P O L Y N O M I A L S

REMAINDER & FACTOR THEOREMS

THE REMAINER THEOREM

If a polynomial $f(x)$ is divided by a linear polynomial $(ax + b)$, then the remainder is given by $f\left(-\frac{b}{a}\right)$.

- If a function $f(x)$ is divided by $x - 3$, the remainder is $f(3)$
- If a function $f(x)$ is divided by $2x - 5$, the remainder is $f\left(\frac{5}{2}\right)$
- If a function $f(x)$ is divided by $5x + 1$, the remainder is $f\left(-\frac{1}{5}\right)$

Example

If $2x^2 + 5x - 1$ is divided by $x + 4$, determine the remainder.

Solution

Remainder $= f(-4)$

$f(-4) = 2(-4)^2 + 5(-4) - 1 = 11$

$\therefore$ the remainder is 11.

THE FACTOR THEOREM

If $f(x)$ is a polynomial, and $f\left(-\frac{b}{a}\right) = 0$, then $ax + b$ is a factor of $f(x)$.

Example

Determine if $x - 1$ is a factor of $f(x) = 3x^4 + 3x^2 - 5x - 1$.

Solution

Remainder $= f(1)$

$f(1) = 3(1)(1)^4 + 3(1)^2 - 5(1) - 1 = 0$

$\therefore x - 1$ is a factor of $f(x)$

The FACTOR THEOREM is mostly used to:

1. **Factorize cubic (third degree) expressions**
2. **Find solutions (*x* – values) of cubic equations**
3. **Find the *x* – intercepts of the graph of the cubic function**

EXERCISE 1: (Remainder & Factor theorems)

1. For each of the following, determine the remainder if

1.1 $f(x) = x^3 + x^2 - 1$ is divided by $2x + 3$
1.2 $g(x) = x^3 + 4x^2 - 11x - 31$ is divided by $x + 5$
1.3 $p(x) = 2x^3 - x^2 + 3x - 8$ is divided by $s(x) = x - 2$
1.4 $h(x) = 8x^4 - 4x^2 - 5$ is divided by $(2x - 1)$

2. Given: $f(x) = x^3 - 2x^2 - 4x + 3$

2.1 Use the remainder theorem and determine the remainder if $f(x)$ is divided by
2.1.1 $x + 1$
2.1.2 $x - 3$
2.2 What may be concluded from the previous two calculations?

3. Prove that $x + 1$ is a factor of $f(x) = x^3 + 5x^2 - 17x - 21$.

4. Given: $g(x) = x^3 + px + 6$

Determine the value of p if $(2 - x)$ is a factor of g.

5. Given: $p(x) = x^3 + ax^2 + bx + 6$

5.1 If $x - 2$ is a factor of p, show that $2a + b = -7$.
5.2 Hence, determine the values of a and b if it is further given that the remainder is 48 when p is divided by $x - 3$.

Your knowledge of the Remainder and Factor theorems will mainly be used to factorise cubic functions, in order to determine the x – intercepts of the graphs.

DIFFERENT METHODS TO FACTORISE A CUBIC POLYNOMIAL (3^{RD} DEGREE)	
METHOD AND DESCRIPTION OF STEPS	**EXAMPLES**
SUM AND DIFFERENCE OF CUBES	A) $f(x) = x^3 + 27$ $= (x+3)(x^2 - 3x + 9)$ Cannot factorise further B) $f(x) = 8x^3 - 1$ $= (2x-1)(4x^2 + 2x + 1)$ Cannot factorise further
FACTORISE BY GROUPING • **Group terms in two pairs** • **Take out common factor from each pair** • **Two sets of brackets now become common factor** • **Factorise bracket further if possible**	$f(x) = \mathbf{x^3 + 3x^2 - 4x - 12}$ $= x^2(x+3) - 4(x+3)$ $= (x+3)(x^2 - 4)$ $= (x+3)(x+2)(x-2)$
FACTORISE BY INSPECTION • **Find one linear factor using factor theorem** • **Find other factor (quadratic expression) by inspection**	$f(x) = 2x^3 - 2x^2 - 10x - 6$ $f(-1) = 2(-1)^3 - 2(-1)^2 - 10(-1)$ $- 6 = 0$ $\therefore (x+1)$ is a factor $f(x) = (x+1)(ax^2 + bx + c)$ Now find these coefficients Start with a and c: $1 \times a = 2 \quad \therefore a = 2$ $1 \times c = -6 \therefore c = 6$ You now need to find b: Multiply the two brackets; the two x^2-terms need to give you $-2x^2$: $f(x) = (x+1)(2x^2 + bx + 6)$ $bx^2 + 2x^2 = -2x^2 \quad \therefore b = -4$ $\therefore f(x) = (x+1)(2x^2 - 4x + 6)$ $= (x+1)(2x+2)(x-3)$
SYNTHETIC OR LONG DIVISION • **Find one linear factor using factor theorem** • **Find other factor (quadratic expression) by long division or synthetic division** **(SEE NEXT PAGE)**	$f(x) = 2x^3 - 2x^2 - 10x - 6$ $f(-1) = 2(-1)^3 - 2(-1)^2 - 10(-1)$ $- 6 = 0$ $\therefore (x+1)$ is a factor $f(x) = (x+1)(ax^2 + bx + c)$ Find a, b, c using synthetic division

SYNTHETIC DIVISION

$f(-1) = 0$, so $(x + 1)$is a factor

$f(-1) = 0$ $f(x) = 2x^3 - 2x^2 - 10x - 6$

-1	2	-2	-10	-6
		-2	4	6
	2	-4	-6	0

This method is called SYNTHETIC division, because we don't really divide.
We actually multiply and add.

Note the following:

- The x-value of -1 that gave us the factor $(x + 1)$ is written on the LHS
- The coefficients of the cubic polynomial are written in the top row
- The first coefficient, 2, is carried down to the last row
- Now starting from the left:
  MULTIPLY along the dotted arrow
  and write the ANSWER in the block one row up and one column right
- Now ADD DOWN in the column (the two values underneath each other)

repeat steps

- You MUST get 0 in the last block
- The 3 values in the bottom row are the coefficients of the quadratic factor.

So, $f(x) = (x + 1)(2x^2 - 4x - 6)$

You can now complete the factorising:

$f(x) = (x + 1)(2x + 2)(x - 3)$

EXERCISE 2: (Factorise cubic expressions and functions)

1 Factorise the following expressions completely:

a $27x^3 - 8$

b $5x^3 + 40$

c $x^3 + 3x^2 + 2x + 6$

d $4x^3 - x^2 - 16x + 4$

e $4x^3 - 2x^2 + 10x - 5$

f $x^3 + 2x^2 + 2x + 1$

g $x^3 - x^2 - 22x + 40$

h $x^3 + 2x^2 - 5x - 6$

i $3x^3 - 7x^2 + 4$

j $x^3 - 19x + 30$

k $x^3 - x^2 - x - 2$

2 Solve for x:

a $x^3 + 2x^2 - 4x = 0$

b $x^3 - 3x^2 - x + 6 = 0$

c $2x^3 - 12x^2 - x + 6 = 0$

d $2x^3 - x^2 - 8x + 4 = 0$

e $x^3 + x^2 - 2 = 0$

f $x^3 = 16 + 12x$

g $x^3 + 3x^2 = 20x + 60$

3 Show that $x - 3$ is a factor of $f(x) = x^3 - x^2 - 5x - 3$ and hence solve $f(x) = 0$.

4 Show that $2x - 1$ is a factor of $g(x) = 4x^3 - 8x^2 - x + 2$ and hence solve $g(x) = 0$.