$6p = 10$

$p = \frac{5}{3}$

c) $m = \frac{-13+6}{-1-0} = \frac{-7}{-1} = 7$

$y + 6 = 7(x - 0)$

$y = 7x - 6$

$\frac{p}{2} = 7p - 6$

$p = 14p - 12$

$-13p = -12$

$p = \frac{12}{13}$

Exercise 4.2

1. a) $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{-4-4} = \frac{2}{-8} = -\frac{1}{4}$

$\therefore \tan\theta = -\frac{1}{4}$

$\tan^{-1}\left(-\frac{1}{4}\right)$

Reference angle is $14{,}04^\circ$

$\therefore \theta = 180^\circ - 14{,}04^\circ$

$\theta = 165{,}96^\circ$

b) $\theta = 164{,}05^\circ$

c) $\theta = 30{,}96^\circ$

d) $\theta = 18{,}43^\circ$

e) $\theta = 45^\circ$

f) $\theta = 71{,}57^\circ$

2. a) $m = 3$

$\theta = \tan^{-1}(3)$

$\theta = 71{,}57^\circ$

b) $\theta = 108{,}43^\circ$

c) $\theta = 56{,}31^\circ$

d) $\theta = 149{,}04^\circ$

e) $\theta = 59{,}09^\circ$

f) $\theta = 110{,}32^\circ$

Exercise 4.3

1. a) $m = 1$

   Equation of the line through the point $(-1\ ;\ 1)$ with $m = 1$ is given by:

   $y = mx + c$

   $1 = 1(-1) + c$

   $c = 2$

   $\therefore\ y = x + 2$

   b) $y = x - 3$

   c) $y = x + 1$

   d) $y = -x$

   e) $y = -x - 3$

   f) $y = -x + 3,2$

   g) $y = -0,25x + 10,525$

   h) $y = 4x - 13$

2. a) Parallel; the gradients are equal

   b) Not parallel; the gradients are not equal

   c) Not parallel, the gradients are not equal

   d) Not parallel; the gradients are not equal

   e) Not parallel; the gradients are not equal

3. The lines whose equations are given by $y = 3x - 6$ and $y = 3x + 1\ 000$ because they have equal gradients.

4. Some of the possible answers:

   $y = \frac{3}{7}x$ and $y = \frac{3}{7}x - 1$

5. Parallel lines have equal gradients:

   a) $\frac{-5-k}{2+3} = \frac{4-6}{-2-3}$

   $\frac{-5-k}{5} = \frac{2}{5}$

   $-5 - k = 5 \times \frac{2}{5}$

   $-5 - k = 2$

   $k = -7$

   b) $k = 4$

   c) $k = 3,24$

   d) $k = \frac{3}{2}$

   e) $k = 2,25$

6. The points $(0\ ;\ -5)$ and $(0\ ;\ 2)$ are on the y-axis. The point $(3\ ;\ -4)$ has to be on a line parallel to the y-axis. Any point whose $x$co-ordinate is 3 will lie on the line passing through the point $(3\ ;\ -4)$.

7. $m_{AB} = m_{DC}$

$\frac{1+1}{6-3} = \frac{y-2}{x+4}$

$\frac{2}{3} = \frac{y-2}{x+4}$

$2x+8 = 3y-6$

$x = \frac{3y-14}{2}$ .... (1)

$m_{BC} = m_{AD}$

$\frac{-1-2}{3+4} = \frac{1-y}{6-x}$

$-\frac{3}{7} = \frac{1-y}{6-x}$

$-18+3x = 7-7y$ .... (2)

Substitute (1) in (2): $-18+3\left(\frac{3y-14}{2}\right) = 7-7y$

$-36+9y-42 = 14-14y$

$23y = 92$

$y = 4$

$\therefore\ x = \frac{3(4)-14}{2} = -1$

$D(-1;4)$

8. D (3 ; 0)

$A(1;1)$, $B(x;y)$, $C(-5;5)$ and $D(-7;6)$

Exercise 4.4

1. a) $-\frac{2}{5}\times 2,5 = -1$; the lines are perpendicular to each other.

   b) $1\times -1 = -1$; the lines are perpendicular to each other.

   c) $2\times -2 = -4$; the lines are not perpendicular to each other.

   d) $\frac{2}{3}\times\frac{3}{2} = 1$; the lines are not perpendicular to each other.

   e) $\frac{2}{3}\times -\frac{3}{2} = -1$; the lines are perpendicular to each other.

2. a) the product of the gradients of perpendicular lines is -1.

$\frac{h-1}{8-0}\times\frac{6+1}{4-5} = -1$

$\frac{h-1}{8}\times\frac{7}{1} = -1$

$\frac{7h-7}{8} = -1$

$7h-7 = 8$

$7h = 15$

$h = \frac{15}{7}$

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b) $h = \frac{5}{7}$

c) $h = -12$

d) $h = -\frac{1}{3}$

3. $m_{AC} = \frac{-1-1}{2+4} = -\frac{1}{3}$

$m_{AB} = \frac{y-1}{0+4} = 3[m_{AB} \times m_{AC} = -1]$

$y - 1 = 12$

$y = 13$

4. $x = -\frac{5}{3}$

Exercise 4.5

1. a) $MB = ME = 5,5 \text{ m}$

$AB^2 = AM^2 + MB^2$

$2,7^2 + 5,5^2$

$AB = \sqrt{2,7^2 + 5,5^2}$

$CD = \frac{1}{2}AB = 3,1 \text{ mAB} = 6,1 \text{ m}$

$ND^2 = CD^2 - CN^2$

$ND = \sqrt{3,1^2 - 2,4^2}$

$ND \approx 2 \text{ m}$

b) Yes

c) $DB = 11 - 4$
7 m

2. a) Length of the ladder $= \sqrt{2^2 + 1^2}$

$= 2,2 \text{ m}$

b) 2 m

c) $\theta = \tan^{-1}(2)$

$= 63,43°$

Consolidation exercise

1. a) $m_{AB} = -2$ and $m_{CD} = -\frac{3}{2}$

The lines AB and CD are neither parallel nor perpendicular to each other.

b) AB is perpendicular to CD

c) AB is neither parallel nor perpendicular to CD

d) AB is perpendicular to CD

2. a) $y = -2x + 11$ (AB)

$y = -\frac{3}{2}x - 2$ (CD)

b) $y = -2x + 11$ (AB)

$y = \frac{1}{2}x + 1$ (CD)

c) $y = \frac{1}{3}x - \frac{7}{3}$ (AB)

$y = 3x - 17$ (CD)

d) $y = \frac{2}{5}x$ (AB)

$y = -\frac{5}{2}x - 13$(CD)

3. Midpoint of AB: $\left(\frac{0+8}{2}; \frac{-1+3}{2}\right) = (4; 1)$

$m_{AB} = -\frac{1}{2}$

Equation of the line: $y - 1 = 2(x - 4)$

$y = 2x - 7$

4. a) $m = -2$

$y - 0 = -2(x - 1)$

$y = -2x + 2$

b) $3x - 2y + 4 = 0$

$y = \frac{3}{2}x + 2$

The line parallel to $y = \frac{3}{2}x + 2$ has a gradient of $\frac{3}{2}$

$y = \frac{3}{2}x + c$

$3 = \frac{3}{2}(0) + c$

$c = \frac{3}{2}$

The equation of the line parallel to $y = \frac{3}{2}x + 2$and cutting the y = *axis* at 3 is:

$y = \frac{3}{2}x + \frac{3}{2}$

c) $3x - 2y + 4 = 0$

$y = \frac{3}{2}x + 2$

The line perpendicular to$y = \frac{3}{2}x + 2$ has a gradient of $-\frac{2}{3}$

$y = -\frac{2}{3}x + c$

$0 = -\frac{2}{3}(2) + c$

$c = \frac{4}{3}$

So the equation the line perpendicular to $y = \frac{3}{2}x + 2$ cutting the x-axis at 2 is $y = -\frac{2}{3}x + \frac{4}{3}$

5. a) $x = -1$

b) $y = 8$

c) $y = 3x + 11$

d) $y = -\frac{1}{3}x + \frac{23}{3}$

CHAPTER 5 Functions and Graphs

Exercise 5.1

1. a) $f(x) = x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	16	9	4	1	0	1	4	9	16

3. a)

		$f(x) = x^2$	$h(x) = 2x^2$	$g(x) = 3x^2$
(i)	Value of	$a = 1$ $b = 0$ $c = 0$	$a = 2$ $b = 0$ $c = 0$	$a = 3$ $b = 0$ $c = 0$
(ii)	x-intercepts	(0;0)	(0;0)	(0;0)
(iii)	y-intercept	(0;0)	(0;0)	(0;0)
(iv)	Co-ordinates of T.P	(0; 0)	(0;0)	(0;0)
(v)	Axis of symmetry	$x = 0$	$x = 0$	$x = 0$
(vi)	Domain	$x \in R$ or $(-\infty; \infty)$	$x \in R$ or $(-\infty; \infty)$	$x \in R$ or $(-\infty; \infty)$
(vii)	Range	y≥0 or (0; ∞)	y≥0 or (0; ∞)	y≥0 or (0; ∞)

b) (i) The graphs are the same in the following aspects;
x-intercepts; y-intercepts; co-ordinates of the TP; Axis of symmetry; Domain and range

(ii) The graphs are different in the following aspects;
The steepness and the width of the graphs

4. Increasing the value of a makes the graph narrower and steeper

Exercise 5.2

1. a) $f(x) = x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	16	9	4	1	0	1	4	9	16

b) $f(x) = \frac{1}{2}x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	8	4.5	2	0.5	0	0.5	2	4.5	8

c) $f(x) = \frac{1}{3}x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	5.33	3	1.33	0.33	0	0.33	1.33	3	5.33

3. As the value of a decreases, the graphs get wider and less steep.

Exercise 5.3

b)

		$f(x) = x^2$	$f(x) = -x^2$	$f(x) = -2x^2$	$f(x) = -3x^2$
(i)	Value of	$a = 1$ $b = 0$ $c = 0$	$a = -1$ $b = 0$ $c = 0$	$a = -2$ $b = 0$ $c = 0$	$a = -3$ $b = 0$ $c = 0$
(ii)	x-intercepts	(0;0)	(0;0)	(0;0)	(0;0)
(iii)	y-intercept	(0;0)	(0;0)	(0;0)	(0;0)
(iv)	Co-ordinates of T.P	(0; 0)	(0; 0)	(0;0)	(0;0)
(v)	Axis of symmetry	$x = 0$	$x = 0$	$x = 0$	$x = 0$
(vi)	Domain	$x \in R$ or $(-\infty; \infty)$	$x \in R$ or $(-\infty; \infty)$	$x \in R$ or $(-\infty; \infty)$	$x \in R$ or $(-\infty; \infty)$
(vii)	Range	$y \geq 0$ or $(0; \infty)$	$y \leq 0$ or $(0; \infty)$	$y \leq 0$ or $(0; \infty)$	$y \leq 0$ or $(0; \infty)$

Exercise 5.4

1. a) $f(x) = -x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	–16	–9	–4	–1	0	–1	–4	–9	–16

b) $f(x) = -\frac{1}{2}x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	–8	–4.5	–2	–0.5	0	–0.5	–2	–4.5	–8

c) $f(x) = -\frac{1}{3}x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	–5.33	–3	–1.33	–0.33	0	–0.33	–1.33	–3	–5.33

3. a)

		$f(x) = x^2$	$f(x) = \frac{1}{2}x^2$	$f(x) = \frac{1}{3}x^2$
(i)	Value of	$a = 1$ $b = 0$ $c = 0$	$a = \frac{1}{2}$ $b = 0$ $c = 0$	$a = \frac{1}{3}$ $b = 0$ $c = 0$
(ii)	x-intercepts	(0;0)	(0;0)	(0;0)
(iii)	y-intercept	(0;0)	(0;0)	(0;0)
(iv)	Co-ordinates of T.P	(0; 0)	(0;0)	(0;0)
(v)	Axis of symmetry	$x = 0$	$x = 0$	$x = 0$
(vi)	Domain	$x \in R$ or $(-\infty; \infty)$	$x \in R$ or $(-\infty; \infty)$	$x \in R$ or $(-\infty; \infty)$
(vii)	Range	$y \leq 0$ or $(0; \infty)$	$y \leq 0$ or $(0; \infty)$	$y \leq 0$ or $(0; \infty)$

Exercise 5.6

1. a) $s(x) = -2x^2 - 3x$

x	–4	–3	–2	–1	0	1	2	3	4
$s(x)$	–20	–9	–2	1	0	–5	–14	–27	–44

b) $r(x) = -2x^2 - 2x$

x	–4	–3	–2	–1	0	1	2	3	4
$r(x)$	–24	–12	–4	0	0	–4	–12	–24	–40

c) $p(x) = -2x^2 - x$

x	–4	–3	–2	–1	0	1	2	3	4
$p(x)$	–28	–15	–6	–1	0	–3	–10	–21	–36

d) $f(x) = -2x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$f(x)$	–32	–18	–8	–2	0	–3	–8	–18	–32

e) $q(\mathbf{x}) = -2x^2 + x$

x	–4	–3	–2	–1	0	1	2	3	4
$p(x)$	–36	–21	–10	–3	0	–1	–6	–15	–28

f) $r(x) = -2x^2 + 2x$

x	–4	–3	–2	–1	0	1	2	3	4
$r(x)$	–40	–24	–12	–4	0	0	–4	–12	–24

g) $w(x) = -2x^2 + 3x$

x	–4	–3	–2	–1	0	1	2	3	4
$w(x)$	–44	–27	–14	–5	0	1	–2	–9	–20

Exercise 5.12

4. The value of p shifts the graph of $f(x) = -2x^2$horizontally. If p is positive, the graph shifts p units to the right and if p is negative, the graph shifts p units to the left.

Exercise 5.14

1. a) $f(x) = 2x^2$

x	–4	–3	–2	–1	0	1	2	3	4
$w(x)$	32	18	8	–5	0	1	–2	–9	–20

b) $g(x) = 2(x - 1)^2 - 1$

x	–4	–3	–2	–1	0	1	2	3	4
$w(x)$	49	31	17	7	1	–1	1	7	17

c) $h(x) = 2(x-1)^2 - 2$

x	–4	–3	–2	–1	0	1	2	3	4
$w(x)$	48	30	16	6	0	–2	0	6	16

3.

		$f(x) = 2x^2$	$g(x) = 2(x-1)^2 - 1$	$h(x) = 2(x-1)^2 - 2$
	Value of	$a = 1$ $p = 0$ $q = 0$	$a = 2$ $p = -1$ $q = -1$	$a = 2$ $p = -1$ $q = -2$
a)	x-intercepts	(0;0)	(1.71; 1)	(0.29; 0)
b)	y-intercept	(0;0)	(0;1)	(0;0)
c)	Co-ordinates of T.P	(0; 0)	(1; –1)	(1; –2)
d)	Axis of symmetry	$x = 0$	$x = 1$	$x = 1$
e)	Domain	$x \in R$ or $(-\infty; \infty)$	$x \in R$ or $(-\infty; \infty)$	$x \in R$ or $(-\infty; \infty)$
f)	Range	$y \geq 0$ or $(0; \infty)$	$y \geq -1$ or $(-1; \infty)$	$y \geq -2$ or $(-2; \infty)$

Exercise 5.16

4. The negative value of q shifts the graph of $f(x) = -2x^2$ downwards

5.

	x–intercept(s)	y-intercept	Axis of symmetry	Co-ordinates of the turning point	Domain	Range	Function
a)	$x = -3$ or $x = 1$	$y = -6$	$x = -1$	$(-1; -2)$	$x \in R$	$y \geq -2$	$y = \frac{1}{2}(x+1)^2 - 2$
b)	$x = -6$ or $x = -2$	$y = -24$	$x = -4$	$(-4; 8)$	$x \in R$	$y \leq 8$	$y = -2(x+4)^2 + 8$
c)	$x = -4$ or $x = 4$	$y = -16$	$x = 0$	$(0; -16)$	$x \in R$	$y \geq -16$	$y = x^2 - 16$
d)	$x = -2$ or $x = 5$	$y = 10$	$x = -\frac{3}{2}$	$\left(-\frac{3}{2}; \frac{49}{4}\right)$	$x \in R$	$y \leq \frac{49}{4}$	$y = -x^2 + 3x + 10$
e)	$x = -3$ or $x = 1$	$y = -6$	$x = -1$	$(-1; -8)$	$x \in R$	$y \geq -8$	$y = 2(x+1)^2 - 8$
f)	$x = -4$ or $x = -2$	$y = -16$	$x = -3$	$(-3; 2)$	$x \in R$	$y \leq 2$	$y = -2(x+3)^2 + 2$

Exercise 5.17

1. a) The new equation is:

   $g(x) = (x+5)^2$

   b) The co-ordinates of the turning point of g are (–5;0)

2. a) The new equation is:

   $g(x) = (x+5)^2 + 2$

   b) $(-5; 2)$

3. a) The new equation is:

   $g(x) = (x+5)^2 - 0,25$

   b) TP $(-5; -0,25)$

Exercise 5.21

1. a) $g(x) = 3^x + 2$

x	–6	–5	–4	–3	–2	1	2	3	4	5	6
$g(x)$	2.00	2.00	2.01	2.04	2.11	5	11	29	83	245	731

b) $q(x) = 4^x - 1$

x	–6	–5	–4	–3	–2	1	2	3	4	5	6
$q(x)$	-0.9998	-0.999	-0.996	-0.984	-0.938	3	15	63	255	1023	4095

c) $y = 3{,}2^x + 1{,}5$

x	–6	–5	–4	–3	–2	1	2	3	4	5	6
y	1.547	1.594	1.688	1.875	2.25	7.5	13.5	25.5	49.5	97.5	193.5

d)

	–6	–5	–4	–3	–2	1	2	3	4	5	6
	–2.997	–2.992	–2.975	–2.926	–2.778	3	15	51	159	4831455	

3.

	Function	Intercepts	Asymptotes	Domain	Range
a)		(0;1)	$y = 0$	$x \varepsilon R$ or $(-\infty; \infty)$	$y>0$ or $y \varepsilon (0; \infty)$
		(0;3)	$y = 2$	$x \varepsilon R$ or $(-\infty; \infty)$	$y>2$ or $y \varepsilon (2; \infty)$
b)		(0;1)	$y = 0$	$x \varepsilon R$ or $(-\infty; \infty)$	$y>0$ or $y \varepsilon (0; \infty)$
		(0;0)	$y = -1$	$x \varepsilon R$ or $(-\infty; \infty)$	$y>-1$ or $y \varepsilon (-1; \infty)$
c)		(0;0)	$y = 0$	$x \varepsilon R$ or $(-\infty; \infty)$	$y>0$ or $y \varepsilon (0; \infty)$
		(0;3)	$y = 0$	$x \varepsilon R$ or $(-\infty; \infty)$	$y>0$ or $y \varepsilon (0; \infty)$
		(0;4.5)	$y = 1.5$	$x \varepsilon R$ or $(-\infty; \infty)$	$y>1.5$ or $y \varepsilon (1.5; \infty)$
d)		(0;0)	$y = 0$	$x \varepsilon R$ or $(-\infty; \infty)$	$y>0$ or $y \varepsilon (0; \infty)$
		(0;0)	$y = 0$	$x \varepsilon R$ or $(-\infty; \infty)$	$y>0$ or $y \varepsilon (0; \infty)$
		(0;–1)	$y = -3$	$x \varepsilon R$ or $(-\infty; \infty)$	$y>-3$ or $y \varepsilon (-3; \infty)$

Exercise 5.22

1. a)

	–6	–5	–4	–3	–2	1	2	3	4	5	6
	–1/64	–1/32	–1/16	–1/8	–1/4	–2	–4	–8	–16	–32	–64

b)

–6	–5	–4	–3	–2	1	2	3	4	5	6
2.00	2.00	1.99	1.96	1.89	–1	–7	–25	–79	–241	–727

c)

–6	–5	–4	–3	–2	1	2	3	4	5	6
–1.00	–1.00	–1.00	–1.02	–1.06	–5	–17	–65	–257	–1025	–4097

d)	–6	–5	–4	–3	–2	1	2	3	4	5	6
	1.45	1.41	1.31	1.13	0.75	–4.5	–10.5	–22.5	–46.5	–94.5	–190.5

e)	–6	–5	–4	–3	–2	1	2	3	4	5	6
	–3.00	–3.01	–3.02	–3.07	–3.22	–9	–21	–57	–165	–489	–1461

3.

	Function	Intercepts	Asymptotes	Domain	Range
a)		(0;1)	$y = 0$	$x \in R$ or $(-\infty; \infty)$	$y>0$ or $y \in (0; \infty)$
		(0;–1)	$y = 0$	$x \in R$ or $(-\infty; \infty)$	$y>0$ or $y \in (0; \infty)$
b)		(0;1)	$y = 0$	$x \in R$ or $(-\infty; \infty)$	$y>0$ or $y \in (0; \infty)$
		(0;1)	$y = 2$	$x \in R$ or $(-\infty; \infty)$	$y<2$ or $y \in (-\infty; 2)$
c)		(0;1)	$y = 0$	$x \in R$ or $(-\infty; \infty)$	$y>0$ or $y \in (0; \infty)$
		(0;–2)	$y = -1$	$x \in R$ or $(-\infty; \infty)$	$y<-1$ or $y \in (-\infty; 1)$
d)		(0;1)	$y = 0$	$x \in R$ or $(-\infty; \infty)$	$y>0$ or $y \in (0; \infty)$
		(0;–3)	$y = 0$	$x \in R$ or $(-\infty; \infty)$	$y<0$ or $y \in (-\infty; 0)$
		(0;1.5)	$y = 1,5$	$x \in R$ or $(-\infty; \infty)$	$y<1.5$ or $y \in (-\infty; 1.5)$
e)		(0;1)	$y = 0$	$x \in R$ or $(-\infty; \infty)$	$y>0$ or $y \in (0; \infty)$
		(0;–2)	$y = 0$	$x \in R$ or $(-\infty; \infty)$	$y<0$ or $y \in (-\infty; 0)$
		(0;–5)	$y = -3$	$x \in R$ or $(-\infty; \infty)$	$y<-3$ or $y \in (-\infty; -3)$

Exercise 5.23

2.

	Function	Domain	Range	Horizontal asymptote
a)		$x \in R$ or $(-\infty;\infty)$	$y>3$ or $y \in (3; \infty)$	$y = 3$
b)		$x \in R$ or $(-\infty;\infty)$	$y>3$ or $y \in (3; \infty)$	$y = 3$
c)		$x \in R$ or $(-\infty;\infty)$	$y>3$ or $y \in (3; \infty)$	$y = 3$
d)		$x \in R$ or $(-\infty;\infty)$	$y>-3$ or $y \in (-3; \infty)$	$y = -3$
e)		$x \in R$ or $(-\infty;\infty)$	$y>-3$ or $y \in (-3; \infty)$	$y = -3$
f)		$x \in R$ or $(-\infty;\infty)$	$y>-3$ or $y \in (-3; \infty)$	$y = -3$

Exercise 5.24

2.

	Function	Domain	Range	Lines of symmetry	Asymptote(s)
a)		$x \in R; x \neq 0$	$y \in R; y \neq 0$	$y = x$ and $y = -x$	$x = 0$ $y = 0$
b)		$x \in R; x \neq 0$	$y \in R; y \neq 0$	$y = x$ and $y = -x$	$x = 0$ $y = 0$
c)		$x \in R; x \neq 0$	$y \in R; y \neq 0$	$y = x$ and $y = -x$	$x = 0$ $y = 0$
d)		$x \in R; x \neq 0$	$y \in R; y \neq 0$	$y = x$ and $y = -x$	$x = 0$ $y = 0$
e)		$x \in R; x \neq 0$	$y \in R; y \neq 0$	$y = x$ and $y = -x$	$x = 0$ $y = 0$
f)		$x \in R; x \neq 0$	$y \in R; y \neq 0$	$y = x$ and $y = -x$	$x = 0$ $y = 0$
g)		$x \in R; x \neq 0$	$y \in R; y \neq 0$	$y = x$ and $y = -x$	$x = 0$ $y = 0$
h)		$x \in R; x \neq 0$	$y \in R; y \neq 0$	$y = x$ and $y = -x$	$x = 0$ $y = 0$
i)		$x \in R; x \neq 0$	$y \in R; y \neq 0$	$y = x$ and $y = -x$	$x = 0$ $y = 0$

CHAPTER 6 Euclidean (Circle) Geometry

Exercise 6.1

1.

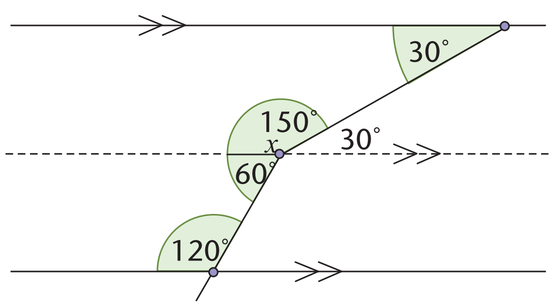
	Complement	Supplement	Difference between supplement and complement
17°	73°	163°	90°
80°	10°	100°	90°
29°	61°	151°	90°
171°	–81°	9°	90°
5°	85°	175°	90°
23°	67°	157°	90°
37°	53°	143°	90°
113°	–23°	67°	90°
90°	0°	90°	90°
0°	90°	180°	90°

2. 90°, the difference between them is always 90°
3. a) $x + 50^\circ + 70^\circ = 180^\circ$(∠'s on a straight line)
   $\therefore x = 60^{\circ\circ}$

   b) $3x = 180^\circ$(∠'s on a straight line)
   $\therefore x = 60^\circ$

Exercise 6.2

1. a) Construct as per diagram below



$\therefore x = 150^\circ + 60^\circ = 210^\circ$

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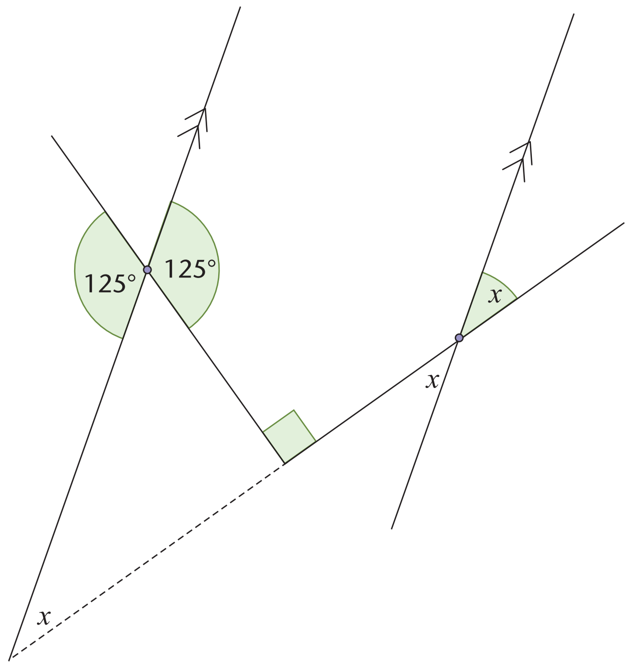
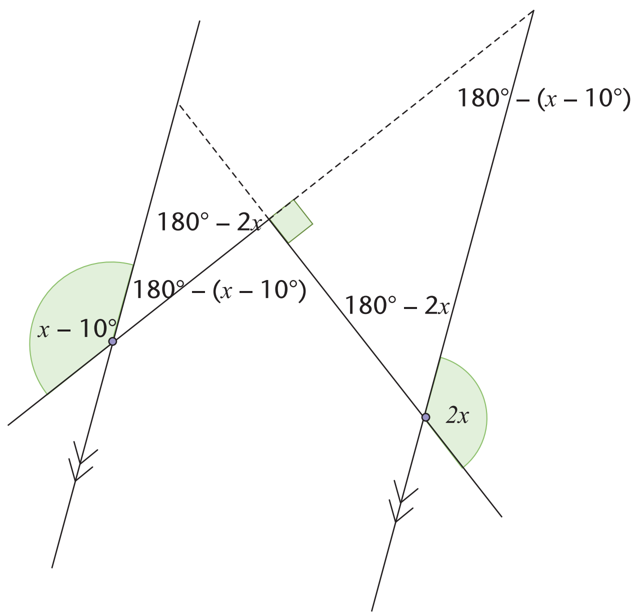
b) Construct as per diagram below



$$x + 55^{\circ} = 90^{\circ}(\text{complementary } \angle\text{'s})$$
$$\therefore x = 35^{\circ}$$

c) Construct as per diagram below



$$180^{\circ} - x + 10^{\circ} + 180^{\circ} - 2x = 90^{\circ}(\text{complementary } \angle\text{'s})$$
$$-3x = 280^{\circ}$$
$$\therefore x = 93{,}33^{\circ}$$

2. $b = 42^\circ$ (vert opp $\angle$'s $=$)

$c = 180^\circ - 42^\circ = 138^\circ$ ($\angle$'s on a straight line)

$y = b = 42^\circ$ (alt $\angle$'s $=$)

$r = y = 42^\circ$ (vert opp $\angle$'s $=$)

$(r + w) + 69^\circ = 180^\circ$ (co-interior $\angle$'s)

$w + 42^\circ + 69^\circ = 180^\circ$

$\therefore w = 69^\circ$

$a = w = 69^\circ$ (alt $\angle$'s $=$)

$t = w = 69^\circ$ (vert opp $\angle$'s $=$)

$x = 69^\circ$ ($\angle$'s on a straight line)

$s = x = 69^\circ$ (vert opp $\angle$'s $=$)

$z = 42^\circ$ ($\angle$'s on a straight line)

$v = z = 42^\circ$ (alt $\angle$'s $=$)

$u = 138^\circ$ ($\angle$'s on a straight line)

$q = u = 138^\circ$ (vert opp $\angle$'s $=$)

Exercise 6.3

1. a) $\hat{C}_1 = 30^\circ$ (alt $\angle$'s $=$)

b) $\hat{A}_1 = \frac{180^\circ - 30^\circ}{2} = 75^\circ$ (sum of $\angle$'s add up to 180°)

c) $\hat{B} = 75^\circ$ (base $\angle$'s isoceles $\Delta =$)

d) $\hat{D}_1 = 52{,}5^\circ$ (alt $\angle$'s $=$)

e) $\hat{D}_2 = 52{,}5^\circ$ (base $\angle$'s isoceles $\Delta =$)

f) $\hat{E}_1 = 75^\circ$ (sum of $\angle$'s of Δ add up to 180°)

g) $\hat{F}_1 = 42{,}8^\circ$ (sum of $\angle$'s of Δ add up to 180°)

2. $B\hat{A}F + \hat{E}_1 = 105^\circ + 75^\circ = 180^\circ$

$\therefore AB \parallel ED$ (co-interior $\angle$'s are supplementary)

3. Parallelogram, because opposite angles are equal

4. $x = 24^\circ$ (alt $\angle$'s $=$)

$y + 24^\circ + 112^\circ = 180^\circ$ (sum of $\angle$'s of Δ)

$\therefore y = 44^\circ$

Exercise 6.4

1. Circle and circular object have curved edges
2. Tyres
3. Subcompact cars, Compact cars, mid-size cars, full-size cars
4. To allow easy and smooth access.

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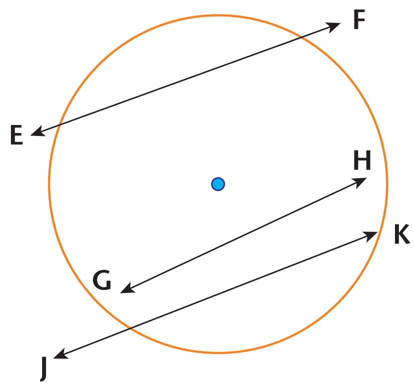
Exercise 6.5

1. a) radius of circle $K = 7$ units
   b) PM and PN
   c) NKP
   d) $D = 2 \times r = 2 \times 7 = 14$ units
   e) QMR
   f) ST

2.



3. a) widths = l – m = 7 – 3,2 = 3,2 mm
   b) concentric circles

Exercise 6.6

1. a) Learners own measurements. Learners should discover that $DE = EB$. Point E is called a midpoint of DB)
   b) $A\hat{E}D = A\hat{E}B = 90^\circ$ (The line segment AE is perpendicular to DB)
   c) A line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord)
2. Yes. A line drawn from the centre of a circle to the midpoint of a chord is perpendicular to the chord)

Exercise 6.7

1. Distance between MN and PQ $= \sqrt{5^2 - 4^2} + \sqrt{5^2 - 3^2}$
   $= 3 + 7$
   $= 10$ cm

2. $x = \sqrt{10^2 - 6^2}$
   $x = 8$ cm
   $SV = 2 \times \sqrt{172 - 82}$
   $= 30$ cm

$$ST = \frac{SV - TU}{2}$$

$$= \frac{30 - 12}{2}$$

$$= 9 \text{ cm}$$

3. a) $r = \sqrt{OS^2 + AS^2}$

$r = \sqrt{6^2 + 8^2}$

$\therefore r = 10 \text{ cm}$

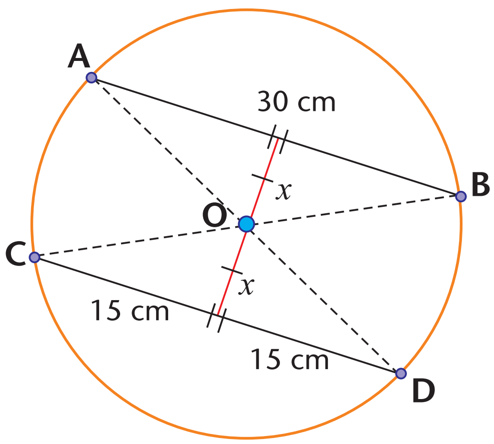
b) $CD = 2 \times \sqrt{10^2 - 8^2}$

$CD = 2 \times 6$

$CD = 12 \text{ cm}$

Exercise 6.8

1.



$x = \sqrt{17^2 - 15^2}$

$\therefore x = 8 \text{ cm}$

This is true for any circle

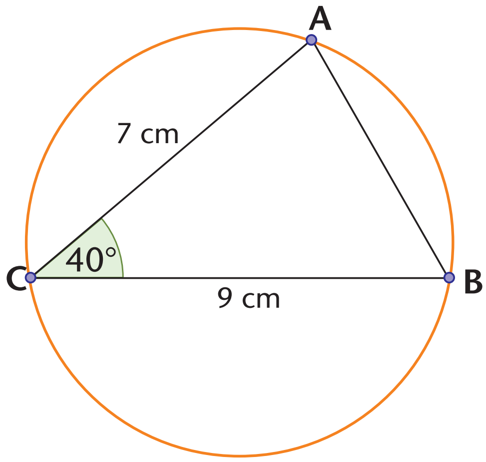
2. $QT = MS = TR = SN = \sqrt{13^2 - 5^2} = 12 \text{ cm}$

This is true for any circle

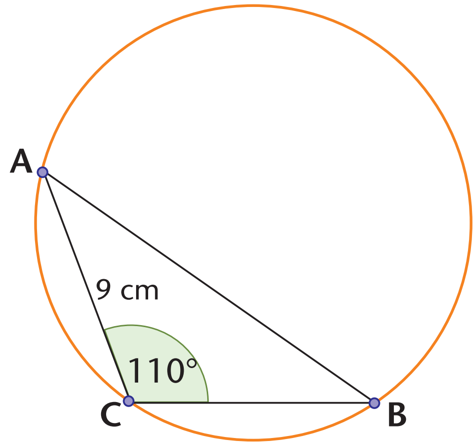
3. Yes

Exercise 6.9

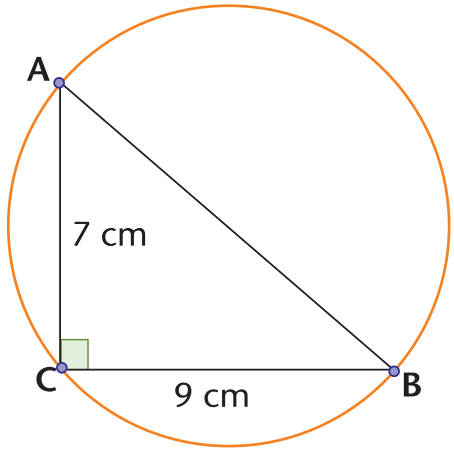
a)



b)



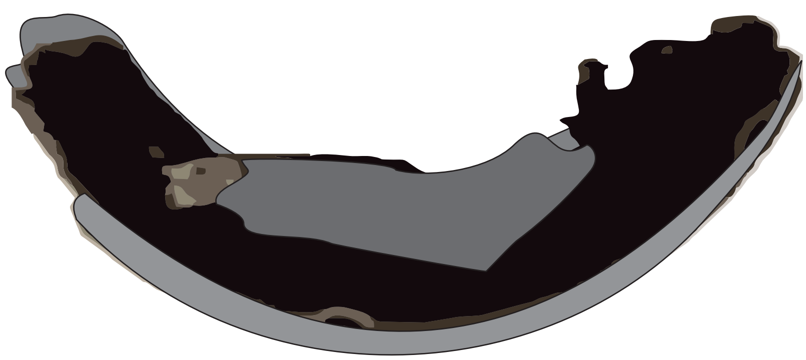
c)



Exercise 6.10

1. The radius is the same the height of the roof as measured from the horizontal length of the roof)

2.



Exercise 6.11

1. a) In circle *M*, point *M* all point *L*, *J* and *K* lie on the circle with radii *JM* and *MK*. In circle *I*, point *H*, *F* and *G* lie on the circle and $FE \neq EG$ (*E* is not the centre of the circle) In circle *O*, point *K* does not lie of the circle)

   b) Learners compare their responses with classmates.

2.

	Name of a circle	Angle at the centre	Angle at the circumference
1.	Circle M	120°	60°
2.	Circle N	60°	30°
3.	Circle P	200°	100°
4.	Circle Q	88°	44°
5.	Circle I	120°	47°
6.	Circle O	120°	69°

   a) yes, $x = 2y$

   b) The angle subtended at the centre of a circle by an arc is two times the angle subtended by the same arc at the circumference of a circle)

   c) No.

Exercise 6.12

a) $x = \frac{104^\circ}{2} = 52^\circ$($\angle$'s at centre is $2 \times \angle$ at circ)

b) $x = \frac{360^\circ - 228^\circ}{2} = 66^\circ$($\angle$'s at centre is $2 \times \angle$ at circ)

c) $x = \frac{360^\circ - 130^\circ}{2} = 115^\circ$($\angle$'s at centre is $2 \times \angle$ at circ)

d) $x = 52^\circ$(alt $\angle$'s =)

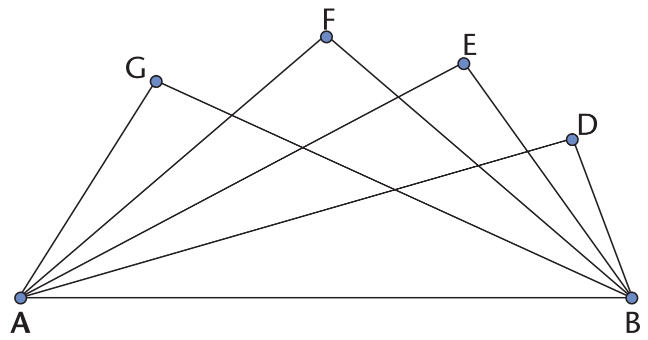
$y = BAC = \frac{1}{2}52^\circ = 26^\circ$($\angle$'s at centre is $2 \times \angle$ at circ)

e) $x = 2B\hat{C}A = 2(2y^\circ) = 4y^\circ$

f) $x = \frac{1}{2}(A\hat{O}C) = \frac{1}{2}(180^\circ - (2 \times 42^\circ)) = 48^\circ$

Exercise 6.13

a)



b) The vertices of a triangle are equidistant from the *circumcentre)*

Exercise 6.14

a) $2x + 60^\circ = 180^\circ$ (sum of $\angle$'s of Δ)

$\therefore x = 60^\circ$

$\hat{C} = 90^\circ$ (angle subtended by diameter)

$y = 90^\circ - 35^\circ = 55^\circ$(comp $\angle$s)

$z = 2x = 120^\circ$($\angle$ at centre is $2 \times \angle$ at circ)

b) $z = \frac{1}{2} \times 102^\circ = 51^\circ$ ($\angle$ *at centre* is $2 \times \angle$ *at circ*)

$x + y = 90^\circ$($\angle$ subtended by diameter)

$\hat{A} = 90^\circ - 51^\circ = 39^\circ$(comp $\angle$s)

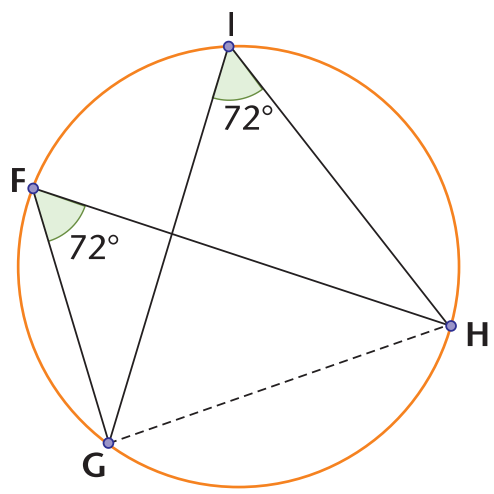
$x = 180^\circ - (102^\circ + 39^\circ)$

$x = 39^\circ$

$y - z = 51^\circ$ (base $\angle$s of Δ)

Exercise 6.15

a) Refer to the diagram below



b) $G\hat{F}H = G\hat{I}H$

c) Refer to the diagram above

d) Yes

e) Should get the same results

Exercise 6.16

a) $y = 42^\circ$ ($\angle$s subtended by the same arc MN)

$x = 35^\circ$($\angle$s subtended by the same arc KL)

b) $TRQ = 34^\circ y = 42^\circ$($\angle$s subtended by the same arc QT)

$butx = TRQ = 34\,^\circ y = 42^\circ$(alt $\angle$=)

c) $x = 44^{\circ}$($\angle$s subtended by the same chord RS)

$y = \hat{S}_1$($\angle$s subtended by the same chord QT)

but $\hat{S}_1 + 98^{\circ} + x = 180^{\circ}$(sum of $\angle$ of Δ)

$\therefore y = 38^{\circ}$

d) $x = 50^{\circ}$($\angle$s at centre is $2 \times \angle$ at circ)

$x + y = 90^{\circ}$($\angle$ subtended by diameter DF)

$y = 40^{\circ}$

$z = y = 40^{\circ}$($\angle$ subtended by chord EF)

e) $z = w$ and $x = y$($\angle$s subtend by the same chord)

$z = w = x + 34^{\circ}$(ext $\angle$s of Δ)

$w + x + 108^{\circ}$(sum of $\angle$s of Δ)

$x + 34^{\circ} + x + 108^{\circ} = 180^{\circ}$

$2x = 38^{\circ}$

$\therefore x = 19^{\circ}$

$y = 19^{\circ}$

$z = w = 19^{\circ} + 34^{\circ} = 53^{\circ}$

f) $p = 19^{\circ}$($\angle$s subtended by the same chord)

$r = 180^{\circ} - (19^{\circ} + 120^{\circ}) = 41^{\circ}$(sum of $\angle$s of Δ)

$q = 180^{\circ} - 120^{\circ} + 19^{\circ} = 41^{\circ}$(sum of $\angle$s of Δ)

$r + s = 90^{\circ}$($\angle$s subtended by diameter DF)

$s = 49^{\circ}$

$t = 2 \times s$

$= 2 \times 49^{\circ}$

$t = 98^{\circ}$

Exercise 6.17

a) $\hat{O}_1 = 220^{\circ}$($\angle$ at centre is $2 \times \angle$ at circ)

$\hat{O}_2 = 128^{\circ}$(revolution)

$\hat{C} = 70^{\circ}$

$\hat{A} + \hat{C} = 110^{\circ} + 70^{\circ} = 180^{\circ}$

b) $\hat{O}_1 = 232^{\circ}$($\angle$ at centre is $2 \times \angle$ at circ)

$\hat{O}_2 = 128^{\circ}$(revolution)

$\hat{C} = 64^{\circ}$

$\hat{A} + \hat{C} = 116^{\circ} + 64^{\circ} = 180^{\circ}$

c) $\hat{O}_1 = 268^{\circ}$($\angle$ at centre is $2 \times \angle$ at circ)

$\hat{O}_2 = 92^{\circ}$(revolution)

$\hat{C} = 46^{\circ}$

$\hat{A} + \hat{C} = 134^{\circ} + 46^{\circ} = 180^{\circ}$

d) $\hat{O}_1 = 144^\circ$($\angle$ at centre is $2 \times \angle$ at circ)

$\hat{O}_2 = 216^\circ$(revolution)

$\hat{C} = 108^\circ$

$\hat{A} + \hat{C} = 72^\circ + 108^\circ = 180^\circ$

e) $\hat{O}_1 = 2x$($\angle$ at centre is $2 \times \angle$ at circ)

$\hat{O}_2 = 360 - 2x$(revolution)

$\hat{C} = 180^\circ - x$

$\hat{A} + \hat{C} = x + 180^\circ - x = 180^\circ$

Exercise 6.18

a) $x = 108^\circ$ (opp $\angle$ of cyclic quad)

$y = 78^\circ$ (opp $\angle$ of cyclic quad)

b) $x = 110^\circ$ (opp $\angle$ of cyclic quad)

$y = 180^\circ - (46^\circ + 79^\circ) = 55^\circ$ (sum of $\angle$ of Δ)

c) $y = 60^\circ$ (opp $\angle$ of cyclic quad)

$2x = 180^\circ - 120^\circ$(base $\angle$s of Δ = ; sum of $\angle$s of Δ)

$\therefore x = 10^\circ$

$2z = 180^\circ - y$(base $\angle$s of Δ = ; sum of $\angle$s of Δ)

$2z = 120^\circ$

$\therefore z = 60^\circ$

d) $B = 105^\circ$(opp $\angle$ of cyclic quad)

$x = 75^\circ$(co-int $\angle$s)

e) $x = 102^\circ$(opp $\angle$ of cyclic quad)

$4y = 180^\circ$(opp $\angle$ of cyclic quad)

$\therefore y = 45^\circ$

f) $y = 90^\circ$(given)

$z = 33^\circ$(complementary $\angle$s)

$x = 123^\circ$(opp $\angle$s of cyclic quad)

g) $x = 32^\circ$($\angle$s subtended by chord WZ)

$Z\hat{W}Y = 56^\circ$($\angle$s subtended by chord ZY)

$y + Z\hat{W}Y + x = 180^\circ$(sum of $\angle$s of Δ)

$y + 56^\circ + 32^\circ = 180^\circ$

$\therefore y = 92^\circ$

h) $x = 87^\circ$(opp $\angle$s of cyclic quad)

$E\hat{B}C = 87^\circ$($\angle$s on a straight line)

$\therefore z = 103^\circ$(opp $\angle$s of cyclic quad)

$DEB = 103^\circ$(alt $\angle$s =)

$y = 87^\circ$(opp $\angle$s of cyclic quad)

Exercise 6.19

1. $E\hat{B}C = \hat{C}$(ext ∠s of cyclic quad)

   $3x = \hat{C}$

   but, $\hat{C} + \hat{D} = 180^\circ$(co-int ∠s)

   $3x + x + 68^\circ = 180^\circ$

   $4x = 112^\circ$

   $\therefore x = 28^\circ$

2. a) $x = 110^\circ$(ext ∠s of cyclic quad)

   b) $a = 87^\circ$(ext ∠s of cyclic quad)

   c) $a = 27^\circ$(ext ∠s of cyclic quad)

   $b = 90^\circ$(∠ subtended by diameter EH)

   $d = 180^\circ - 127^\circ = 53^\circ$(∠s on a straight line)

   $c = 90^\circ - 53^\circ = 37^\circ$(complementary ∠s)

   d) $x = 103^\circ$(ext ∠s of cyclic quad)

   $y = 103^\circ$(opp ∠s of cyclic quad)

   e) $x = 110^\circ$(ext ∠s of cyclic quad)

   $y = 115^\circ$(ext ∠s of cyclic quad)

   f) $a = 83^\circ$(ext ∠s of cyclic quad)

   $b = 79^\circ$(ext ∠s of cyclic quad)

Exercise 6.20

1. a) $x^2 = 20^2 + 15^2$ (tangent ⊥ radius)

   $\therefore x = \sqrt{20^2 + 15^2}$

   $\therefore x = 25$ units

   b) $(x + 30)^2 = 40^2 + 30^2$ (Pythagoras' thm, tangent ⊥ radius)

   $x + 30 = 50$

   $\therefore x = 20$ units

   c) $x^2 = 10^2 - 8^2$ [(tangent ⊥ radius); Pythagoras' thm]

   $\therefore x = \sqrt{100 - 64}$

   $\therefore x = 6$ units

2. a) $x = 51^\circ$(tan ⊥ rad)

   b) $O\hat{P}Q = 33^\circ$(base ∠s of Δ)

   $b + O\hat{P}Q = 90^\circ$(tan ⊥ rad)

   $b + 33^\circ = 90^\circ$

   $b = 57^\circ$

c) $2c = 180^\circ - 64^\circ$(base $\angle$s of Δ; sum of $\angle$s of Δ)

$c = 58^\circ$

Exercise 6.21

a) $\hat{D}_3 = a$ (base $\angle$ of Δ)

$2a = 180^\circ - 59^\circ$

$\therefore a = 60,5^\circ a + b = 90^\circ$ (tangent $\perp$ radius)

$b = 90^\circ - 60,5^\circ$

$\therefore b = 29,5^\circ$

$c = 180^\circ - (2 \times 29,5^\circ)$ (sum of $\angle$s of Δ)

$\therefore$ c = 121°

b) $x = 30^\circ$(base $\angle$s of Δ)

$y = 120^\circ$(sum of $\angle$s of Δ)

$z = 60^\circ$(sum of $\angle$s of Δ)

Exercise 6.22

a) $x = 70^\circ$(tan-chord thm)

$y = 50^\circ$(tan-chord thm)

b) $x = 97^\circ$(tan-chord thm)

$y = 50^\circ$(tan-chord thm)

c) $x = 50^\circ$(tan-chord thm)

$\hat{H} = 71^\circ$(opp $\angle$s of a cyclic quad)

$y = 180^\circ - (71 + 50^\circ)$

$\therefore y = 51^\circ$

d) $\hat{K} = x$ (base $\angle$s of Δ)

$\therefore x = 63^\circ$(tan-chord thm)

$y = 180^\circ - (2 \times 63^\circ)$(sum of $\angle$s of Δ)

$\therefore y = 54^\circ$

e) $x = 33^\circ$ (tan-chord thm)

$y = x = 33^\circ$($\angle$s subtended by chord QN)

f) $y = 7x$(tan-chord thm)

$x = \frac{y}{7}$

CHAPTER 7 Circles, angles and angular movement

Exercise 7.1

1. a) $A = \pi r^2 = 3,14 \times \dfrac{13,12}{2}^2 = 135,13\ \text{cm}^2$

   circumference $= 2\pi r = 2 \times 3,14 \times \dfrac{13,12}{2} = 41,20\ \text{cm}$

   b) $A = \pi r^2 = 3,14 \times 3,87^2 = 47,03\ \text{cm}^2$

   circumference $= 2\pi r = 2 \times 3,14 \times 3,87 = 24,30\ \text{cm}$

   c) $A = \pi r^2 = 3,14 \times \dfrac{24,6}{2}^2 = 475,05\ \text{cm}^2$

   circumference $= 2\pi r = 2 \times 3,14 \times \dfrac{24,6}{2} = 77,24\ \text{cm}$

   d) $A = \pi r^2 = 3,14 \times \dfrac{3,14}{4}^2 = 1,93\ \text{cm}^2$

   circumference $= 2\pi r = 2 \times 3,14 \times \dfrac{3,14}{4} = 4,93$

   e) Area $= 3,14 \times (2\sqrt{5})^2 = 62,8\ \text{cm}^2$

   circumference $= 2 \times 3,14 \times 2\sqrt{5} = 28,09\ \text{cm}$

   f) Area $= 3,14 \times \left(\dfrac{\sqrt[3]{5}}{2}\right)^2 = 2,30\ \text{cm}^2$

   circumference $= 2 \times 3,14 \times \dfrac{\sqrt[3]{5}}{2} = 5,37\ \text{cm}$

2. area of topping $= \pi \left(\dfrac{38,72}{2}\right)^2 = 1176,90\ \text{cm}^2$

Exercise 7.2

a) $30° \times \left(\dfrac{\pi \text{ radians}}{180°}\right)$

$\therefore\ 30° = \dfrac{\pi}{6}$ radians

b) $60° \times \left(\dfrac{\pi \text{ radians}}{180°}\right)$

$\therefore\ 60° = \dfrac{\pi}{3}$ radians

c) $90° \times \left(\dfrac{\pi \text{ radians}}{180°}\right)$

$\therefore\ 90° = \dfrac{\pi}{2}$ radians

d) $180° \times \left(\dfrac{\pi \text{ radians}}{180°}\right)$

$\therefore\ 180° = \pi$ radians

e) $330° \times \left(\dfrac{\pi \text{ radians}}{180°}\right)$

$\therefore\ 330° = \dfrac{11\pi}{6}$ radians

f) $270° \times \left(\dfrac{\pi \text{ radians}}{180°}\right)$

$\therefore\ 270° = \dfrac{3\pi}{2}$ radians

Exercise 7.3

1. Perimeter $= \frac{1}{2}[2\pi(3\sqrt{2})] + \frac{3}{2} \times (2\pi\sqrt{2}) = 3\sqrt{2}\pi + 3\sqrt{2}\pi = 6\sqrt{2}\pi$

2. a) It turns $s = r\theta$ Therefore it covers $= \frac{2,5}{2} \times 90° \times \frac{\pi}{180} = 1,96$ m

   b) $s = r\theta$ $4,5 \text{ m} = \frac{2,5 \text{ m}}{2}\theta$ $\theta = 3,6 \times \frac{180}{\pi}$; $\theta = 206°$

Exercise 7.4

1. a) arc length $= r\theta = 1 \times \frac{90 \times \pi}{180} = 1,57$ units

   $A = \frac{r^2\theta}{2} = 1^2 \times \frac{90 \times \pi}{360} = 0,79$ units²

   b) arc length $= r\theta = 4 \times \frac{45 \times \pi}{180} = 3,14$ units

   $A = \frac{r^2\theta}{2} = 4^2 \times \frac{45 \times \pi}{360} = 6,28$ units²

   c) arc length $= r\theta = 6 \times \frac{120° \times \pi}{180} = 12,57$ units

   $A = \frac{r^2\theta}{2} = 6^2 \times \frac{120° \times \pi}{360} = 12\,\pi$ units²

   d) arc length $= r\theta = 4 \times \frac{90° \times \pi}{180} = 2\pi$ units

   $A = \frac{r^2\theta}{2} = 4^2 \times \frac{90° \times \pi}{360} = 4\pi$ units²

2. $A = \frac{r^2\theta}{2} = 6^2 \times \frac{60° \times \pi}{360} = 6\,\pi$ units²

3. $A = \frac{r^2\theta}{2} = 160\pi r = \sqrt{160}\,\frac{r^2\theta}{2} = 40\pi \text{ cm}^2$ $\theta = \frac{40\pi \times 2}{160} = \frac{1}{2}\pi$ radians $\theta = 90°$

   arc length $= r\theta = 4\sqrt{10} \times \frac{\pi}{2}$ radians $= \sqrt{10}\,\pi$ cm

4. $A = \frac{r^2\theta}{2} = \frac{7\pi}{2}$; $\theta = 315 \times \frac{\pi}{180}$ $r^2 = \frac{7\pi}{2 \times \frac{7}{4}\pi} \times 2r = 2$ units

5. a) $\frac{r^2\theta}{2} = 15\pi \text{ cm}^2$; $\theta = 50° \times \frac{\pi}{180°}$ $r^2 = \frac{30\pi}{\frac{5\pi}{18}} r = 10,4$ cm

   b) $\frac{r^2\theta}{2} = 20\pi \text{ cm}^2$; $\theta = 290° \times \frac{\pi}{180°}$ $r^2 = \frac{40\pi}{\frac{29\pi}{18}} r \approx 5$ cm

   c) $\frac{r^2\theta}{2} = 15\pi \text{ cm}^2$; $\theta = \frac{\pi}{6}$ $r^2 = \frac{30\pi}{\frac{\pi}{6}}$ $r = 13,4$ cm

   d) $\frac{r^2\theta}{2} = 21\pi \text{ cm}^2$; $\theta = \frac{7\pi}{4}$ $r^2 = \frac{21\pi}{\frac{7\pi}{4}}$ $r = 3,5$ cm

6. a) Perimeter $= 3r\theta + 3(6) = 3(3\pi) + 18\,cm = 46,3$ cm

   b) Perimeter $= 4r\theta = 4(7\pi) = 28\pi$ cm

   c) Perimeter $= 2(8 \text{ cm}) + 2(2 \text{ cm}) + 4r\theta = 16 \text{ cm} + 4 \text{ cm} + 4\left(1 \times \frac{90° \times \pi}{180}\right) = 26,3$ cm

   d) Perimeter $= 2\left(\frac{7\pi}{6}\right) + 2(\pi) + 4\left(\frac{\pi}{12} \times \frac{\pi}{2}\right) = \frac{7\pi}{3} + 2\pi + \frac{\pi^2}{6} = 15,3$ cm

Exercise 7.6

1. $4h^2 - 4dh + x^2 = 0$

   $4(0,3^2) - 4(4)(0,3) + x^2 = 0$

   $x^2 = 4,44$

   $x = 2,11$ cm

2. $4h^2 - 4dh + x^2 = 0$

   $4h^2 - 4(10)h + 7^2 = 0$

   $h = \frac{40 \pm \sqrt{(-40)^2 - 4(4)(49)}}{8}$

   $h = \frac{40 + \sqrt{816}}{8}$ or $h = \frac{40 - \sqrt{816}}{8}$

   $h = 17,14$ cm or $h = 2,86$ cm

3. $d = 6 + 12 = 18$

   $h = 6$

   $4h^2 - 4dh + x^2 = 0$

   $4(6)^2 - 4(18)(6) + x^2 = 0$

   $144 - 432 + x^2 = 0$

   $x^2 = 288$

   $x = 16,97$

4. $4h^2 - 4dh + x^2 = 0$

   $4(2)^2 - 4d(2) + 7^2 = 0$

   $16 - 8d + 49 = 0$

   $8d = 65$

   $d = 8,13$ cm

Exercise 7.7

1. $w = 2\pi \times 5000$ radians per minute $= 10\ 000\pi \times 60$ radians pe second
   $= 600\ 000\ \pi$ radians per second
2. $v = \pi Dn = \pi \times 20,4 \times 1$ radian per minute $= 20,4\pi$ radian per minute
3. $\omega = 2\pi \times 45$ revolutions per minute $= 90\pi \times 60 = 5\ 400\pi$ radians per second

   $v = \pi Dn = 60\pi \times 12\ cm \times 45$ revolutions per minute $= 32\ 400\pi$ cm per second
4. a) $\omega = 2\pi \times 1,5$ radians per minute $= 3\pi$ radians per minute

   b) $v = \pi Dn = \pi \times 100\ m \times 1,5$ revolutions per minute $= 150\ \pi$ cm per minute

Consolidation exercise

1. $\frac{\pi}{15} \times \frac{180}{\pi} = 12°$

2. $45° \times \frac{\pi}{180} = \frac{\pi}{4}$radians

3. $\sin\left(\frac{18\pi}{5}\right) = 0,196.$

4. a) $l = r\theta = 9 \text{ m} \times \frac{3\pi}{5} \approx 17 \text{ m}$

   b) $l = r\theta 14,9 \text{ cm} = 2,98rr = 5 \text{ cm}$

   c) $l = r\theta\frac{49\pi}{9} \text{ m} = 7\theta\theta = \frac{7\pi}{9}$ radians

5. $10x = 360°;\ x = 36°\text{CD} = 12 \text{ m} \times 3x \times \frac{\pi}{180}\text{CD} = 12 \text{ m} \times 108° \times \frac{\pi}{180}\text{CD} = \frac{36}{5}\pi \text{ m}$

6. $l = rx53 \text{ m} - 34 \text{ m} = 17 \times x \quad x = \frac{19}{17}$ radians

7. Area $= \frac{9^2 \times 0,97}{2}$Area $= 39,285 \text{ cm}^2$

8. $\hat{C} = 2\pi - 1,46 = 4,82543 \text{ m}^2 = \frac{r^2 \times 4,82}{2}r = 15 \text{ m}$

9. Area of segment = area of sector – area of triangle

   $= \frac{19^2 \times 2,98}{2} - \frac{1}{2} \times 19 \times 19 \times \sin\left(2,98 \times \frac{180}{\pi}\right) = 537,89 - 29,045 = 508,85 \text{ m}^2$

10. Perimeter of minor sector $= r + r + 2,22r = 4,22r4,22r = 80 \text{ m}r \approx 19 \text{ m}$

    area of sector $= \frac{19^2 \times 2,22}{2} = 400,71 \text{ m}^2$

    area of triangle $= \frac{19 \times 19 \sin\left(2,22 \times \frac{180}{\pi}\right)}{2} = 143,78$

    Area of segment $400,71\ m^2 - 143,78 = 256,93\ m^2$

11. $66 \text{ cm} = 0,00066 \text{ km}v = \pi Dn = \pi \times 0,00066 \text{ km} \times 125$ revolutions per minute

    $= \frac{33}{400}\pi$ km per minute

CHAPTER 8 Trigonometry

Exercise 8.1

1. a) $\cos\alpha = \sqrt{1 - t^2}$

   b) $\tan\alpha = \frac{t}{\sqrt{1 - t^2}}$

2. a) (i) $OP^2 = (-2)^2 + (-2)^2$

   $OP = 2\sqrt{2}$

   (ii) $OP^2 = (4)^2 + (-3)^2$

   $OP = 5$

   (iii) $OP^2 = (-3)^2 + (2)^2$

   $OP = \sqrt{13}$

   (iv) $OP^2 = (4)^2 + (5)^2$

   $OP = \sqrt{41}$

   b)

$OP = 2\sqrt{2}$	$\sin\theta = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$	$\cos\theta = \frac{-2}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$	$\tan\theta = \frac{-2}{-2} = 1$
$OP = 5$	$\sin\theta = -\frac{3}{5}$	$\cos\theta = \frac{4}{5}$	$\tan\theta = -\frac{3}{4}$
$OP = \sqrt{13}$	$\sin\theta = -\frac{2}{\sqrt{13}}$	$\cos\theta = -\frac{3}{\sqrt{13}}$	$\tan\theta = -\frac{3}{2}$
$OP = \sqrt{41}$	$\sin\theta = \frac{5}{\sqrt{41}}$	$\cos\theta = \frac{4}{\sqrt{41}}$	$\tan\theta = \frac{5}{4}$

3. $\sin\theta = -\frac{7}{25}$

   a) $\cos\theta = -\frac{24}{25}$

   b) $\tan\theta = \frac{-7}{-24} = \frac{7}{24}$

Exercise 8.2

a) $C = 42{,}03°$, $BD = 4{,}91$ units, $CD = 4{,}82$ units

b) $Y = 45{,}05°$, $Z = 102{,}02°$, $XY = 6{,}5$ units

c) $R = 10{,}04°$, $QR = 6{,}14$ units, $PR = 9{,}6$ units

d) $V = 28{,}16°$, $U = 120{,}63°$, $TV = 8{,}57$ units

Exercise 8.3

a) $B = 53{,}13°$, $C = 96{,}87°$, $c = 7{,}94$ cm

b) $B = 90°$, $C = 60°$, $c = 5\sqrt{3}$ cm

c) $B = 56{,}44°$, $C = 93{,}56°$, $c = 11.98°$

Exercise 8.4

1. $\frac{\sin Q}{43,8} = \frac{\sin 130°}{51,25}$

   $\sin Q = \frac{43,8 \sin 130°}{51,25}$

   $Q = 41°; R = 1°$

   $r = 43,89 \text{ cm}$

3. The wires are 5,18 m long.
5. BC = 5,18 cm

   Perimeter = 25,18 cm

Exercise 8.5

1. a) $AC^2 = (3,7)^2 + (4,14)^2 - 2(3,7)(4,14)\cos 96,65°$

   $AC = 5,86 \text{ units}$

   $\frac{\sin A}{4,14} = \frac{\sin 96,65°}{5,86}$

   $\sin A = \frac{4,14 \sin 96,65°}{5,86}$

   $\sin A = 0,70$

   $A = 44,43°$

   $C = 180° - (95,65° + 44,43°)$

   $C = 39,92°$

2. a) $C = 44,42°$, $B = 78,49°$, $A = 57,09°$

   b) $b = 34,04$ m, $A = 23,33°, 35,67°$

4. $BD^2 = 6^2 + 10^2 - 2(6)(10)\cos 70°$

   $BD = AC = 9,74 \text{ cm}$

5. $AD = 1,36 \text{ m}$

Exercise 8.6

1. a) Area of $\Delta KML = \frac{1}{2}(4,84)(6,17)\sin 36,97°$

   $= 8,98$ square units.

   b) 7,01 square units

   a) Area of $LMN = \frac{1}{2}(3)(5)\sin 80°$

   $= 7,39 \text{ cm}^2$

   b) $12\sqrt{2} \text{ cm}^2$

   c) $12,5 \text{ m}^2$

3. a) $319,45 \text{ cm}^2$

   b) $7,04 \text{ cm}^2$

   c) $22,66 \text{ cm}^2$

4. $(4,9)^2 = (5,5)^2 + (8,5)^2 - 2(5,5)(8,5)\cos A$

$24,01 = 30,25 + 72,25 - 93,5\cos A$

$\cos A = 32,92°$

Area of $\Delta ABC = \frac{1}{2}(5,5)(8,5)\sin 32,92°$

$= 12,7\ \text{cm}^2$

5. Area of the stand $= 68,52\ \text{m}^2$

Exercise 8.15

1.

		Amplitude	Period	Phase shift
a)	$y = 3\cos 2x$	3	$\frac{360°}{2} = 180°$	No phase shift.
b)	$y = 2\sin(x - 60°)$	2	$\frac{360°}{1} = 360°$	The graph of $y = \sin x$ is shifted 60°to the right along the x – axis.
c)	$y = -2\sin\frac{x}{2}$	2	$\frac{360°}{\frac{1}{2}} = 720°$	No phase shift.
d)	$y = \frac{3}{2}\cos(x + 30°)$	$\frac{3}{2}$	$\frac{360°}{1} = 360°$	The graph of $y = \cos x$is shifted 30°to the left along the x – axis.

2. a) (i) $k = 4$ (ii) $b = 2$ (iii) $p = 0$

$y = 4\cos 2x$

c) $-4 \leq$
$y \leq 4$

d) $0° \leq x \leq 360°$

e) $(90°;\ -4)$

f) $(0°;\ 4)$ and $(180°;\ 4)$

Exercise 8.16

x	0°	30°	45°	60°	90°
$\sin x$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\cos x$	1	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

Exercise 8.17

1.

	Equation	Reference angle	Quadrants where the function has +/- values	All solutions
a)	$\cos x = \frac{1}{2}$	60°	First and fourth	$x = 60°;\ 300°$
b)	$\tan x = 1$	45°	First and third	$x = 45°;\ 225°$
c)	$\sin x = \frac{\sqrt{3}}{2}$	60°	First and second	$x = 60°;\ 120°$
d)	$\cos x = \frac{1}{\sqrt{2}}$	45°	First and fourth	$x = 45°;\ 315°$
e)	$\tan x = \sqrt{3}$	60°	First and third	$x = 60°;\ 240°$
f)	$\sin x = -\frac{1}{2}$	30°	Third and fourth	$x = 210°;\ 330°$
g)	$\tan x = -\frac{1}{\sqrt{3}}$	30°	Second and fourth	150°; 330°
h)	$2\sin^2 x = 1$	45°	Positive in first and second Negative in third and fourth	45°; 135° 225°; 315°

2. a) $x = \sin^{-1}(0,7)$

$x = 44,4°$

$x = 180° - 44,4°$

$x = 135,3°$

b) $x = 160,3°$ or $340,7°$ (reference angle is $19,3°$)

c) $x = 64,5°$ or $295,5°$

d) $x = 42,3°$ or $222,3°$

e) $x = 233,1°$ or $306,9°$

f) $x = 139,5°$ or $220,5°$

g) $\sin\theta = -\frac{1}{2}$

$\theta = 210°$ or $330°$

h) $x = 120°$ or $300°$

i) $\theta = 150°$ or $210°$

j) $\theta = 45°$ or $135°$

k) $x = 45°$ or $135°$ or $225°$ or $315°$

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Exercise 8.18

1. a) $\sin^2 x$
   b) $\frac{\cos\theta}{\sin\theta}$
   c) $\frac{1}{\sin^2\beta}$
   d) 1
   e) $\frac{\tan^2\theta}{\sec^2\theta}$
   f) $\sin\beta$
   g) $\sin x + 1$
   h) $\cos^2 A$
   i) $\tan A$
   j) $\cos A$

Exercise 8.19

a) $-\cos\beta$
b) $-\tan 70°$
c) $\sin 50°$
d) $\sin(180°-35°) \;=\; \sin 35°$
e) $\tan(180°-75°) \;=\; -\tan 75°$
f) $\cos(180°-1°) \;=\; -\cos 1°$
g) $-\cos 60° = -\frac{1}{2}$
h) $-\tan 45° = -1$
i) $\sin 30° = \frac{1}{2}$
j) $\sin 60° = \frac{\sqrt{3}}{2}$
k) $-\cos 45° = -\frac{1}{\sqrt{2}}$
l) $-\tan 30° = -\frac{1}{\sqrt{3}}$
m) $-\tan 60° = -\sqrt{3}$
n) $\sin 45° = \frac{1}{\sqrt{2}}$
o) $-\cos 30° = -\frac{\sqrt{3}}{2}$

Exercise 8.20

a) $-\sin\theta$
b) $\tan 20°$
c) $\cos 255° = \cos(180°+75°) \;=\; -\cos 75°$
d) $\tan 18°$
e) $-\sin 27°$
f) $-\cos 30° = -\frac{\sqrt{3}}{2}$
g) $-\sin 30° = -\frac{1}{2}$
h) $\tan 45° = 1$
i) $-\sin 45° = -\frac{1}{\sqrt{2}}$
j) $-\cos 45° = -\frac{1}{\sqrt{2}}$
k) $-\sin 60° = -\frac{\sqrt{3}}{2}$
l) $\tan 60° = \sqrt{3}$
m) $-\cos 60° = -\frac{1}{2}$

Exercise 8.21

a) $\cos 10°$
b) $-\sin 50°$
c) $\tan(360°-40°) \;=\; -\tan 40°$
d) $\cos 55°$
e) $-\sin 78°$
f) $-\tan 30° = -\frac{1}{\sqrt{3}}$
g) $-\sin 30° = -\frac{1}{2}$
h) $\cos 30° = \frac{\sqrt{3}}{2}$
i) $-\sin 60° = -\frac{\sqrt{3}}{2}$
j) $\cos 60° = \frac{1}{2}$
k) $-\tan 60° = -\sqrt{3}$
l) $\cos 45° = \frac{1}{\sqrt{2}}$
m) $-\tan 45° = -1$
n) $-\sin 45° = -\frac{1}{\sqrt{2}}$
o) $\cos 85°$

Exercise 9.1

1. a) Year 1:

$A = 2\ 400\left(1 + \frac{5,3}{100}\right) = R2\ 527,20$

Year 2:

$A = 2\ 527,20\left(1 + \frac{5,9}{100}\right) = R2\ 676,30$

It costs R2 676,30

b) Year 1:

$A = 820\left(1 + \frac{5,3}{100}\right) = R863,46$

Year 2:

$A = 863,46\left(1 + \frac{5,9}{100}\right) = R914,40$

It costs R914,40

c) Year 1:

$A = 119\left(1 + \frac{5,3}{100}\right) = R125,31$

Year 2:

$A = 125,31\left(1 + \frac{5,9}{100}\right) = R132,70$

It costs R132,70

2. $A = 8\left(1 + \frac{5,7}{100}\right)^5 = R10,56$

3. $A = 25\left(1 + \frac{7,5}{100}\right)^{10} = R51,53$

Exercise 9.2

1. a) $A = 8\ 000(1 + 0,08 \times 5)$

$A = R11\ 200$

b) $A = 8\ 000(1 + 0,08)^5$

$A = R11\ 754,62$

2. $A = 70\ 000(1 + 0,11)^7$

$A = R145\ 331,21$

3. $A = 120\ 000(1 + 0,15 \times 3)$

$A = R174\ 000$

4. a) $A = 4\ 500(1 + 0,135 \times 9)$

$A = R9\ 967,50$

b) $A = 4\ 500(1 + 0,135)^9$

$A = R14\ 066,15$

c) Compound interest is better because it yield more interest.

5. a) $A = 3\ 000(1 + 0,067 \times 4)$

$A = R3\ 804$

$I = A - P = R804$

b) $A = 3\ 000(1 + 0,054)^4$

$A = R3\ 702,40$

$I = A - P = R702,40$

6. a) $A = 25\ 000(1 + 0,164 \times 3)$

$A = R37\ 300$

b) $\frac{37\ 300}{36} = R1\ 036,11$

7. $P = \frac{30\ 591}{(1 + 0,15)^8} = R10\ 000,25$

8. $i = \frac{\frac{7\ 700}{440} - 1}{8} = 9,34\%$

9. $i = \sqrt[10]{\frac{450\ 000}{150\ 000}} - 1 = 11,61\%$

10. $P = \frac{850\ 000}{\{(1 + 0,0825)^3(1 + 0,125)^9} = R232\ 146,70$

11. $3\ 500 = P(1+i)^5$

$12\ 500 = P(1+i)^{10}$

Let $(1+i)^5 = x$, then

$3\ 500 = Px$ and $12\ 500 = Px^2$

$12\ 500 = P\left(\frac{3\ 500}{P}\right)^2$

$P = \frac{3\ 500^2}{12\ 500}$

$3\ 500 = Px$

So: $x = \frac{3\ 500}{P}$

$P = R980$

$(1 + i)^5 = \frac{3\ 500}{980}$

$i = \sqrt[5]{\frac{3\ 500}{980}} - 1$

$i = 28,99\%$

Exercise 9.3

1. a) $P = \frac{75\ 000}{1-3i}; \quad P = \frac{15\ 000}{1-7i}$

$i = \frac{60\ 000}{480\ 000} = 0,125$

$i = 12,5\%$

b) $P = \frac{75\ 000}{1-3i};$

$P = \frac{75\ 000}{1-3(0,125)}$

$P = \text{R}120\ 000$

2. A = 320 000 (1 – 0,15 × 4)

A = R128 000

3. A = 120 000 (1 – 0,0 5 × 8)

A = R72 000

4. a) A = 400 000 (1 – 4 × i)

0 = 400 000 (1 – 4i)

$i = \frac{400000}{1600000}$

$i = 0,25$

$i = 25\%$

b)

Year	Value at beginning of year	Depreciated value	Value at end of year
1	400 000	100 000	300 000
2	300 000	100 000	200 000
3	200 000	100 000	100 000
4	100 000	100 000	0

Exercise 9.4

1. a)

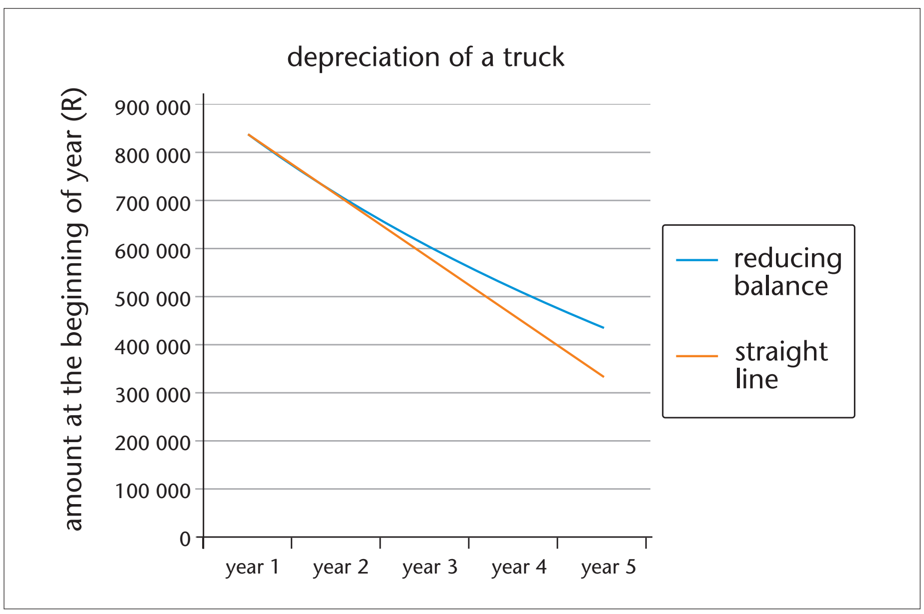
Straight line depreciation

Year	Book value	Depreciation	Value end of year
1	840 000	126 000	714 000
2	714 000	126 000	588 000
3	588 000	126 000	462 000
4	462 000	126 000	336 000
5	336 000	126 000	210 000

Reducing balance depreciation

Year	Book value	Depreciation	Value end of year
1	840 000	126 000	714 000
2	714 000	107 100	606 900
3	606 900	91 035	515 865
4	515 865	77 379,95	438 485,25
5	438 485,25	65 772,79	372 712,46

b)



Exercise 9.5

1. $A = 225\ 000(1-0{,}066)^4$

   $A = \text{R}171\ 226{,}12$

2. $149\ 500 = P(1-0{,}035)^3$

   $P = \dfrac{149\ 500}{(1-0{,}035)^3}$

   $P = \text{R}166\ 363{,}96$

3. $A = 145\ 000(1-0{,}14\times 4)$

   $A = \text{R}63\ 800$

4. $10\ 000 = 22\ 000(1-i)^5$

   $i = 1-\sqrt[5]{\dfrac{10\ 000}{22\ 000}}$

   $i = 14{,}6\%$

5. a) A = 1 259 people; $i = 0{,}08$; n = 5

   $A = 1\ 259(1-0{,}08)^5$

   = 829 people

   b) $A = 1\ 259(1-0{,}08\times 5)$

   = 755 people

Exercise 9.6

1. a) $i_e = \left[\left(1+\frac{0{,}09}{4}\right)^4-1\right]\times 100$

   $i_e = 11\%$

   $i_e = 9{,}31\%$

   b) $i_e = \left[\left(1+\frac{0{,}11}{2}\right)^2-1\right]\times 100$

   $i_e = 11{,}30\%$

   c) $i_e = \left[\left(1+\frac{0{,}095}{12}\right)^{12}-1\right]\times 100$

   $i_e = 9{,}92\%$

   d) $i_e = \left[\left(1+\frac{0{,}123}{365}\right)^{365}-1\right]\times 100$

   $i_e = 13{,}09\%$

   e) $i_e = [(1+0{,}11)-1]\times 100$

   $i_e = 11\%$

2. a) $i^{12} = 0{,}086$

   $i = \sqrt[12]{0{,}086}$

   $i = 0{,}82$

   b) $i^2 = 0{,}086$

   $i = \sqrt{0{,}086}$

   $i = 0{,}29$

   c) $\dfrac{i^{365}}{365} = \sqrt[365]{0{,}075+1}-1$

   $i^{365} = 365\left(\sqrt[365]{1{,}075}-1\right)\times 100$

   $i^{365} = 7{,}23\%$

Exercise 9.7

1. a) $i_e = \left[\left(1+\frac{0{,}103}{12}\right)^{12}-1\right]\times 100$

   $i_e = 10{,}8\%$

   b) $i_e = \left[\left(1+\frac{0{,}104}{4}\right)^4-1\right]\times 100$

   $i_e = 10{,}81\%$

   c) $i_e = \left[\left(1+\frac{0{,}1}{365}\right)^{365}-1\right]\times 100$

   $i_e = 10{,}52\%$

   d) $i_e = \left[\left(1+\frac{0{,}102}{52}\right)^{52}-1\right]\times 100$

   $i_e = 10{,}73\%$

   e) $i_e = [(1+0{,}107)-1]\times 100$

   $i_e = 10{,}7\%$

   10,4% p.a. compounded quarterly is best because it has an effective rate of 10,81%.

2. a) $A = 12\ 000\left(1+\frac{0{,}14}{12}\right)^{60}$

   $A = \text{R}24\ 669{,}35$

   b) $A = 12\ 000\left(1+\frac{0{,}14}{12}\right)^{120}$

   $A = \text{R}50\ 714{,}72$

   c) $A = 12\ 000\left(1+\frac{0{,}14}{12}\right)^{180}$

   $A = \text{R}104\ 258{,}74$

   d) The investment approximately doubles up every five years.

   12% p.a. compounded monthly for two years.

3. $A = 4\ 300\left(1+\frac{0,0875}{2}\right)^{\frac{3}{2}\times 2}$

$A = \text{R}4\ 889,43$

4. Monthly / Semi-annually

Monthly: $i_e = \left(1+\frac{0,07}{12}\right)^{12}-1$

$i_e = 7,64$

Semi-annually: $i_e = \left(1+\frac{0,075}{2}\right)^{2}-1$

$i_e = 7,23$

7,5% compounded semi-annually is a better option

5. $A = 1\ 000\left(1+\frac{0,14}{4}\right)^{12}$

$A = \text{R}1\ 511,07$

6. $25\ 073 = 4\ 500(1+i)^{12}$

$i = \sqrt[12]{\frac{25\ 073}{4\ 500}}-1$

$i = 0,15389 \times 100 = 15,38\%$

7. $15\ 565 = P\left(1+\frac{0,25}{12}\right)^{84}$

$P = \frac{15\ 565}{\left(1+\frac{0,25}{12}\right)^{84}}$

$P = \text{R}2\ 753,86$

8. $A = 3\ 500\left(1+\frac{0,144}{4}\right)^{\frac{6}{12}\times 4} + 1\ 000\left(1+\frac{0.16}{12}\right)^{\frac{9}{2}\times 12}$

$A = \text{R}9\ 144,76$

9. Value of the car after 5 years is: $A = 289\ 900\left(1-\frac{0,1}{1}\right)^{5} = \text{R}171\ 183,05$

Value of new car after 5 years: $A = 289\ 900\left(1+\frac{0,12}{1}\right)^{5} = \text{R}510\ 902,85$

Tintswalo's savings after 5 years: $A = 50\ 000\left(1+\frac{0,134}{12}\right)^{5} = \text{R}97\ 349,71$

She will need: $\text{R}510\ 902,85 - (\text{R}171\ 183,05 + \text{R}97\ 349,71) = \text{R}242\ 370,07$

Exercise 9.8

1. $A = 28\ 000\left(1+\frac{0,093}{12}\right)^{48}\left(1+\frac{0,11}{4}\right)^{24}$

$A = \text{R}77\ 777,99$

2. $\text{R}1\ 927\ 713,61 = P(1+0,09)^{8}\left(1+\frac{0,11}{12}\right)^{12\times 12}$

$P = \frac{R1\ 927\ 713,61}{(1,09)^{8}\left(1+\frac{0,11}{12}\right)^{144}} = \text{R}260\ 000$

3. $A = 16\ 000\left(1+\frac{0,055}{12}\right)^{36}\left(1+\frac{0,062}{365}\right)^{4\times 365}$

$A = \text{R}24\ 122,68$

4. $A = 3\ 500\left(1+\frac{0,08}{12}\right)^{48}\left(1+\frac{0,09}{2}\right)^{5\times 2}$

$A = R7\ 477,29$

5. $21\ 546,67 = 10\ 800\left(1+\frac{0,075}{4}\right)^{20}\left(1+\frac{x\%}{4}\right)^{16}$

$\frac{x\%}{4} = \sqrt[16]{\frac{21\ 546,67}{(10\ 800)\left(1+\frac{0,075}{4}\right)^{20}}} - 1$

$x = 0,08058 = 8,06\%$

Exercise 9.9

1. $A = 120\ 000\left(1+\frac{0,08}{12}\right)^{72} + 6\ 000\left(1+\frac{0,08}{12}\right)^{48}$

$A = R201\ 874,26$

2. $A = 18\ 000\left(1+\frac{0,06}{2}\right)^{28} + 2\ 500\left(1+\frac{0,06}{2}\right)^{20} - 3\ 000\left(1+\frac{0,06}{2}\right)^{12}$

$A = R41\ 120,69$

He will still need R23 579

3. $415\ 550 = P\left(1+\frac{0,1465}{2}\right)^{20}\left(1+\frac{0,094}{4}\right)^{24}\left(1+\frac{0,086}{12}\right)^{108}$

$P = \frac{415\ 550}{\left(1+\frac{0,1465}{2}\right)^{20}\left(1+\frac{0,094}{4}\right)^{24}\left(1+\frac{1+0,086}{12}\right)^{108}}$

$P = 27\ 140,87$

4. Original loan = R24 000+ R120 000= R120 000 = 144 000

5. $9\ 725,60 = 3\ 500\left(1+\frac{x\%}{4}\right)^{\frac{6}{12}\times 4}\left(1+\frac{0,16}{12}\right)^{4,5\times 12} + 1\ 000\left(1+\frac{0,16}{12}\right)^{4,5\times 12}$

$\frac{x\%}{4} = \sqrt[0,5]{\frac{9\ 725,6 - 1\ 000\left(1+\frac{0,16}{12}\right)^{4,5\times 12}}{3\ 500}} - 1$

$x = 15,26\%$

Consolidation Exercise

1. $A = 12\ 800(1+0,15)^5$

$A = R25\ 745,37$

2. $I = Pin$

$I = 8\ 900 \times 0,07 \times 5$

$I = R3\ 115$

3. $A = Pin$

$A = 60\ 000 \times 0,1275 \times 11$

$A = R84\ 150$

4. $A = 16\ 900(1+0,0775)^5$

$A = R24\ 545,57$

5. Simple interest

$I = Pin$

$I = 3\ 000 \times 0,073 \times 1 + 9\ 600 \times 0,052 \times 1$

$I = \text{R}219 + \text{R}499,20$

$I = \text{R}718,20$

Compound interest

$A = 3\ 000(1 + 0,073) + 9\ 600(1 + 0,052)$

$A = \text{R}13\ 318,20$

$I = A - P = 13\ 318,20 - 12\ 600$

$I = \text{R}718,20$

6. $400 = P(1+i)^4$

$20\ 000 = P(1+i)^{12}$

Let $(1+i)^4 = x$, then

$4000 = Px$ and $20\ 000 = Px^3$

$20\ 000 = P\left(\frac{4\ 000}{P}\right)^3$

$P = \sqrt{\frac{4\ 000^3}{20\ 000}}$

$P = \text{R}1\ 788,85$

$x = \frac{4\ 000}{P}$

$P = \text{R}1\ 788,85$

$(1 + i)^4 = \frac{4\ 000}{1\ 788,85}$

$i = \sqrt[4]{\frac{4\ 000}{1\ 788,85}} - 1$

$i = 22,3\%$

7. a) $A = P(1 - in)$

$A = 185\ 000(1 - 0,06 \times 7)$

$A = \text{R}107\ 300$

b) $A = P(1 - i)^n$

$A = 185\ 000(1 - 0,06)^7$

$A = \text{R}119\ 968,35$

8. $142\ 500 = 950\ 000(1 - 5i)$

$\frac{142\ 500}{950\ 000} = 1 - 5i$

$i = \left(1 - \frac{142\ 500}{950\ 000}\right) \times \frac{1}{5}$

$i = 17\%$

9. a) if there is continuous drought and 5% of the water is continuously released, the dam can eventually dry up

b) amount of water after 15 days $= 17\ 000\ 000\ 000 - 15 \times 0,05 \times 17$ billion litres

$= 4\ 250\ 000\ 000$ litres (4 billion 250 million litres)

c) After 30 days, half the remaining water was released.

Half 4 250 000000 = 2 125 000 000 litres lost in 30 days.

In one day: $\frac{1\ 225\ 000\ 000}{30} = 70\ 833\ 333,33\text{litres}$.

Percentage of water released in 1 day $= \frac{70\ 833\ 333,33}{4\ 250\ 000\ 000} \times 100 = 1,7\%$

10.

	Calculation	Accumulated amount	Effective annual interest rate
Daily	$\left[\left(1+\frac{0,09}{365}\right)^{i_e 365}-1\right]\times 100$	$A = 45\ 000\left(1+\frac{0,09}{365}\right)^{365\times\frac{30}{12}}$ A = R56 352,96	i = 9,41%
Weekly	$i_e = \left[\left(1+\frac{0,09}{52}\right)^{52}-1\right]\times 100$	$A = 45\ 000\left(1+\frac{0,09}{52}\right)^{52\times\frac{30}{12}}$ A = R56 343,56	i = 9,41%
Monthly	$i_e = \left[\left(1+\frac{0,09}{12}\right)^{12}-1\right]\times 100$	$A = 45\ 000\left(1+\frac{0,09}{12}\right)^{12\times\frac{30}{12}}$ A = R56 307,23	i = 9,38%
Quarterly	$i_e = \left[\left(1+\frac{0,09}{4}\right)^{4}-1\right]\times 100$	$A = 45\ 000\left(1+\frac{0,09}{4}\right)^{4\times\frac{30}{12}}$ A = R56 078,19	i = 9,31%
Semi-annually	$i_e = \left[\left(1+\frac{0,09}{2}\right)^{2}-1\right]\times 100$	$A = 45\ 000\left(1+\frac{0,09}{2}\right)^{2\times\frac{30}{12}}$ A = R56 352,96	i = 9,20%
Yearly	$i_e = \left[\left(1+\frac{0,09}{1}\right)^{1}-1\right]\times 100$	$A = 45\ 000(1+0,09)^{\frac{30}{12}}$ A = R55 818,58	i = 9%

11. a) $A = 40\ 000\left(1+\frac{0,09}{12}\right)^{12}$

A= R43 752,28

b) $A = 40\ 000\left(1+\frac{0,092}{4}\right)^{4}$

A = R43 808,92

c) $A = 40\ 000(1+0,093)$

A = R43 720,00

Botle-Buhle should take 9,2 % compounded quarterly as it yields better interest.

12. $A = 145\ 000\left(1+\frac{0,09}{2}\right)^{4}\left(1+\frac{0,04}{4}\right)^{24} - 20\ 000\left(1+\frac{0,04}{4}\right)^{16} + 15\ 000\left(1+\frac{0,04}{4}\right)^{8}$

A = R212 347,69

13. $6\ 000\left(1+\frac{i}{4}\right)^{28} + 10\ 000\left(1+\frac{i}{4}\right)^{20} + 50\ 000\left(1+\frac{i}{4}\right)^{4} - 15\ 000\left(1+\frac{i}{4}\right)^{16} - 58\ 602,87 = 0$

Let $\left(1+\frac{i}{4}\right)^{4} = x$, then

$6\ 000x^7 + 10\ 000x^5 + 50\ 000x - 15\ 000x^4 - 58\ 602,87 = 0$

$x = 1,25984$

$i =$

CHAPTER 10 Mensuration

Exercise 10.1

1. 600 cm^2
2. 9 cm
3. 78 cm^2
4 338 m^2

Exercise 10.2

1. 7 225,67 cm^2
2. 282,74 cm^2
3. 93,46 cm^2
4. 63 cm

Exercise 10.3

1. 31,25 m^2
2. 103,49 cm^2

Exercise 10.4

1. 597,66 m^2
2. a) 989,60 cm^2
   b) 197 920 cm^2
   c) $\frac{\text{R}109,90}{\text{m}^2}$; R72 090, 50

Exercise 10.5

a) 255 600 981,2 km^2

b) 74 000 000 km^2

Exercise 10.6

1. a) $SA = 2 \times \frac{1}{2} \times 8 \times 5 + 2 \times \frac{1}{2}(14 + 22) \times 5$
   b) 2 514 tiles
2. a) circle, square-based pyramid and cuboid
   b) 6 413 cm^2

Exercise 10.7

1. a) Original SA = $36\sqrt{3}$ mm^2
   k = 2: New SA = $144\sqrt{3}$ mm^2
   k = 3: New SA = $324\sqrt{3}$ mm^2
   k = k: New SA = $36k^2\sqrt{3}$ mm^2

b) Original SA = 356,27 m^2

$8 \times r$: New SA = 16 815,31 m^2

$8 \times r$: New SA = 1 104,4 m^2

2. a) Original surface area = 113,097 m^2

New surface area = 1 017,876 m^2 = $9 \times$ original SA

b) Original SA = 9π m^2

New SA = 324π m^2 = $36 \times$ original SA

3. Original surface area = $2ab + 2(a + b)c$

New surface area = $20ab + 12(5a + 2b)c$

4. New surface area = $k^2 \times [2ab + 2(a + b)c]$

Exercise 10.8

1. V = 188,04 m^3
2. 5,5 cm
3. a) $\frac{\sqrt{287}}{2}$ cm

   b) $\approx$ 720 cm^3

   c) length = width = 17 cm; height = 30 cm

Exercise 10.9

1. 1,64 cm
2. a) 2,79 m^3

   b) 2,790 l

   c) 0,093 s
3. a) length = 20 cm; width = 20 cm and height = 11 cm

   b) 4 400 cm^3

Exercise 10.10

1. 48 cm^3
2. 30,78 m

Exercise 10.11

1. 37,70 cm^3
2. 15 m
3. 1,5 cm

Exercise 10.12

1. $\approx$ 10 m
2. 11,22 m^3
3. 64 000 pellets
4. 908,97 mm^3

Exercise 10.13

1. a) cylinders
   b) 753,98 cm^3
2. 117 cm
3. 1,7 cm^3
4. a) 4 cm
   b) 324 cm^3

Exercise 10.14

1. a) $V\text{ (original)} = \frac{125}{6\sqrt{2}}\text{ mm}^3$

   $V\text{ (new)} = 2^3 \times \frac{125}{6\sqrt{2}}\text{ mm}^3 = 2^3 \times V\text{ (original)}$

   b) $V\text{ (original)} = \pi \times 6,4^2 \times 2,8$

   $7 \times r{:}\, V(\text{new}) = 7^2 \times \pi \times 6,4^2 \times 2,8 = 7^2 \times V(\text{original})$

   $7 \times H{:}\, V(\text{new}) = 7 \times V(\text{original})$

   c) $V\text{ (original)} = \frac{1}{3} \times 4^2 \times 5$

   $3 \times r{:}\, V(\text{new}) = 3^2 \times \frac{1}{3} \times 4^2 \times 5 = 3^2 \times V(\text{original})$

   $3 \times H{:}\, V(\text{new}) = 3 \times \frac{1}{3} \times 4^2 \times 5 = 3 \times V(\text{original})$

   d) $V\text{ (original)} = \frac{4}{3} \times \pi \times (1,5)^3$

   $6 \times r{:}\, V(\text{new}) = 6^3 \times V(\text{original})$

   $3 \times r{:}\, V(\text{new}) = 3^3 \times V(\text{original})$

2. $V = L \times B \times H$

   $V(\text{new}) = 5^3 \times V$

3. $V = \pi \times r^2 \times H$

   $V(\text{new}) = 3^2 \times 5 \times V$

Exercise 10.15

1. Area = 46 000 m^2= 4,6 ha
2. a) a = 1 000 m
   b) m_1 = 3,5 m; m_2 = 4,25 m; m_3 = 4,75 m; m_4 = 5,5 m; m_5 – 5,75 m; m_6 = 5 m; m_7 = 4,75 m; m_8 = 4,5 m; m_9 = 3,5 m; m_{10} = 1,5 m
   c) 385 000 000 m^2 = 38 500 ha
   d) R2 152 150 000,00

Notes