

# POLYNOMIALS

## REMAINDER & FACTOR THEOREMS

### THE REMAINDER THEOREM

If a polynomial  $f(x)$  is divided by a linear polynomial  $(ax + b)$ , then the remainder is given by  $f\left(-\frac{b}{a}\right)$ .

- If a function  $f(x)$  is divided by  $x - 3$ , the remainder is  $f(3)$
- If a function  $f(x)$  is divided by  $2x - 5$ , the remainder is  $f\left(\frac{5}{2}\right)$
- If a function  $f(x)$  is divided by  $5x + 1$ , the remainder is  $f\left(-\frac{1}{5}\right)$

#### Example

If  $2x^2 + 5x - 1$  is divided by  $x + 4$ , determine the remainder.

#### Solution

Remainder =  $f(-4)$

$$f(-4) = 2(-4)^2 + 5(-4) - 1 = 11$$

∴ the remainder is 11.

### THE FACTOR THEOREM

If  $f(x)$  is a polynomial, and  $f\left(-\frac{b}{a}\right) = 0$ , then  $ax + b$  is a factor of  $f(x)$ .

#### Example

Determine if  $x - 1$  is a factor of  $f(x) = 3x^4 + 3x^2 - 5x - 1$ .

#### Solution

Remainder =  $f(1)$

$$f(1) = 3(1)(1)^4 + 3(1)^2 - 5(1) - 1 = 0$$

∴  $x - 1$  is a factor of  $f(x)$

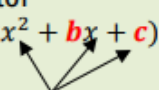
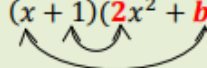
**The FACTOR THEOREM is mostly used to:**

1. Factorize cubic (third degree) expressions
2. Find solutions ( $x$  – values) of cubic equations
3. Find the  $x$  – intercepts of the graph of the cubic function

**EXERCISE 1: (Remainder & Factor theorems)**

1. For each of the following, determine the remainder if
  - 1.1  $f(x) = x^3 + x^2 - 1$  is divided by  $2x + 3$
  - 1.2  $g(x) = x^3 + 4x^2 - 11x - 31$  is divided by  $x + 5$
  - 1.3  $p(x) = 2x^3 - x^2 + 3x - 8$  is divided by  $s(x) = x - 2$
  - 1.4  $h(x) = 8x^4 - 4x^2 - 5$  is divided by  $(2x - 1)$
2. Given:  $f(x) = x^3 - 2x^2 - 4x + 3$ 
  - 2.1 Use the remainder theorem and determine the remainder if  $f(x)$  is divided by
    - 2.1.1  $x + 1$
    - 2.1.2  $x - 3$
  - 2.2 What may be concluded from the previous two calculations?
3. Prove that  $x + 1$  is a factor of  $f(x) = x^3 + 5x^2 - 17x - 21$ .
4. Given:  $g(x) = x^3 + px + 6$   
 Determine the value of  $p$  if  $(2 - x)$  is a factor of  $g$ .
5. Given:  $p(x) = x^3 + ax^2 + bx + 6$ 
  - 5.1 If  $x - 2$  is a factor of  $p$ , show that  $2a + b = -7$ .
  - 5.2 Hence, determine the values of  $a$  and  $b$  if it is further given that the remainder is 48 when  $p$  is divided by  $x - 3$ .

Your knowledge of the Remainder and Factor theorems will mainly be used to factorise cubic functions, in order to determine the  $x$  – intercepts of the graphs.

DIFFERENT METHODS TO FACTORISE A CUBIC POLYNOMIAL (3 <sup>RD</sup> DEGREE)	
METHOD AND DESCRIPTION OF STEPS	EXAMPLES
<b>SUM AND DIFFERENCE OF CUBES</b>	<p>A) <math>f(x) = x^3 + 27</math>  <math>= (x + 3)(x^2 - 3x + 9)</math>            Cannot factorise further</p> <p>B) <math>f(x) = 8x^3 - 1</math>  <math>= (2x - 1)(4x^2 + 2x + 1)</math>            Cannot factorise further</p>
<b>FACTORISE BY GROUPING</b> <ul style="list-style-type: none"> <li>Group terms in two pairs</li> <li>Take out common factor from each pair</li> <li>Two sets of brackets now become common factor</li> <li>Factorise bracket further if possible</li> </ul>	$f(x) = x^3 + 3x^2 - 4x - 12$ $= x^2(x + 3) - 4(x + 3)$ $= (x + 3)(x^2 - 4)$ $= (x + 3)(x + 2)(x - 2)$
<b>FACTORISE BY INSPECTION</b> <ul style="list-style-type: none"> <li>Find one linear factor using factor theorem</li> <li>Find other factor (quadratic expression) by inspection</li> </ul>	$f(x) = 2x^3 - 2x^2 - 10x - 6$ $f(-1) = 2(-1)^3 - 2(-1)^2 - 10(-1) - 6 = 0$ $\therefore (x + 1)$ is a factor $f(x) = (x + 1)(ax^2 + bx + c)$  Now find these coefficients Start with <b>a</b> and <b>c</b> : $1 \times a = 2 \therefore a = 2$ $1 \times c = -6 \therefore c = -6$ You now need to find <b>b</b> : Multiply the two brackets; the two $x^2$ -terms need to give you $-2x^2$ : $f(x) = (x + 1)(2x^2 + bx - 6)$  $bx^2 + 2x^2 = -2x^2 \therefore b = -4$ $\therefore f(x) = (x + 1)(2x^2 - 4x - 6)$ $= (x + 1)(2x + 2)(x - 3)$
<b>SYNTHETIC OR LONG DIVISION</b> <ul style="list-style-type: none"> <li>Find one linear factor using factor theorem</li> <li>Find other factor (quadratic expression) by long division or synthetic division (SEE NEXT PAGE)</li> </ul>	$f(x) = 2x^3 - 2x^2 - 10x - 6$ $f(-1) = 2(-1)^3 - 2(-1)^2 - 10(-1) - 6 = 0$ $\therefore (x + 1)$ is a factor $f(x) = (x + 1)(ax^2 + bx + c)$ Find <b>a</b> , <b>b</b> , <b>c</b> using synthetic division

SYNTHETIC DIVISION

$f(-1) = 0$  , so  $(x + 1)$  is a factor

$f(-1) = 0$

$f(x) = 2x^3 - 2x^2 - 10x - 6$

$-1$	$2$	$-2$	$-10$	$-6$
	$2$	$-4$	$-6$	$0$
	$2$	$-2$	$4$	$6$

This method is called SYNTHETIC division, because we don't really divide.  
We actually multiply and add.

Note the following:

- The  $x$ -value of  $-1$  that gave us the factor  $(x + 1)$  is written on the LHS
- The coefficients of the cubic polynomial are written in the top row
- The first coefficient,  $2$ , is carried down to the last row
- Now starting from the left:  
MULTIPLY along the dotted arrow  
and write the ANSWER in the block one row up and one column right
- Now ADD DOWN in the column (the two values underneath each other)
- You MUST get 0 in the last block
- The 3 values in the bottom row are the coefficients of the quadratic factor.

repeat  
steps

So,  $f(x) = (x + 1)(2x^2 - 4x - 6)$

You can now complete the factorising:

$f(x) = (x + 1)(2x + 2)(x - 3)$

**EXERCISE 2: (Factorise cubic expressions and functions)**

**1** Factorise the following expressions completely:

a  $27x^3 - 8$

b  $5x^3 + 40$

c  $x^3 + 3x^2 + 2x + 6$

d  $4x^3 - x^2 - 16x + 4$

e  $4x^3 - 2x^2 + 10x - 5$

f  $x^3 + 2x^2 + 2x + 1$

g  $x^3 - x^2 - 22x + 40$

h  $x^3 + 2x^2 - 5x - 6$

i  $3x^3 - 7x^2 + 4$

j  $x^3 - 19x + 30$

k  $x^3 - x^2 - x - 2$

**2** Solve for  $x$ :

a  $x^3 + 2x^2 - 4x = 0$

b  $x^3 - 3x^2 - x + 6 = 0$

c  $2x^3 - 12x^2 - x + 6 = 0$

d  $2x^3 - x^2 - 8x + 4 = 0$

e  $x^3 + x^2 - 2 = 0$

f  $x^3 = 16 + 12x$

g  $x^3 + 3x^2 = 20x + 60$

**3** Show that  $x - 3$  is a factor of  $f(x) = x^3 - x^2 - 5x - 3$  and hence solve  $f(x) = 0$ .

**4** Show that  $2x - 1$  is a factor of  $g(x) = 4x^3 - 8x^2 - x + 2$  and hence solve  $g(x) = 0$ .