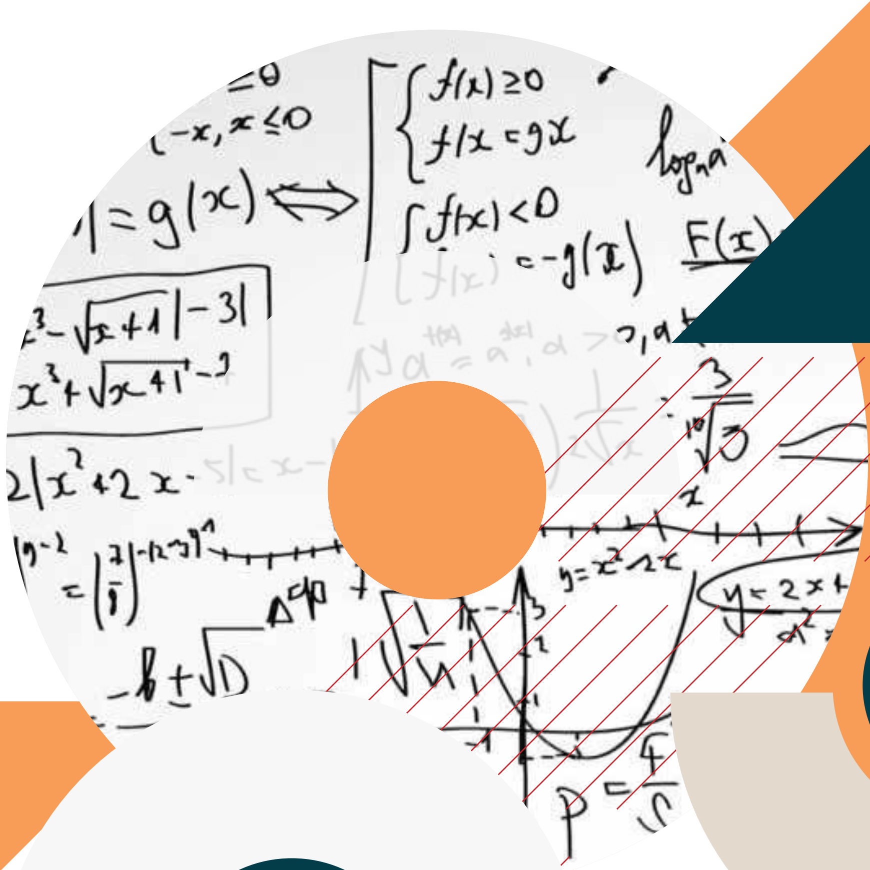
GRADE 12

# Technical Mathematics

## SELF STUDY GUIDE
## BOOK 1

*1. ALGEBRA*

*2. FUNCTIONS AND GRAPHS*



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## (i) INTRODUCTION

The declaration of COVID-19 as a global pandemic by the World Health Organisation led to the disruption of effective teaching and learning in many schools in South Africa. The majority of learners in various grades spent less time in class due to the phased-in approach and rotational/ alternate attendance system that was implemented by various provinces. Consequently, the majority of schools were not able to complete all the relevant content designed for specific grades in accordance with the Curriculum and Assessment Policy Statements in most subjects.

As part of mitigating against the impact of COVID-19 on the current Grade 12, the Department of Basic Education (DBE) worked in collaboration with subject specialists from various Provincial Education Departments (PEDs) developed this Self-Study Guide. The Study Guide covers those topics, skills and concepts that are located in Grade 12, that are critical to lay the foundation for Grade 12. The main aim is to close the pre-existing content gaps in order to strengthen the mastery of subject knowledge in Grade 12. More importantly, the Study Guide will engender the attitudes in the learners to learning independently while mastering the core cross-cutting concepts.

## (ii) HOW TO USE THIS SELF STUDY GUIDE?

- This study guide covers three topics, namely Algebra as well as Functions and Graphs.

- In 2021, there are three Technical Mathematics Books. This one is Book 1. Book 2 covers Differential Calculus and Integration while Book 3 covers Trigonometry and Euclidean Geometry.

- For each topic, sub-topics are listed followed by the weighting of the topic in the paper where it belongs. This booklet covers the three topics mentioned which belong to Technical Mathematics Paper 1

- Definitions of concepts are provided for your understanding

- Concepts are explained first so that you understand what action is expected when approaching problems in that particular concept.

- Worked examples are done for you to follow the steps that you must follow to solve the problem.

- Exercises are also provided so that you have enough practice.

- Selected Exercises have their solutions provided for easy referral/ checking your correctness.

- More Exam type questions are provided.

## 1 ALGEBRA

- Quadratic Equations and Inequalities
- Nature of roots
- Exponents and surds
- Logarithms
- Complex numbers

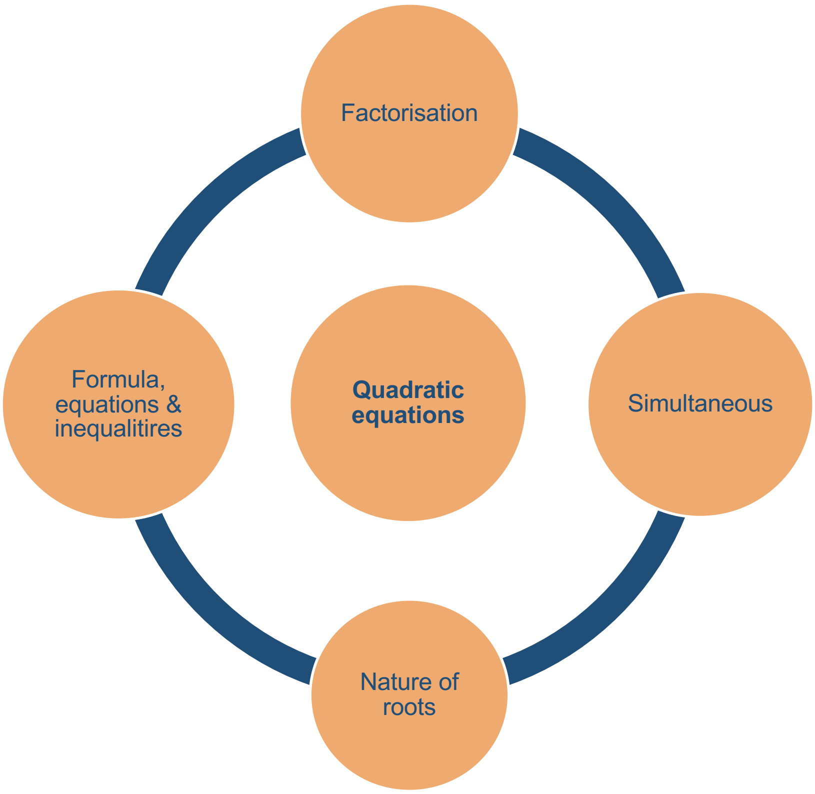
Mark allocation

| Grade 10 | Grade 11 | Grade 12 |
|---|---|---|
| $60 \pm 3$ | $90 \pm 3$ | $50 \pm 3$ |

### Quadratic Equations

A quadratic equation has at most two solutions or roots

Quadratic equations Mind map



**Factorisation** is a method of writing numbers as the product of their factors or divisors.

Determining the roots/solution by **factorisation,** do the following steps:

- The equation should be in the standard form: $ax^2 + bx + c = 0$
- Factorise the quadratic equation
- Write the equation with factors
- Solve the equation
- Write down the final answer
- Take note that there are equations with two/three terms with variables and have to first take out a common factor and find values of *x*.

- Common factors
- Difference of two squares
- Trinomials

**Examples:**
Factorise the following:

**1** $6xy + 15yz$

Step 1: $6xy = 2 \times 3 \times x \times y$

Step 2: common factor of the expression is 3*y*

$\therefore 6xy + 15yz = (2 \times 3 \times x \times y) + (5 \times 3 \times y \times z)$

$= 3y(2x + 5z)$

**2** $x^2 + 10x + 25$

$= (x+5)(x+5)$

**3** $9x^2 - 25y^2$

$= (3x)^2 - (5y)^2$

$= (3x+5y)(3x-5y)$

**Practice questions**

Factorise the following:

**1** $15x + 25$ (2)

**2** $3x^2 - 9$ (3)

**3** $6x - 4 + 6x^2$ (3)

**4** $7x^2 - 28$ (3)

Solve for *x*:

**Quadratic formula**: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

- Using the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ on quadratic equations given with the form $ax^2 + bx + c = 0$, where *a* is the co-efficient of $x^2$, *b* is the co-efficient of *x* and *c* is the constant term.
- It can be used to solve equations that can be factorised and those that cannot be factorised.

**Steps to follow:**

- Identify numerical values of *a*, *b* and *c*.
- Copy formula from the **FORMULA SHEET**: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Substitute the numerical values of *a*, *b* and *c* into the quadratic formula
- Solve for the values of *x*.

**Examples**

Solve for *x* using the quadratic formulae. Correct to two decimal places where necessary.

Factorise the following:

**1** $x^2 - 2x - 5 = 0$

$a = 1,\ b = -2$ and $c = -5$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ quadratic formula

$- \times - =$

$x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-5)}}{2(1)}$ substitution

$x = 3{,}45$ or $x = -1{,}45$ *x*-values

**2** $x^2 + 4x - 14 = 7$

$x^2 + 4x - 21 = 0$ standard form

$a = 1,\ b = 4$ and $c = -21$

$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ quadratic formula

$x = \dfrac{-4 \pm \sqrt{(4)^2 - 4(1)(-21)}}{2(1)}$ substitution

$x = 3 \; or \; x = -7$ *x*-values

**3**

$$3x^2 - 6x = -10$$

$$3x^2 - 6x + 10 = 0$$ standard form

$a = 3$, $b = -6$ and

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-6 \pm \sqrt{(-6)^2 - 4(3)(10)}}{2(3)}$$ substitution

$$x = \frac{-6 \pm \sqrt{-84}}{6}$$

Undefined/no solution in real number system

**OR**

The solution can be presented in complex number form:

$$x = \frac{-6 + \sqrt{84}\,i}{6} \quad \text{or} \quad x = \frac{-6 - \sqrt{84}\,i}{6}$$

if $\sqrt{-}$, then the equation will not yield an answer. there are no real solutions. Solution can be written in complex number form.

## Inequalities

| Inequality sign | Meaning |
|---|---|
| **<** | Less than |
| $\leq$ | Less than or equal to |
| $\neq$ | Not equal to |
| **>** | Greater than |
| $\geq$ | Greater than or equal to |
| $\bullet$ | Value is included |
| $\circ$ | Value not included |
| ( | Excluded |
| ] | Included |
| ( ] | First value excluded and the last value included |

A statement that one quantity is less or greater than the other.

Solving inequalities is very much like solving equations. The difference is: when you solve the equation, you get the point but when you solve inequalities, you get an interval. If $x \in \mathbb{Z}$ (implies that *x* is an element of integers), the set of {.... −3; −2; − 1; 0; 1; 2; 3; ...}.

If $x \in {}^{\sim}$ , then *x* is any number on the number line.

When you are solving an inequality see which set of numbers the solution belongs to.

**Examples:**

Solve for *x* in the following linear inequalities:

**1**

$x + 5 \geq 8$

$x + 5 - 5 \geq 8 - 5$ isolate *x* by subtracting 5 both sides

$\therefore x \geq 3$

**2**

$x - 3 < 5$

$x - 3 + 3 < 5 + 3$ isolate x by adding 3 both sides

$\therefore x < 8$

**3**

$-2x > -8$

$\frac{-2x}{-2} > \frac{-8}{-2}$ divide both sides by -2

$\therefore x < 4$

If we multiply or divide inequalities by a negative number, then the inequality sign

**4**

$\frac{x}{2} \geq -\frac{5}{4}$

$\frac{x}{2} \times 2 \geq -\frac{5}{4} \times 2$ multiply by 2 both sides

$x \geq -\frac{10}{4}$

$\therefore x \geq -\frac{5}{2}$

**OR**

$\therefore x \in \left[-\frac{5}{2};\infty\right)$ solution include all numbers greater than $-\frac{5}{2}$, including $-\frac{5}{2}$

## Solving quadratic inequalities

Quadratic inequality are inequalities in the form:

- $ax^2 + bx + c < 0,\ a \neq 0$
- $ax^2 + bx + c > 0,\ a \neq 0$
- $ax^2 + bx + c \leq 0,\ a \neq 0$
- $ax^2 + bx + c \geq 0,\ a \neq 0$

Key points on solving inequalities

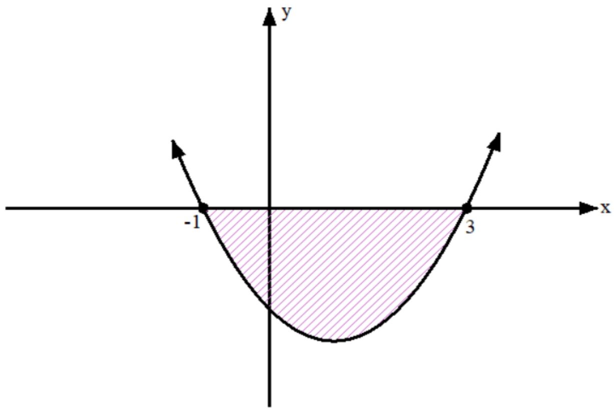
- Factorize and identify critical values.
- Use number line/graphs or any other method to make conclusion (interval)

**Examples**:
Solve for *x* and represent answers graphically

**1** $x^2 - 2x - 3 < 0$

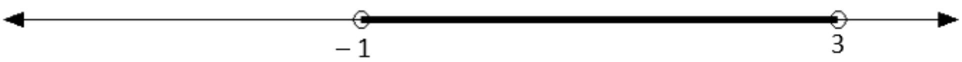
$(x - 3)(x + 1) < 0$ Factors

Critical values: $3$ and – 1




$\therefore -1 < x < 3$ Identify an area where the graph is below the *x*-axis

Number line representation



**2** $x^2 + 5x + 4 \geq 0$

$(x+1)(x+4) \geq 0$ Factors

Critical values: $-4$ and $-1$

$x \leq -4$ or $x \geq -1$ Area where the graph is above the $x$-axis

**3** $x^2 - 3x > 0$

$x(x-3) > 0$ Factors

Critical values: 0 and 3

$x < 0$ or $x > 3$ Area where the graph is above the $x$-axis

## Practice questions

Solve for *x* and represent your solution graphically:

| | | |
|---|---|---|
| **1** | $x(3-x) > 0$ | (4) |
| **2** | $2x \leq 6 - x^2$ | (6) |
| **3** | $(x-2)(3x-1) > 1$ | (7) |
| **4** | $5x(x-3) \leq 0$ | (4) |
| **5** | $-3 < x-1 \leq 2$ | (2) |
| **6** | $x^2 + 2 < -3x$ | (6) |
| **7** | $x^2 - 4 > 0$ | (5) |
| **8** | $-x^2 - 4x + 5 \geq 0$ | (5) |

## Nature of roots

- Formula: $\Delta = b^2 - 4ac$, where *a* is the coefficient of $x^2$, *b* is the coefficient of *x* and *c* is the constant term.
- The solutions of an equation are the roots of the same equation
- $\Delta$ value and description of the roots:

| $\Delta$ value | description of the roots |
|---|---|
| $\Delta = 0$ | Real, rational and equal |
| $\Delta > 0$, perfect square | Real, rational and equal |
| $\Delta > 0$, not a perfect square | Real, irrational and unequal |
| $\Delta \geq 0$ | Real |
| $\Delta < 0$ | Non-real or unreal |

Steps to follow when determining the nature of roots:

- Identify the values of: *a*, *b* and *c* from the given standard equation $ax^2 + bx + c = 0$
- Write the discriminant formula: $\Delta = b^2 - 4ac$
- Substitute the numerical values of *a*, *b* and *c*.
- Use a calculator to find the numerical value.
- Use the numerical value to determine the nature of roots.

**Examples**

Describe the nature of roots of the following equations, without solving the equations:

**1** $4x^2-4x+1=0$

$a=4, b=-4$ and $c=1$ identifying the values of *a* , *b* and *c*

$\Delta=b^2-4ac$ writing down the discriminant formula

$=(-4)^2-4(4)(1)$ substitution

$=16-16$

$\Delta=0$

$\therefore$ the roots are equal

**2** $2x^2+8x+3=0$

$a=2, b=8$ and $c=3$ identifying the values of *a* , *b* and *c*

$\Delta=b^2-4ac$ writing down the discriminant formula

$\Delta=(8)^2-4(2)(3)$ substitution

$=64-24$

$\Delta=40$

$\therefore$ the roots are real, irrational *and* unequal

**3** $3x+2=5x^2$

$-5x^2+3x+2=0$ standard form

$a=-5, b=3$ and $c=2$ identifying the values of *a* , *b* and *c*

$\Delta=b^2-4ac$ writing down the discriminant formula

$=(3)^2-4(-5)(2)$ substitution

$\Delta=49$

$\therefore$ the roots are rational

**4** $x=\dfrac{-3\pm\sqrt{41}}{6}$

Remember: $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$ and it be written as $x=\dfrac{-b\pm\sqrt{\Delta}}{2a}$

$\therefore\ \Delta=41$

$\therefore$ the roots are real, irrational and unequal

## Practice questions

Describe the nature of roots of the following equations, without solving the equations:

**1** $3 - 2x = x^2$ (3)

**2** $2x = 6 - 4x^2$ (3)

**3** $(2x - 2)(3x - 1) = -6$ (4)

**4** $x = \dfrac{6 \pm \sqrt{-10}}{8}$ (1)

**5** $x = \dfrac{6 \pm \sqrt{51}}{8}$ (2)

**6** $ax^2 + bx + c = 0$ if $a < 0$, $b > 0$ and $c = 0$ (2)

## Determining the unknown if the nature of roots is given

### Examples

**1** For which value(s) of $k$ will the equation $x^2 - 5x + k = 0$ have equal roots?

$x^2 - 4x + k = 0$

$\Delta = b^2 - 4ac$ formula

For equal roots, $\Delta = 0$

$(-4)^2 - 4(1)(k) = 0$ substitution

$16 - 4k = 0$

$\therefore k = 4$

**2** For which value(s) of $k$ will the roots of the equation $6x^2 + 6 = 4kx$ be real and equal

$6x^2 + 6 = 4kx$

$6x^2 - 4kx + 6 = 0$ standard form

For real and unequal roots, $\Delta \geq 0$ condition

$(-4k)^2 - 4(6)(6) > 0$ substitution

$16k^2 - 144 > 0$

$k^2 - 9 = 0$ $\div$ by 16

$(k - 3)(k + 3) > 0$

$\therefore k < -3$ or $k > 3$ applying inequalities

## Practice questions

**1** Given $x^2 + bx + 4 = 0$,determine the values of *b* for which the roots will be equal. (3)

**2** If 2 is the root of $2x^2 + kx = -6$, determine the value of *k*. (3)

**3** Given: $2x^2 + kx + 2 = 0$

3.1 Write down the discriminant in terms of *k*. (2)

3.2 Determine the value(s) of *k* for which the roots of the equation are real and equal.

## Simultaneous equations

- Solving for *x* and *y* simultaneously can be used when calculating the point of intersection in graphs.
- When solving simultaneous equations, use the linear equation (equation 1) to make one of the unknown the subject of the formula (equation 3), then substitute it in the other equation (equation2).
- Simplify the equation 2 after substituting to standard form.
- Factorize/use quadratic formula.
- Write down the values of the variable.
- Substitute the calculated variable from equation to in equation 3 to determine the values of the other variable.

**Examples**

Solve for *x* and *y* simultaneously:

**1** $y + 7 = 2x$ and $x^2 - xy + 3y^2 = 15$

| | |
|---|---|
| $y = 2x - 7 \quad (1)$ | subject of the formula |
| $x^2 - x(2x - 7) + 3(2x - 7)^2 = 15$ | substitution |
| $x^2 - 2x^2 + 7x + 3(4x^2 - 28x + 49) = 15$ | |
| $11x^2 - 77x + 132 = 0$ | standard form |
| $x^2 - 7x + 12 = 0$ | $\div$ by 11 |
| $\therefore (x-3)(x-4) = 0$ | factors |
| $\therefore x = 3$ or $x = 4$ | *x*-values |
| $y = 2(3) - 7$ or $y = 2(4) - 7$ | substitution |
| $\therefore y = -1$ or $y = 1$ | *y*-values |

### Practice questions

Solve for *x* and *y* simultaneously:

**1** $y + 3x = 8$ and $y = x^2 + 2x + 4$ (7)

**2** $y^2 + x^2 - 3xy = 1$ and $x - y = 2$ (7)

**3** $y - x + 1 = 0$ and $y + 7 = x^2 + 2x$ (7)

**4** $x - 2y = 0$ and $x^2 + y^2 = 12$ (7)

**5** $3x - y = 2$ and $2y + 9x^2 = -1$ (7)

### Word problems

To solve word problems, you need to:

- Read the problem carefully.
- Identify and underline key words. Key words will assist in identifying mathematical operations that will be used.
- Use variables to distinguish between items used (let the other be *x* and the other one be *y*) if the word problem has two unknowns.
- Set up the equation or equations, if there are two unknowns.
- Solve the equations simultaneously.
- Test your answer.

**Examples**

**1** 593 people came to watch the UJ kwaito explosion concert. 341 of them were children and there were twice as many women as men. How many men were there?

Let the number of men who watched the concert be x
Since 341 out of 593 people were children, the were 593 – 341 = 252 adults
$\therefore$ the number of women is 252 – x
According to the statement "there are twice as many women as men" we have the equation: 252 – *x* = 2*x* (The smaller number of men is the one multiplied by 2)
Then 3*x* = 252 hence *x* = 84, hence there were 84 men at the concert.

**2** Sizwe made 50 runs in his latest cricket match and this raised his average over a number of matches from 45 to 46. Determine how many runs he must make in the next match to raise his average to 47.

Let the average of n matches be 45

$$\therefore \frac{45n + 50}{n+1} = 46$$ LCD $(n+1)$

$$\therefore (n+1)\left(\frac{45n+50}{n+1}\right) = 46(n+1)$$

$$45n + 50 = 46n + 46$$ subtract 46

$$45n + 4 = 46n$$ subtract 45*n*

$$\therefore n = 4$$

Let x be the number of runs in the next match (the sixth match)

$$\frac{(n+1)46 + x}{6} = 47$$

$$\frac{(4+1)46 + x}{6} = 47$$ substituting $n = 4$

$$\frac{230 + x}{6} = 47$$

$$6 \times \frac{230 + x}{6} = 47 \times 6$$ LCD: 6

$$230 + x = 282$$

$$\therefore x = 52$$

The number of runs needed is 52

### Practice questions

1 Tickets for the school concert were sold at 70c for adults and 30c for children. 430 tickets were sold for an amount of R173 each. How many of each kind of were sold?

2 A tank was full of water, and then 25% of the water was run off. Later ,150 litres more run off and this left the tank at exactly $\frac{1}{3}$ full. Determine the capacity of the tank.

### Exponents, surds and logs

| Exponential laws | Properties of logs |
|---|---|
| $a^0 = 1$ ; $a \neq 0$ | $\log_a xy = \log_a x + \log_a y$ |
| $a^p \times a^q = a^{p+q}$ | $\log_a \frac{x}{y} = \log_a x - \log_a y$ |
| $a^{-p} = \frac{1}{a^p}, a \neq 0$ | $\log_a x^n = n \log_a x$ |
| $a^p \div a^q = a^{p-q}$ | $\log_a x = \frac{\log_b x}{\log_b a}$ |
| $(ab)^n = a^n b^n$ | $\log_a a = 1$ |
| | $\log_a 1 = 0$ |

**Examples**

Simplify the following, without the use of a calculator:

**1**

$$\sqrt{72}-\sqrt{18}+\sqrt{8}$$
$$= \sqrt{36 \times 2} - \sqrt{9 \times 2} + \sqrt{4 \times 2}$$
$$= 6\sqrt{2} - 3\sqrt{2} + 2\sqrt{2}$$
$$= 5\sqrt{2}$$

**2**

$$\frac{\sqrt{12} + \sqrt{27}}{\sqrt{75}}$$
$$= \frac{\sqrt{4 \times 3} + \sqrt{9 \times 3}}{\sqrt{25 \times 3}}$$
$$= \frac{2\sqrt{3} + 3\sqrt{3}}{5\sqrt{3}}$$
$$= \frac{5\sqrt{3}}{5\sqrt{3}}$$
$$= 1$$

### Practice questions

Simplify the following, without the use of a calculator:

**1** $$\frac{\sqrt{48}-\sqrt{12}}{2\sqrt{75}}$$ (4)

**2** $$\sqrt{75}-\left(\sqrt{64-16}\right)$$ (3)

**3** $$\frac{\sqrt{48}-\sqrt{12}}{2\sqrt{75}}$$ (3)

**4** $$\sqrt{32}-\sqrt{72}+\sqrt{18}$$ (2)

**5** $$\frac{\sqrt[3]{8x^3}}{\sqrt{9x^2+16x^2}}$$ (5)

### Examples

Simplify the following, without the use of a calculator:

**1** $$\left(8a^{-3}b^{6}\right)^{0}$$
$$=1$$

**2** $$\frac{5^{b-3}}{5^{b+1}}=5^{b-3-b-1}$$
$$=5^{-4}$$
$$=\frac{1}{5^4}$$
$$=\frac{1}{625}$$

**3** $$\frac{2^{3x-1}.8^{x+1}}{4^{2x-2}}$$
$$=\frac{2^{3x-1}\cdot 2^{3(x+1)}}{2^{2(2x-2)}}$$
$$=\frac{2^{3x-1}\cdot 2^{3x+3}}{2^{4x-4}}$$
$$=2^{3x-1+3x+3-(4x-4)}$$
$$=2^{2x+6}$$

## Practice questions

Simplify the following, without the use of a calculator:

**1** $$\frac{3^{2x-1}\times 27^{x+1}}{9^{2x+2}}$$ (3)

**2** $$\frac{7\times 3^{n+2}}{3^{n+4}-6\times 3^{n+1}}$$ (3)

**3** $$\sqrt{\frac{5^{x+1}-5^{x}}{5^{x-1}}}$$ (3)

**4** $$\frac{5^{1-m}-4\cdot 5^{-m}}{5^{-m}+2\cdot 5^{-m+1}}$$ (4)

**5** $$\frac{5\cdot 3^{n+2}+6\cdot 3^{n}}{3^{n+1}}$$ (3)

**Examples**

Simplify the following, without the use of a calculator:

**1** $(\log x + 4\log y) - 5\log z$

$$= \log\left(\frac{xy^4}{z^5}\right)$$

**2**

$$\frac{\log 6 - \log 2}{\log 9(2\log 5 + \log 4)} = \frac{\log\left(\frac{6}{2}\right)}{\log 9\left(\log 5^2 + \log 4\right)}$$

$$= \frac{\log 3}{\log 3^2\left(\log(25 \times 4)\right)}$$

$$= \frac{\log 3}{2\log 3 \times \log 100}$$

$$= \frac{\log 3}{2\log 3 \times 2}$$

$$= \frac{\log 3}{4\log 3}$$

$$= \frac{1}{4}$$

## Practice Questions

Simplify the following, without the use of a calculator:

**1** $\log_2 8+\log_3 \frac{1}{27}$ (4)

**2** $-\log_3 243+\log_3 1$ (4)

**3** $2\log_2 4 + \log_2 10 - \log_2 5$ (4)

**4** $\log_9 81 + \log_9 1 + \log_2 16 - \log_{25} 0{,}04$ (5)

## Examples

Solve for *x*:

**1**

$$5^{x+1} + 5^x = 6$$
$$5^x \times 5^1 + 5^x = 6$$
$$5^x(5+1) = 6$$
$$5^x(6) = 6$$
$$5^x = 1$$
$$5^x = 5^0$$
$$\therefore x = 0$$

**2**

$$x^{\frac{3}{4}} = 8$$
$$x^{\frac{3}{4}\times\frac{4}{3}} = 8^{\frac{4}{3}}$$
$$x^{\frac{3}{4}\times\frac{4}{3}} = 2^{3\times\frac{4}{3}}$$
$$\therefore x = 16$$

## Practice Questions

Solve for *x*:

**1** $2^{x}+2^{x+1}=6$

**2** $3^{x+1}=\dfrac{1}{27}$

**3** $\dfrac{\left(4^{x}\right)^{2x}\times\sqrt{16^{-3}}}{4^{x}}=\left(4x\right)^{0}$

## Worked examples

Solve for *x*:

**1** $\log_3 x=\log_3 4+\log_3 7$

$\log_3 x=\log_3(4\times 7)$ — applying log property

$\log_3 x=\log_3 28$

$\therefore x=28$ — *x*-value

**2** $\log_5(5x+2)=\log_5 3+2\log_5 x$

$\log_5(5x+2)=\log_5 3x^2$ — applying log property

$(5x+2)=3x^2$

$\therefore 0=3x^2-5x-2$ — standard form

$(3x+1)(x-2)=0$ — factors

$x=-\dfrac{1}{3}$ or $x=2$ — *x*-values

$x\neq-\dfrac{1}{3}$ — definition of log

**3** $\log_5(x+3)-\log_5(x-1)=1$

$\log_5\left(\dfrac{x+3}{x-1}\right)=1$ — applying log property

$\dfrac{x+3}{x-1}=5^1$ — changing to exponential form

$5x-5=x+3$ — simplification

$4x=8$

$\therefore x=2$ — *x*-value

## Practice Questions

Solve for *x*:

**1** $\log_2(3x-2)+\log_2 0{,}5=3$ (5)

**2** $\log_5(5x-3)=\log_5 2+2\log_5 x$ (5)

**3** $2\log(x-2)=6$ (3)

**4** $\log_2(x+62)-\log_2 = 5$ (4)

## Complex numbers

We define a complex number to be the number of the form $a+bi$, where *a* and *b* are real numbers and *i* has the property that $i^2=-1$

It is standard convention to use *z* to represent a complex number $a+bi$. If we are given a complex number in the form $z=a+bi$; the real number *a* is known as the real part of *z* and the real number *b* is known as the imaginary part of *z*.

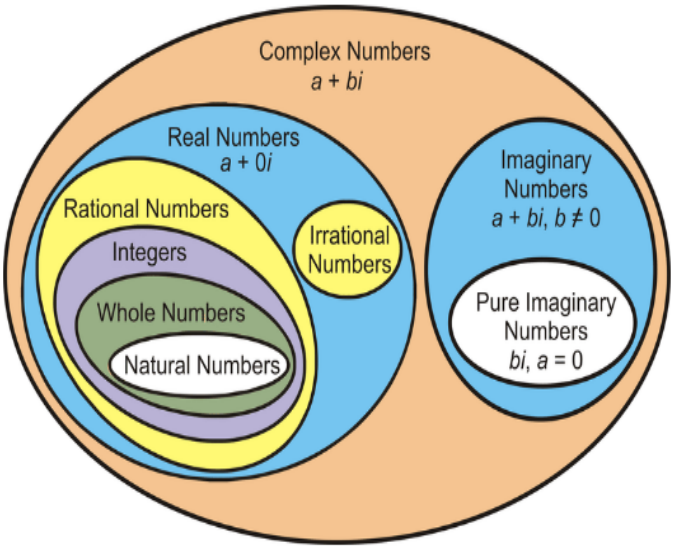
The following are examples of complex numbers:

$3+6i$ $\quad -1.5+3.2i$ $\quad -4i$ $\quad$ 2

The real parts and the imaginary parts for the given complex numbers are shown in the table:

| Complex number | Real part of the complex number | Imaginary part of the complex number |
|---|---|---|
| $3+6i$ | 3 | $6i$ |
| $-1{,}5+3{,}2i$ | $-1{,}5$ | $3{,}2i$ |
| $-4i$ | 0 | $-4i$ |
| 2 | 2 | 0 |

## Relationship between numbers




### Imaginary number

An imaginary number is a number of the form *ib*, where *b* is a real number. The imaginary number *i* has the property $i^2 = -1$. This makes any imaginary number of the form *ib* to have the important property $(ib)^2 = i^2b^2 = (-1)b^2 = -b^2$ (where *b* is a real number).

### Example

### Simplify

| | | |
|---|---|---|
| $4i \times 3i$<br>$= 12i^2$<br>$= 12(-1)$<br>$= -12$ | $-2i \times 3i$<br>$= -6i^2$<br>$= -6(-1)$<br>$= 6$ | $5i \times 5i$<br>$= 25i^2$<br>$= 25(-1)$<br>$= -25$ |

## The square root of a negative number

We can write $\sqrt{-x}$ in terms of $i$ : $\sqrt{-x} = \sqrt{xi^2} = \sqrt{x}\sqrt{i^2} = \sqrt{x}\, i$

For example: 1 $\sqrt{-9} = \sqrt{9} \times \sqrt{-1} = \sqrt{9}\sqrt{i^2} = 3i$

2 $\sqrt{-49} = \sqrt{49} \times \sqrt{-1} = \sqrt{49}\sqrt{i^2} = 7i$

## Addition and subtraction of complex numbers

$(a + ib) + (x + iy) = (\ a + x) + i(b + y)$ and

$(a + ib) - (x - iy) = (a - x) + i(b + y)$

where $a$, $b$, $x$ and $y$ are real numbers.

### Examples

Simplify:

**1** $3 - i + 7 + 2i$

$= 10 + i$

**2** $(2 + 11i) - (-2 + 20i)$

$= 2 + 11i + 2 - 20i$

$= 4 - 9i$

**3** $i^{29} = \left(i^2\right)^{14} \cdot i$

$= (-1)^{14} \cdot i$

$= i$

**4** $i^{103} = \left(i^2\right)^{51} \cdot i$

$= (-1)^{51} \cdot i$

$= - i$

## Complex conjugates

the complex number $z = a + bi$, where $a, b \in R$ is known as the complex conjugate of z.

### Example:

Write down the conjugate of: 1 $z = 4 - 3i$ 2 $z = 3 + i$

Conjugate: Conjugate:

$\bar{z} = 4 + 3i$ $\bar{z} = 3 - i$

### Addition and subtraction of a complex number and its conjugate

$$(a + bi) + i(a - bi)$$
$$= a + bi + ai - bi$$
$$= a + ai$$

$$(a - bi) - (a + bi)$$
$$= a - bi - a - bi$$
$$= -2bi$$

### The product of a complex number and its conjugate

$$(a + bi)(a - bi)$$
$$= a^2 + b^2 i^2$$
$$= a^2 + (-1)b^2$$
$$= a^2 - b^2$$

When we multiply a complex number and its conjugate, we get a positive real number.

### Division of complex numbers

The main idea when dividing complex numbers is to write the quotient in standard form, that is, in the form $z = a + bi$. The denominator(s) of a complex number in standard form are real numbers. We have already established that the product of a complex number and its conjugate is a real number. We will use this established fact when dividing complex numbers.

To divide two complex numbers, we multiply both the numerator and the denominator by the complex conjugate of the denominator.

### Practice Questions

**1** Given: $z_1 = 9 + i$ (3)
Determine the difference of $z_1$ and its conjugate.

**2** Given: $z = 1 - 8i$ (3)
Determine the sum of $z$ and its conjugate.

**3** Simplify: $i^{171}$ (3)

## Multiplication of complex numbers

we use the ordinary rules of algebra together with the property $i^2 = -1$

$$
\begin{aligned}
&(a + ib)(x + iy) \\
&= a(x + iy) + ib(x + iy) \\
&= ax + aiy + ibx + i^2by \\
&= ax + aiy + ibx + (-1)by \\
&= ax + aiy + ibx - by \\
&= (ax - by) + (ay + bx)i
\end{aligned}
$$

### Worked examples

Simplify:

**1**

$$
\begin{aligned}
&(1 + i)(1 - i) \\
&= 1 - i^2 \\
&= 1 - (-1) \\
&= 2
\end{aligned}
$$

**2** Given: $z_1 = 10 - 3i$ and $z_2 = -7 - 9i$

Determine: $z_1 \times z_2$

$$
\begin{aligned}
z_1 \times z_2 &= (10 - 3i)(-7 - 9i) \\
&= -70 - 90i + 21i + 27i^2 \\
&= -70 - 69i + 27(-1) \\
&= -70 - 27 - 69i \\
&= -97 - 69i
\end{aligned}
$$

## Examples

Simplify:

**1** $$\frac{6+7i}{2} = \frac{6}{2} + \frac{7i}{2}$$
$$= 3 + 3{,}5i$$

**2** $$\frac{3}{6i} \times \frac{i}{i} = \frac{3i}{6i^2}$$
$$= \frac{3i}{6(-1)}$$
$$= -\frac{3i}{6}$$
$$= -\frac{1}{2}i$$

**3** $$\frac{2+5i}{3-i} = \frac{2+5i}{3-i} \times \frac{3+i}{3+i}$$
$$= \frac{6+17i+5i^2}{9-i^2}$$
$$= \frac{6+17i+5(-1)}{9-(-1)}$$
$$= \frac{1+17i}{10}$$
$$= \frac{1}{10} + \frac{17}{10}i$$

## Practice Questions

Simplify

**1** $$\frac{2-i}{5+2i}$$

**2** $$\frac{-4+i}{1+3i}$$

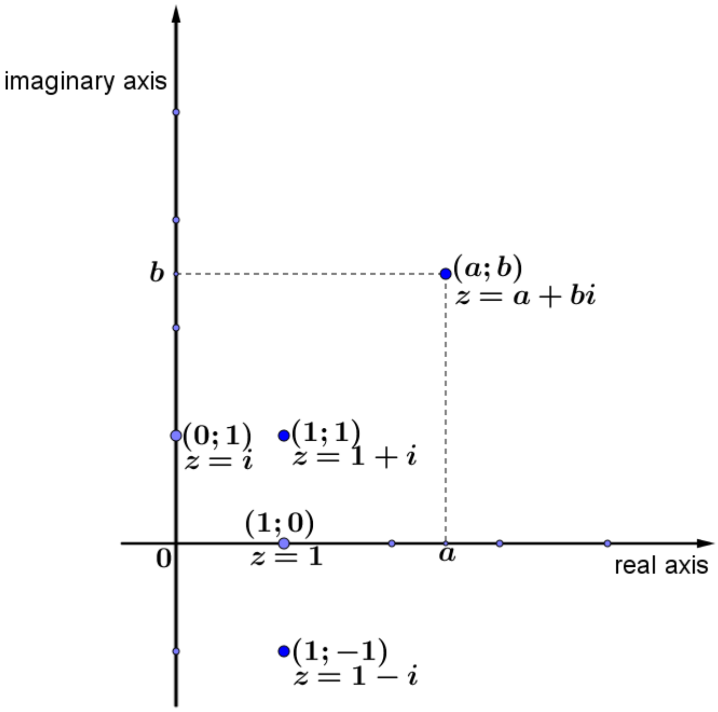
**3** Given: $z_1 = 3 + 2i$ and $z_2 = 1 - i$

Determine $\frac{z_1}{z_2}$

## The Argand diagram

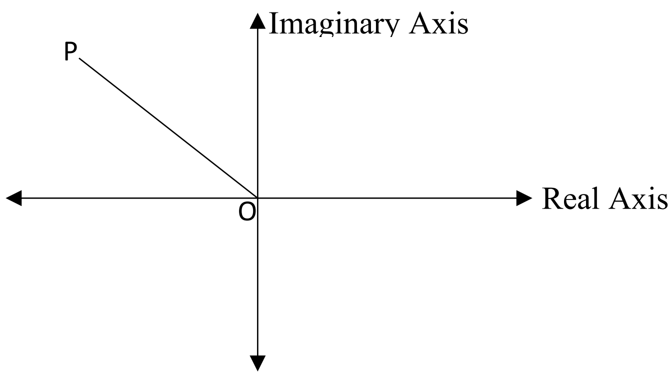
We can represent complex numbers on the complex number plane known as the Argand diagram. The complex plane consists of the real and imaginary axes. When we plot points on the Argand diagram, we must first identify the real and imaginary parts of a given complex number as illustrated in the table below. The real part of the complex number is plotted on the real axis also known as the $x-axis$, and the imaginary part is plotted on the real axis also known as the $y-axis$.

| **Complex number:** $z = a + bi$ | **Ordered pair:** $z = (a;b)$ |
|---|---|
| $z = 1 + i$ | $(a;b) = (1;1)$ |
| $z = 1 - i$ | $(a;b) = (1;-1)$ |
| $z = 1$ | $(a;b) = (1;0)$ |
| $z = i$ | $(a;b) = (0;1)$ |




## The argument of a complex number

Consider the diagram below



There are two important quantities associated with line segment OP:

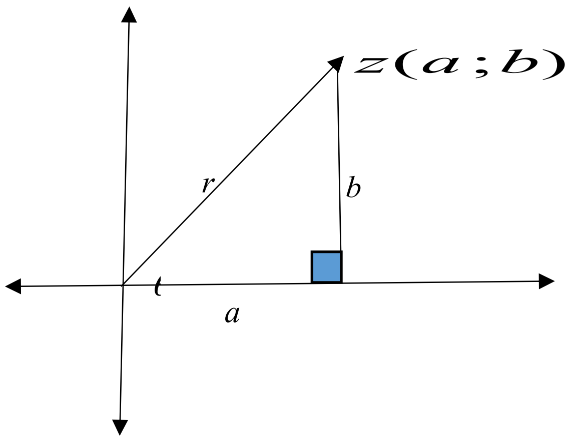
- The length of OP also known as the **modulus** of OP.
- The angle made by the line segment OP with the positive real axis also known as the **argument** $(\theta)$.

The argument of a complex number $z = a + bi$, where $a, b \in R$ and $i^2 = -1$, is the angle formed by the line segment (or vector) and the real axis. We denote the argument of a complex number by argument (*r*).

**The Trigonometric or Polar form of a complex number**

The trigonometric form of a complex number $z = a + bi$ is $z = r\left(\cos\theta + i\sin\theta\right)$ where:

- $a = r\cos\theta$
- $b = r\sin\theta$
- $r = \sqrt{a^2 + b^2}$
- $\tan\theta = \dfrac{b}{a}$




The number $r$ is called the modulus of $z$. It is the distance from the origin to the point.

$\theta$ is called the argument of $z$.

**From the Trigonometric form to the Cartesian form of a complex number**

The standard form $z = a + bi$ of a complex number is also known as the **Cartesian form**. It denotes the complex number in terms of its horizontal and vertical dimensions.

A complex number that takes the form $z = r\left(\cos\theta + i\sin\theta\right)$ **OR** $z = r\underline{|\theta}$ **OR** $z = rcis\theta$ is known as the **Trigonometric form or the Polar form**. It denotes the complex number in terms of length and the angle of its vector.

Converting between the trigonometric from to the Cartesian form involves trigonometry.

## Worked examples

Given: $z = -5 + 2i$

**1** Determine the modulus of $z$.

$|r| = \sqrt{(-5)^2 + (2)^2}$ substitution

$= \sqrt{29}$ modulus

**2** Determine the argument of $z$.

$\tan\theta = -\dfrac{2}{5}$

$\text{Ref. } \angle = 21{,}8^\circ$

$\theta = 180^\circ - 21{,}8^\circ$

$= 158{,}2^\circ$

**3** $z = \sqrt{29}\left(\cos 158{,}2^\circ + i\sin 158{,}2^\circ\right)$

**OR**

$z = \sqrt{29}\,\underline{|158{,}2^\circ}$

**OR**

$z = \sqrt{29}\,cis\,158{,}2^\circ$

## Practice questions

**1.1** Given: $z = -2 + 3i$

**1.1.1** Determine the conjugate of *z*.

**1.1.2** Represent *z* and its conjugate on an Argand diagram.

**1.1.3** Calculate the modulus of *z*.

**1.1.4** Calculate the value of the Argument of *z*.

**1.2** Determine a complex number in the form $z = a + bi$ if $r = 2$ and $\theta = 60°$

**1.3** Determine the complex number $z = a + bi$ where $a, b \in \sim$ , if:

**1.3.1** $|r| = 2$ and $\theta = 60^{\circ}$

**1.3.2** $|r| = 4$ and $\theta = \dfrac{3\pi}{4}$

**1.4** Given: $z = -2 + i$

**1.4.1** Determine the modulus of $z$. (2)

**1.4.2** Sketch $z$ in the Argand diagram plane. (2)

**1.4.3** Determine the argument of $z$. (3)

**1.4.4** Express $z$ in polar form. (2)

## Solving equations with complex numbers with two variables

Two complex numbers $z_1 = a + bi$ and $z_2 = x + yi$, where $a$ and $b$ are $\in$ ~ if $a = x$ and $b = y$

### Worked examples

Solve for $x$ and $y$:

**1**

| | |
|---|---|
| $-2 + i = (x - yi)(1 + i)$ | |
| $= x + xi - yi - yi^2$ | product |
| $= x + xi - yi - y(-1)$ | |
| $= -2 + i = (x + y) + (x - y)i$ | common factor |
| $-2 = x + y$.....(1) and $1 = x - y$....(2) | equations |
| $x = -2 - y$....(3) | subject of the formula |
| $1 = -2 - y - y$ | substitution |
| $-2y = 3$ | |
| $\therefore y = -\dfrac{3}{2}$ | *y*-value |
| $x = -2 - \left(-\dfrac{3}{2}\right)$ | substitution |
| $\therefore x = -\dfrac{1}{2}$ | *x*-value |

**2**

| | |
|---|---|
| $x + yi = \dfrac{7 + i}{2 - i}$ | |
| $7 + i = (x + yi)(2 - i)$ | cross multiplication |
| $= 2x - xi + 2yi - yi^2$ | product |
| $= 2x - xi + 2yi - y(-1)$ | |
| $7 + i = (2x + y) + (2y - x)i$ | common factor |
| $7 = 2x + y$.....(1) and $1 = 2y - x$....(2) | equations |
| $7 - 2x = y$....(3) | subject of the formula |
| $1 = 2(7 - 2x) - x$ | substitution |
| $1 = 14 - 4x - x$ | |
| $-5x = -13$ | |
| $\therefore x = \dfrac{13}{5}$ | *x*-value |
| $7 - 2\left(\dfrac{13}{5}\right) = y$ | substitution |
| $\therefore y = \dfrac{9}{5}$ | *y*-value |

## Practice questions

Simplify:

**1** $x + 2yi = (-2 + 6i)(4 - 7i)$ (4)

**2** $x + 3i = (1 + yi)(2 - 3i)$ (5)

**3** $x + yi = \left(3 + i^{3}\right)\left(4 - 6i^{3}\right)$ (5)

**4** $x + yi = (9 - 7i)(3 + i)$ (5)

**5** $x + yi = (2 - 3i)^{2}$ (4)

## 2 FUNCTIONS AND GRAPHS

**Total marks in Functions and graphs** $=\pm 35$

In this topic, learners should be able to:

- Draw sketch graphs of functions by dual intercept method.

- Identify area in the graph where:
  - is increasing
  - is decreasing
  - above the *x*-axis
  - below the *y*-axis

- Write down the domain and range of graphs

- Do interpretation of drawn sketch graphs by determining:
  - Equations
  - Intercepts and lengths
- Functions and graphs that learners are expected to learn:
  - $y = f(x) = mx + c$ (Straight line)
  - $y = f(x) = ax^2 + bx + c$ and $y = f(x) = a(x + p)^2 + q$ (Parabola)
  - $y = f(x) = \frac{a}{x} + q$ (Hyperbola)
  - $y = f(x) = a \cdot b^x + q$, where $b > 0$ and $b \neq 0$ (Exponential)
  - $x^2 + y^2 = r^2$ (Circle)
  - $y = f(x) = +\sqrt{r^2 - x^2}$ (Semi-circle above *x*-axis)
  - $y = f(x) = -\sqrt{r^2 - x^2}$ (Semi-circle below *x*-axis)

### Glossary and notations used in functions and graphs

- Intercepts: the points where the graph cuts the axes.
- Turning point: is a point where the graph changes direction. Turning point can either be local minima or local maxima.
- Point of intersection: the point where the two functions are equal.
- $f(x) > 0$: area where the function is above the *x*-axis.
- $f(x) < 0$: area where the function is below the *x*-axis.
- $f'(x) > 0$: area where the function is increasing/gradient is positive.
- $f'(x) < 0$: area where the function is decreasing/gradient negative.
- Asymptotes: a line that a function approaches as it heads towards positive or negative infinity.
- Axis of symmetry: a line that divides the graph into two equal parts.

## Strategies for interpretation of graphs

| Type of question | Interpretation |
|---|---|
| $f(x) < 0$<br>$(-\infty;0)$<br>$f(x) \leq 0$<br>$(-\infty;0]$ | • The graph is below the *x*-axis<br>• Value(s) is/are excluded<br>• The graph is below the *x*-axis<br>• Value(s) is/are included. |
| $f(x) > 0$<br>$(0;\infty)$<br>$f(x) \geq 0$<br>$[0;\infty)$ | • The graph is above the *x*-axis<br>• Value(s) is/are excluded<br>• The graph is above the *x*-axis<br>• Value(s) is/are included. |
| $f(x) > g(x)$<br><br>$f(x) \geq g(x)$ | • Graph of $f$ is above the graph of $g$<br>• Value(s) is/are excluded.<br>• Graph of $f$ is above the graph of $g$<br>• Value(s) is/are included. |
| $f(x) \cdot g(x) > 0$<br><br>$f(x) \cdot g(x) \geq 0$ | • Both graphs of $f$ and $g$ are either above or below the *x*-axis<br>• Value(s) are excluded<br><br>• Both graphs of $f$ and $g$ are either above or below the *x*-axis<br>• Value(s) are included |
| $f(x) \cdot g(x) < 0$<br><br>$f(x) \cdot g(x) \leq 0$ | • One of the graphs of $f$ or $g$ is above the *x*-axis and the other is below the *x*-axis<br>• Value(s) is/are excluded.<br><br>• One of the graphs of $f$ or $g$ is above the *x*-axis and the other is below the *x*-axis<br>• Value(s) is/are included. |
| $f(x) - g(x) = 0$ | • $f(x) = g(x)$<br>• Point of intersection |
| $f^{/}(x) > 0$ | • Area where the graph is increasing.<br>• Value(s) is/are excluded |
| $f^{/}(x) < 0$ | • area where the function is decreasing.<br>• Value(s) is/are excluded |

| Notations | Interpretation |
|---|---|
| ( | • Excluded |
| ] | • Included |
| ( ] | • First value excluded and the last value included |

### Sketching of Straight line and Parabola

Points to remember when sketching Straight line and Parabola

| Function | What to find? | How? |
|---|---|---|
| **Straight line**<br>$y = f(x) = mx + c$ | *y*-intercept | • To find *y*-intercept, let *x* = 0 |
| | *x*-intercept | • To find *x*-intercept, let *y* = 0 |
| **Parabola**<br>$y = f(x) = ax^2 + bx + c$ and<br>$y = f(x) = a(x + p)^2 + q$ | *y*-intercept | • To find *y*-intercept, let *x* = 0 |
| | *x*-intercept | • To find *x*-intercept, let *y* = 0<br>• Solve for *x* by factorizing a trinomial or use Quadratic formula. |
| | Turning point | • First find axis of symmetry using $x = -\frac{b}{2a}$<br>• Then substitute the value of *x* obtained in the axis of symmetry in the given function to find corresponding value of *y*.<br>**OR**<br>Use $\left(-\frac{b}{2a};\frac{4ac-b^2}{4a}\right)$<br>**OR**<br>First differentiate *f*, equate the differentiated function to zero and then solve for *x*. Then substitute the value of *x* obtained in the axis of symmetry in the given function to find corresponding value of *y*. |

## Worked examples

Given the functions defined by $f(x) = x^2 - 4x - 5$ and $g(x) = x - 5$

**1.1** Write down the *y*-intercept of *f* and *g*.

$y$-intercept ; $x = 0$

$f(0) = (0)^2 - 4(0) - 5 = -5$

$g(0) = 0 - 5 = -5$

$\therefore$ $y$-intercept for both $f$ and $g$ is $(0;-5)$

**1.2** Determine the *x*-intercepts of *f* and *g*.

$x$-intercept ; $y = 0$

For $g$:

$x - 5 = 0$

$\therefore x = 5$

For $f$:

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$\therefore x = 5$ or $x = -1$

**1.3** Determine the coordinates of the turning point of *f*.

$$x = -\frac{b}{2a}$$

$$= -\frac{(-4)}{2(1)}$$

$x = 2$

$f(2) = (2)^2 - 4(2) - 5$

$f(2) = -9$

$\therefore$ turning point is $(2;-9)$

**OR**

$$\left(-\frac{b}{2a};\frac{4ac-b^2}{4a}\right)$$

$$\left(-\frac{(-4)}{2(1)};\frac{4(1)(-5)-(-4)^2}{4(1)}\right)$$

$\therefore$ turning point is $(2;-9)$

**OR**

$f'(x) = 2x - 4 = 0$

$2x = 4$

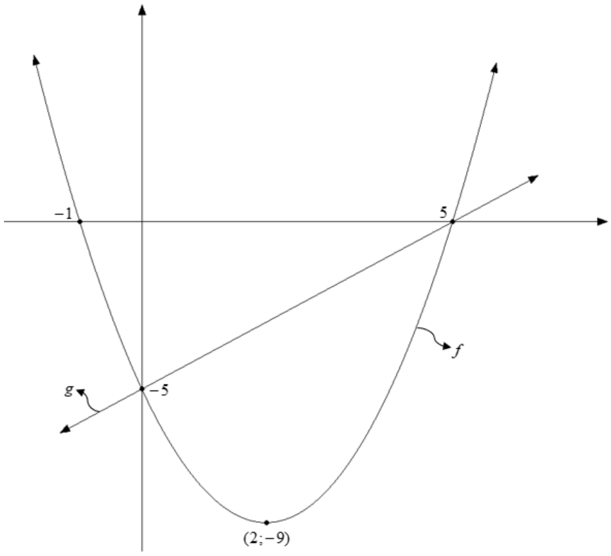
$x = 2$

$f(2) = (2)^2 - 4(2) - 5$

$f(2) = -9$

$\therefore$ turning point is $(2;-9)$

**1.4** Draw sketch graph of $f$ and $g$. Clearly indicate all the intercepts with the axes and turning point.




**1.5** Write down:

**1.5.1** Range of $f$.

$y \geq -9$

**OR**

$[-9;\infty)$

**1.5.2** Domain of $g$.

$x \in \mathbb{R}$

**OR**

$x \in (-\infty;\infty)$

**1.5.3** Coordinates of the point of intersection of $f$ and $g$.

$(0;-5)$

**1.5.4** Value(s) of $x$ for which $f(x) < 0$

$-1 \leq x \leq 5$

**OR**

$x \in [-1;5]$

**1.5.5** Values of $x$ for which $f'(x) > 0$

$x > 2$

**OR**

$x \in (2;\infty)$

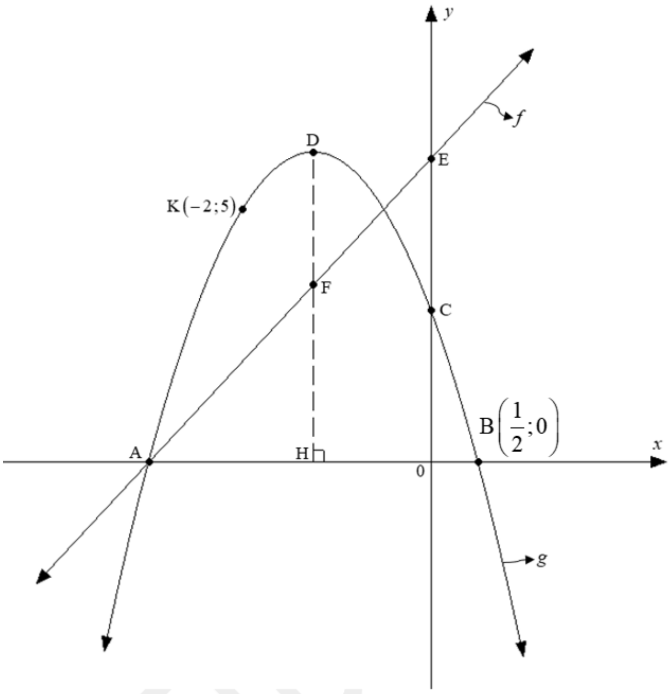
## Worked examples:

The graph below represents functions defined by $f(x) = 2x + 6$ and $g(x) = ax^2 + bx + c$

The point A is the $x$-intercept of both $f$ and $g$ and $B\left(-\frac{1}{2};0\right)$ is the other $x$-intercept of $g$.

C and D are $y$-intercept and turning point of $g$ respectively and $B\left(-\frac{1}{2};0\right)$ is a point on $g$.

E is the $y$-intercept of $f$.
F and $K(-2;5)$ are the points on $f$ and $g$ respectively.
DH is perpendicular to the $x$-axis.




**1** Determine the coordinates of A and E.

$y$-intercept ; $x = 0$

$f(0) = 2(0) + 6$

$\therefore E(0;6)$

$x$-intercept ; $y = 0$

$0 = 2x + 6$

$2x = -6$

$x = -3$

$\therefore A(-3;0)$

**2** Hence, show that the equation of $g$ is $g(x) = -2x^2 - 5x + 3$

$g(x) = a(2x - 1)(x + 3)$

$5 = a(2(-2) - 1)(-2 + 3)$

$5 = -5a$

$a = -1$

$g(x) = -(2x - 1)(x + 3)$

$\quad = -(2x^2 + 5x - 3)$

$\therefore g(x) = -2x^2 - 5x + 3$

**3** Write down the coordinates of C.

$y$-intercept ; $x = 0$

$g(x) = -2(0)^2 - 5(0) + 3 = 3$

$C(0;3)$

**4** Determine the length of AB.

$A(-3;0)$ and $B\left(\frac{1}{2};0\right)$

$AB = 3 + \frac{1}{2}$

$\therefore AB = \frac{7}{2}$ units

**5** Determine the length of DF.

$$x = -\frac{b}{2a}$$

$$= -\frac{(-5)}{2(-2)}$$

$$x = -\frac{5}{4}$$

$$f(2) = -2\left(-\frac{5}{4}\right)^2 - 5\left(-\frac{5}{4}\right) + 3$$

$$f(2) = \frac{49}{8}$$

$$\therefore \mathrm{D}\left(-\frac{5}{4};\frac{49}{8}\right)$$

$$f\left(-\frac{5}{4}\right) = 2\left(-\frac{5}{4}\right) + 6$$

$$\mathrm{F}\left(-\frac{5}{4};\frac{7}{2}\right)$$

$$\mathrm{DF} = \frac{49}{8} - \frac{7}{2}$$

$$\therefore \mathrm{DF} = \frac{21}{8} \text{ units}$$

## Practice Questions

### Question 1

Given the functions defined by $f(x) = x^2 + x - 12$ and $h(x) = x + 4$

**1.1** Determine:

**1.1.1** *y*-intercepts of *f* and *h*. (2)

**1.1.2** *x*-intercept of *h*. (1)

**1.1.3** *x*-intercept of *f*. (2)

**1.1.4** Coordinates of the turning point of *f*. (4)

**1.2** Draw sketch graphs of *f* and *h* on the same system of axes.
Clearly indicate all the intercepts with the axes and turning points. (3)

**1.3** Determine the value(s) for which $f(x) - h(x) = 0$ (6)

**1.4** Use your sketch graph to write down:

**1.4.1** Range of *f*. (2)

**1.4.2** Minimum value of *f*. (1)

**1.4.3** Value(s) of *x* for which $f(x) > 0$
(2)

**1.4.4** Value(s) of *x* for which $f(x) \cdot h(x) > 0$ (2)

### Question 2

Given the function defined by $f(x) = -2(x-1)^2 + 8$

**2.1** Write down coordinates of the turning point of *f*. (2)

**2.2** Determine the intercepts with the axes of *f*. (4)

**2.3** Draw sketch graph of *f*. Clearly indicate all the intercepts

with the axes and the turning point. (3)

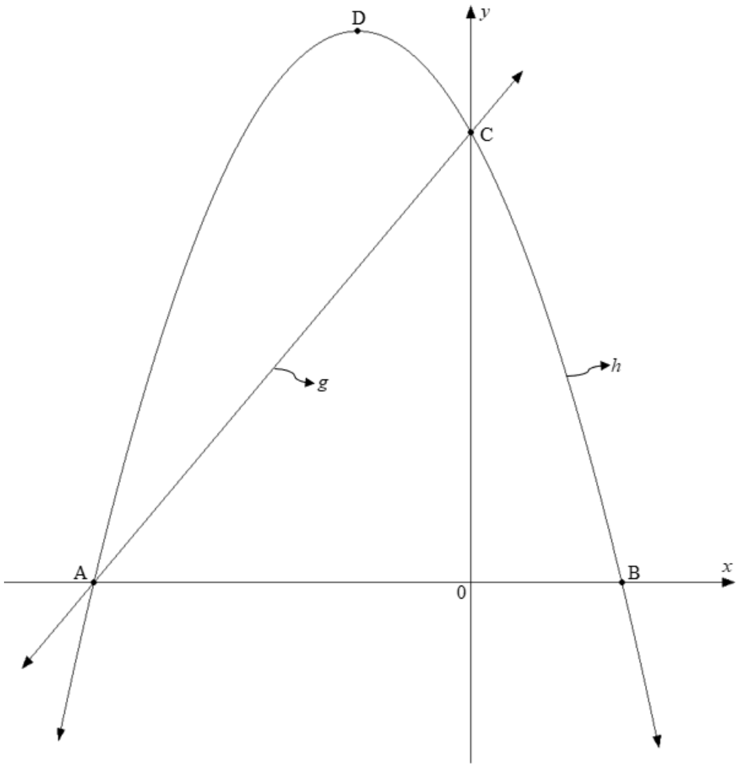
**2.4** Write down the value(s) of *x* for which $f'(x) < 0$ (2)

## Question 3

The graph below represents functions defined by $h(x) = -x^2 - 3x + 10$ and $g(x) = 2x + 10$

The two graphs intersect at points A and C. The points A, B and C are intercepts with the axes

of $h$. D is the turning point of $h$.




Determine:

**3.1** Coordinates of A and B. (3)

**3.2** Coordinates of C. (1)

**3.3** Coordinates of D. (4)

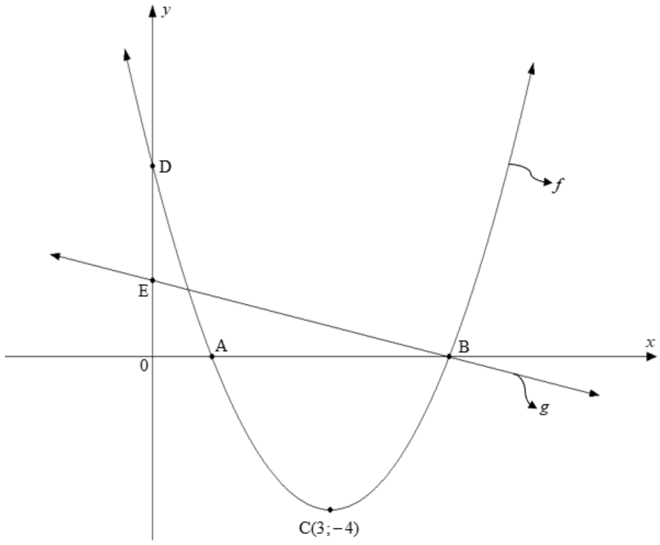
**3.4** Values of $x$ for which $f(x) \cdot g(x) < 0$ (2)

### Question 4

The graphs below represents functions defined by $f(x) = a(x + p)^2 + q$ and

$g(x) \rightarrow 5y + 2x - 10 = 0 \rightarrow g(x) = -\frac{2}{5}x + 2$

The points A, B, and D are the intercepts with the axes of *f*. The point $C(3;-4)$ is the turning point of *f*. E is the *y*-intercept of *g*.




**4.1** Determine coordinates of B and E. (2)

**4.2** Show that $f(x) = (x - 3)^2 - 4$ (3)

**4.3** Determine length of AB (4)

**4.4** Determine length of DE (3)

**4.5** Write down the minimum value of *f*. (1)

## Sketching of Hyperbola

Points to remember when sketching Hyperbola

| Function | What to find? | How? |
|---|---|---|
| Hyperbola<br>$y = f(x) = \dfrac{a}{x} + q$ | *y*-intercept | To find *y*-intercept, let *x* = 0 |
| | *x*-intercept | To find *x*-intercept, let *y* = 0 |
| | Asymptotes | Vertical asymptote is always $x = 0$ as per the Policy Document on this function.<br><br>Asymptote is written as an equation using *x* = …. and *y* = ….<br>For horizontal asymptote, *y* is equated to the value of *q*; i.e. $y = q$ |

### Example 1

Given the function defined by $h(x) = \dfrac{2}{x} - 1$

**1.1** Write down equations of asymptotes of *h*.

**1.2** Determine intercepts with the axes of *h*.

**1.3** Draw sketch graph of *h*. Clearly indicating all intercepts with the axes and asymptotes.

**1.4** Write down the equation of axis of symmetry of *h* with $m > 0$

## Solution to Example 1

**1.1** $x = 0$ and $y = -1$

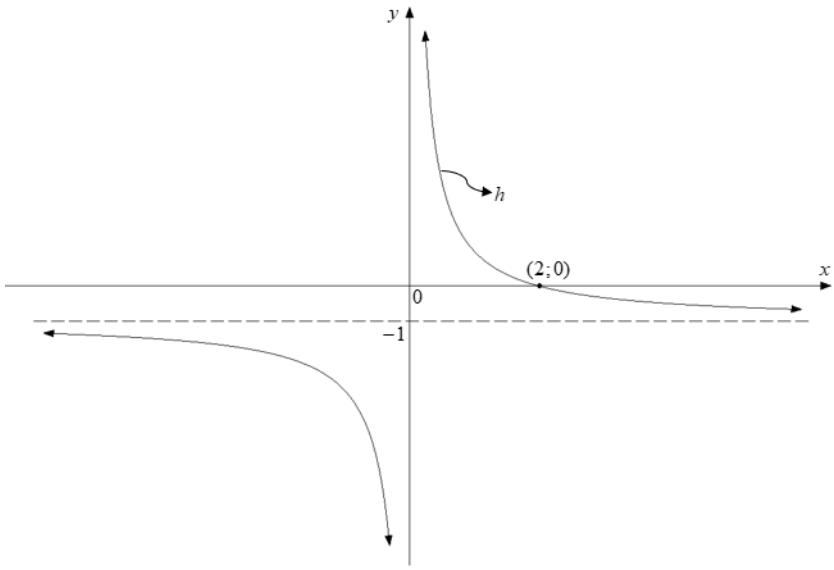
**1.2** $x$-intercept ; $y = 0$

$0 = \frac{2}{x} - 1$

$1 = \frac{2}{x}$

$\therefore x = 2$

**1.3**




**1.4** $m > 0$ implies that $m = +1$

$y = x + c$

Point of intersection of asymptotes is $(0;-1)$

$-1 = 0 + c$

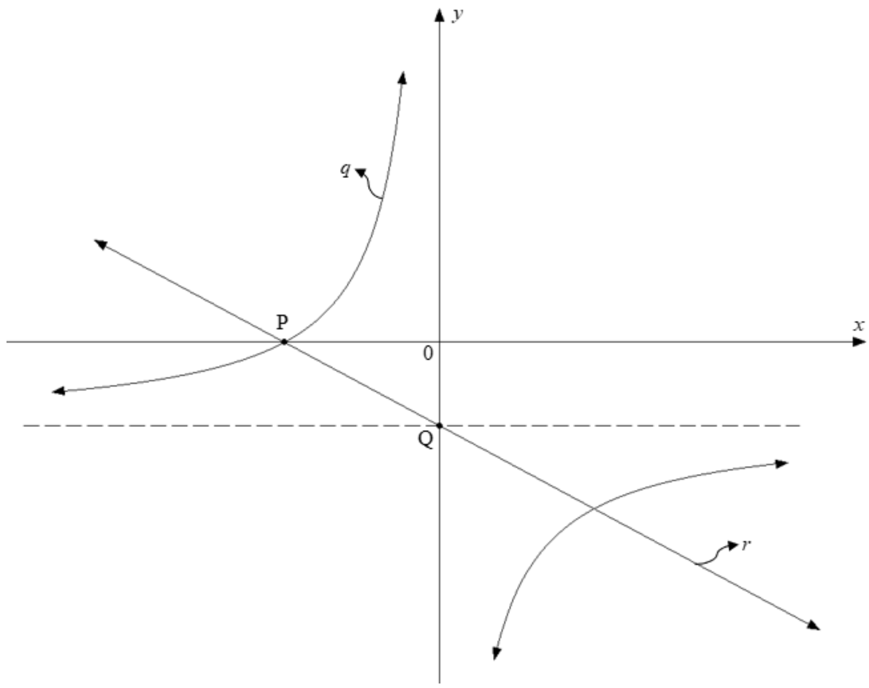
$c = -1$

$\therefore y = x - 1$

### Example 2

The graphs below represents functions defined by $p(x)=\frac{a}{x}+q$ and $r(x)=-x-2$

P is the *x*-intercept of both *p* and *r* and Q is the *y*-intercept of *r*.




**2.1** Determine the coordinates of P and Q.

**2.2** Write down the equation of the horizontal asymptote of *q*.

**2.3** Hence, determine the equation *p*.

### Solution to Example 2

**2.1**

$r(x)=-x-2$

$x$-intercept ; $y=0$

$0=-x-2$

$\therefore x=-2$

$\therefore$ P$(-2;0)$

$y$-intercept ; $x=0$

$y=r(0)=-0-2$

$\therefore y=-2$

$\therefore$ Q$(0;-2)$

**2.2** $y=-2$

**2.3**

$p(x)=\frac{a}{x}-2$

$0=\frac{a}{-2}-2$

$\frac{a}{-2}=2$

$a=-4$

$\therefore p(x)=-\frac{4}{x}-2$

## Practice Questions

### Question 1

Given the function defined by $h(x) = -\dfrac{6}{x} - 1$

**1.1** Write down equations of the asymptotes of $h$. (2)

**1.2** Determine $x$-intercept of $h$. (2)

**1.3** Draw sketch graph of $h$. Clearly indicating all intercepts with the axes and asymptotes. (3)

**1.4** Write down the equation of axis of symmetry of $h$ with $m < 0$ (4)

**1.5** Show that the point $(2;-4)$ lies on the graph of $h$. (2)

**1.6** Write down the range of $h$. (1)

**Question 2**

Given the function defined by $k(x) = \frac{2}{x} + 2$

**2.1** Write down equations of the horizontal asymptotes of *k*. (2)

**2.2** Determine *x*-intercept of *k*. (2)

**2.3** Draw sketch graph of *k*. Clearly indicating all intercepts with the axes and asymptotes. (3)

**2.4** Write down the equation of axis of symmetry of *k* with $m < 0$ (3)

**2.5** Write down the domain of *k*. (2)

**Question 3**

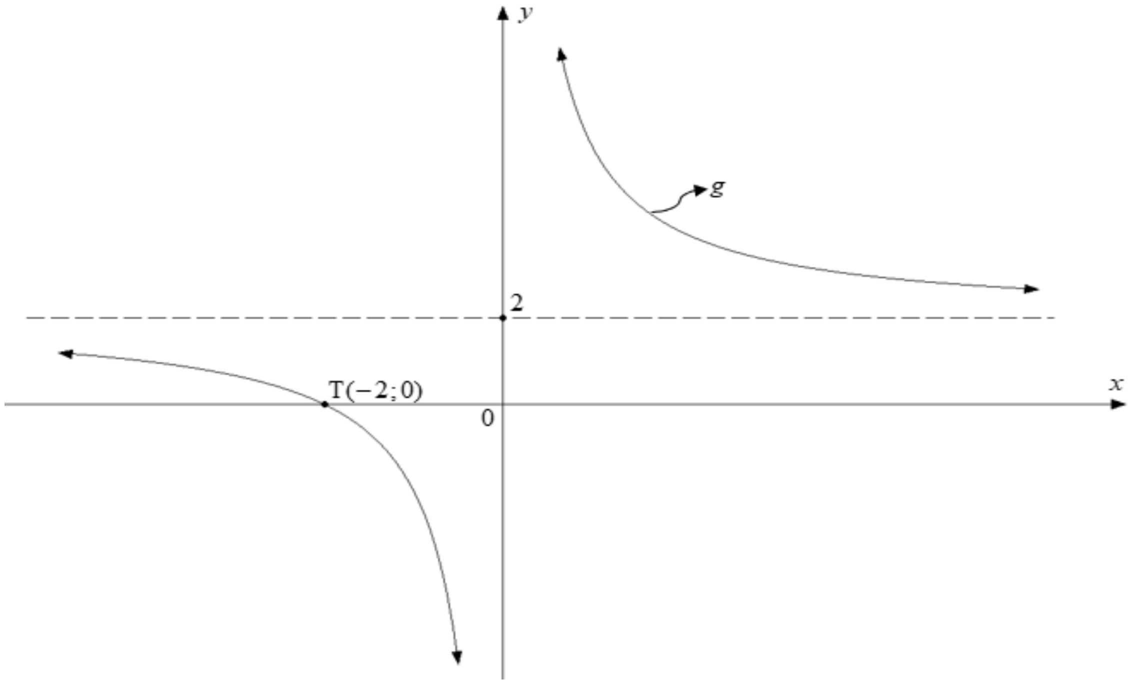
Draw sketch graph $f(x) = \frac{a}{x} + q$ with the following properties:

- $a < 0$
- $f(2) = 0$ (3)
- $y \neq 2$

## Question 4

The drawn graph below represents the function defined by $g(x) = \frac{a}{x} + q$

The graph of $g$ cuts the $x$-axis at the point $\mathrm{T}(-2;0)$.
The dotted line cuts the $y$-axis at 2.




Determine:

**4.1** Numerical value of $q$.

**4.2** Numerical value of $a$.

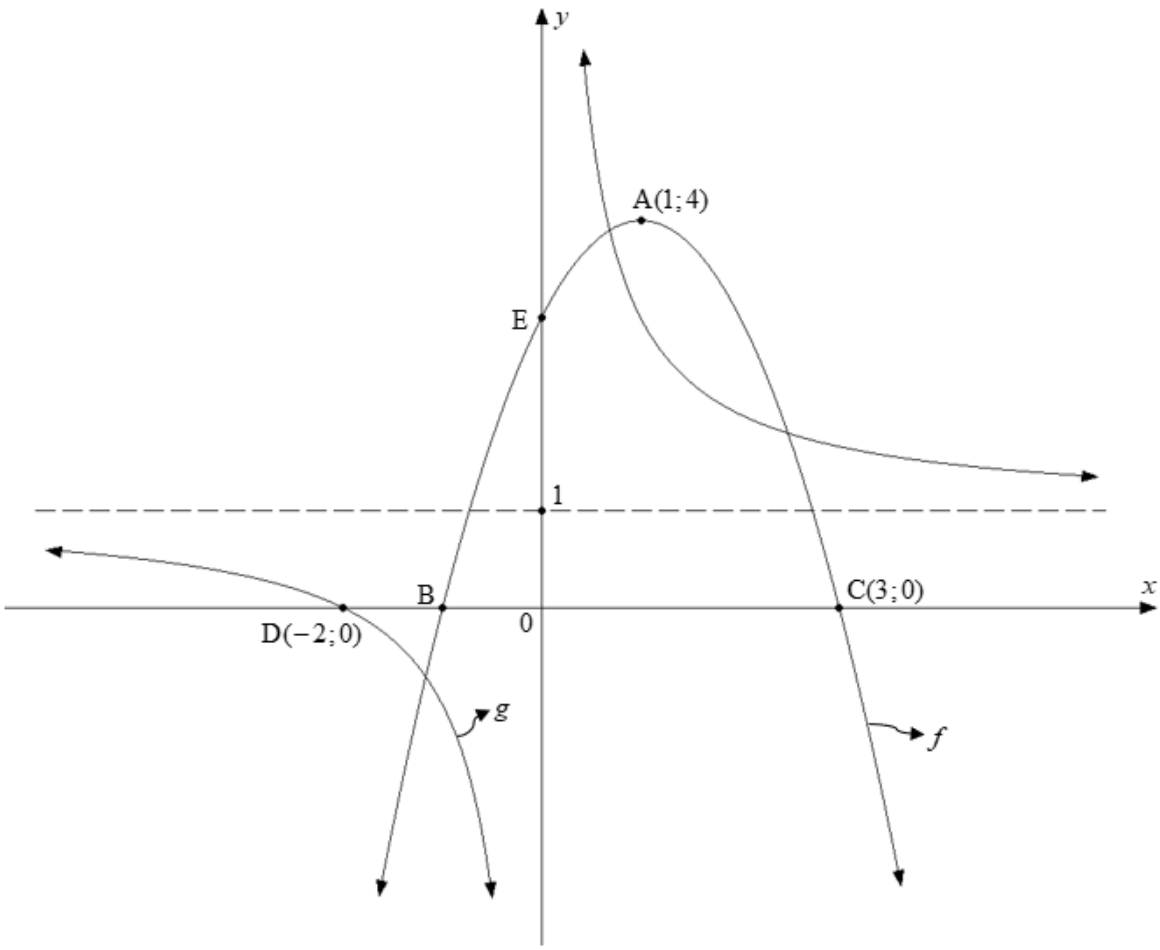
**4.3** The equations of the asymptotes of $g$.

**4.4** The domain of $g$.

**Question 5**

The sketch below is the graphs of the functions defined by $f(x) = a(x + p)^2 + q$ and $g(x) = \dfrac{k}{x} + n$

$A(1;4)$ is the turning point of $f$ which intersects the $x$-axis at B and C.
D and E are $x$-and- $y$-intercepts of $f$ respectively.




**5.1** Write down the coordinates of B. (1)

**5.2** Write down the values of $p$, $q$ and $n$. (3)

**5.3** Determine the equation of $f$. (3)

**5.4** Write down the equation of the axis of symmetry of $f$. (1)

**5.5** Determine the equation of $g$. (2)

## Sketching of Exponential function

Points to remember when sketching Exponential function

| Function | What to find? | How? |
|---|---|---|
| Exponential<br>$y = f(x) = a \cdot b^x + q$,<br>where $b > 0$ and $b \neq 1$ | $y$-intercept | To find $y$-intercept, let $x = 0$ |
| | $x$-intercept | To find $x$-intercept, let $y = 0$ |
| | Asymptotes | Asymptote is written as an equation using<br>$y = ....$<br><br>Horizontal asymptote, $y$ is equated to<br>the value of $q$; i.e. $y = q$ |

### Example 1

Given the function defined by $g(x) = 2^x - 1$

**1.1** Write down the equation of the asymptote of $g$.

**1.2** Determine the $x$-intercept of $g$.

**1.3** Determine the $y$-intercept of $g$.

**1.4** Draw sketch graph of $g$, indicating all the intercepts with the axes and the asymptote.

**1.5** Write down the range of $g$.

### Solution to Example 2

**2.1** $y = -1$

**2.2** $0 = 2^x - 1$

$2^x = 1$

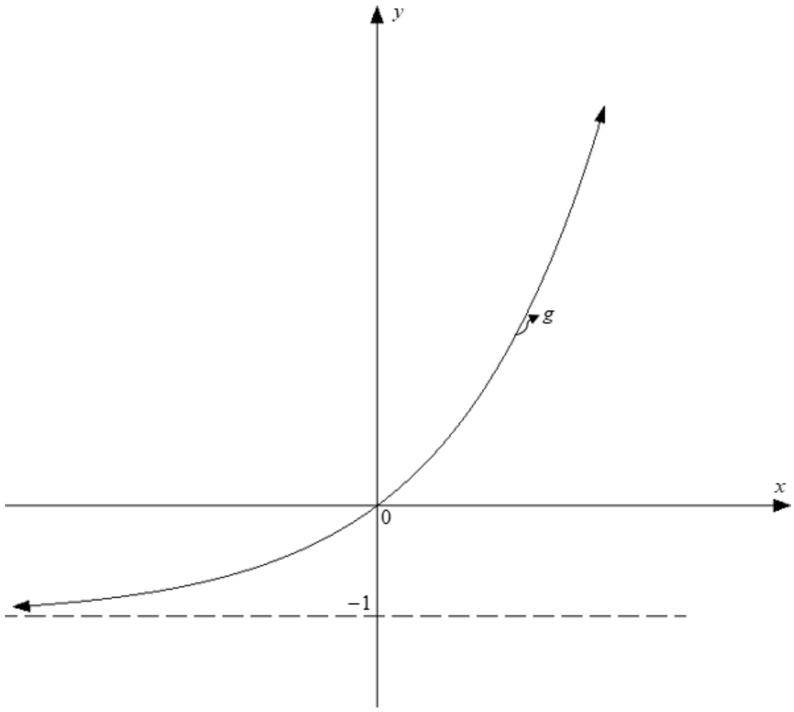
$2^x = 2^0$

$\therefore x = 0$

**2.3** $g(0) = 2^0 - 1$

$\therefore y = 0$

**2.4**




**2.5** $y > -1$

**OR**

$(-1;\infty)$

**Example 2**

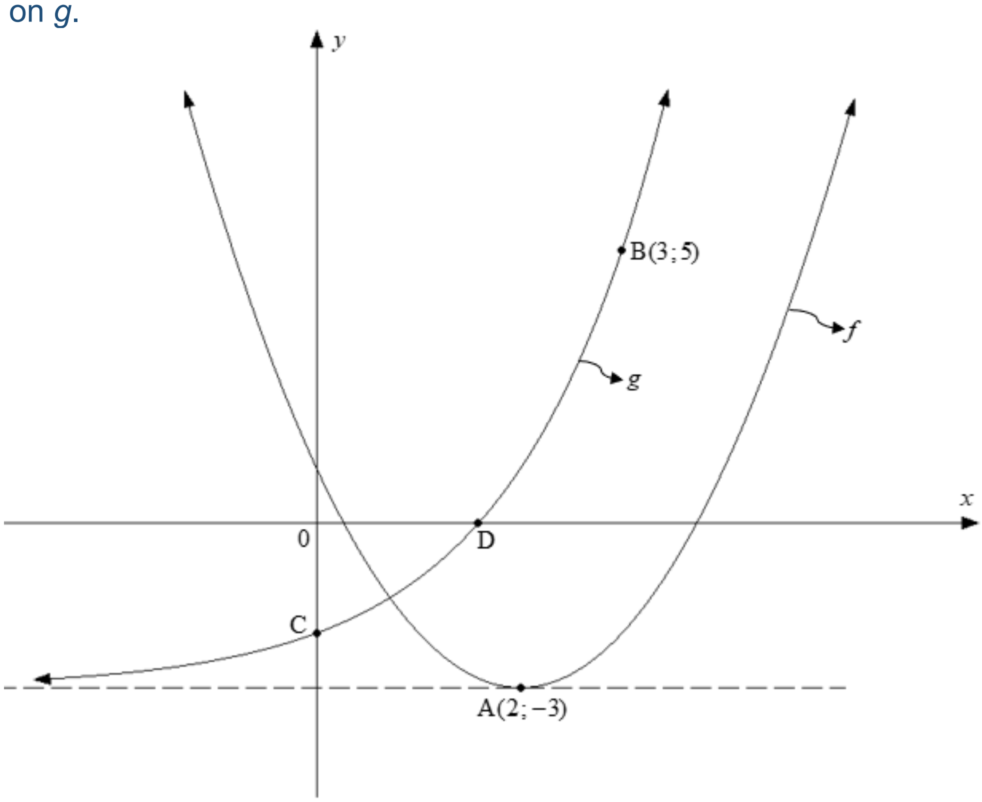
The graphs below represent functions defined by $f(x) = (x - 2)^2 - 3$ and

$g(x) = a^x + q$

$A(2;-3)$ is the turning point of *f*. C and D are *y*-and-*x*-intercepts of *g* respectively and

$B(3;4)$

is a point on *g*.




**2.1** Write down the value of *q*.

**2.2** Hence, determine equation of *g*.

**2.3** Determine coordinates of C and D.

**2.4** Write down:

**2.4.1** Range of *g*.

**2.4.2** Values of *x* for which $g(x) \geq 0$

## Solution to Example 2

**2.1** $q = -3$

**2.2** $g(x) = a^x - 3$

$5 = a^3 - 3$

$8 = a^3$

$2^3 = a^3$

$a = 2$

$g(x) = 2^x - 3$

**2.3** $g(x) = 2^x - 3$

$y$-intercept ; $x = 0$

$y = g(0) = 2^0 - 3$

$y = -2$

$C(0;-2)$

$x$-intercept ; $y = 0$

$0 = 2^x - 3$

$2^x = 3$

$x = \log_2 3$

$x \approx 1,58$

$D(1,58;0)$

**2.4.1** $y > -3$

**OR**

$(-3;\infty)$

**2.4.2** $x \geq 1,58$

**OR**

$[1,58;\infty)$

## Practice Questions

### Question 1

Given the function defined $h(x) = \left(\frac{1}{3}\right)^{x} - 9$

**1.1** Write down the equation of the asymptote of $h$. (1)

**1.2** Determine $y$-and-$x$-intercepts of $h$. (3)

**1.3** Draw sketch graph of $h$. Clearly indicating all intercepts with the axes and asymptotes. (3)

**1.4** Write down:

1.4.1 Domain of $h$. (1)

1.4.2 Range of $h$. (2)

1.4.3 Values of $x$ for which $h(x) \geq 0$ (2)

### Question 2

Given the function defined $h(x) = -2^{x} + 4$

**2.1** Write down the equation of the asymptote of $h$. (2)

**2.2** Determine $y$-and-$x$-intercepts of $h$. (4)

**2.3** Draw sketch graph of $h$. Clearly indicating all intercepts with the axes and asymptotes. (3)

**2.4** Write down:

2.4.1 Domain of $h$. (2)

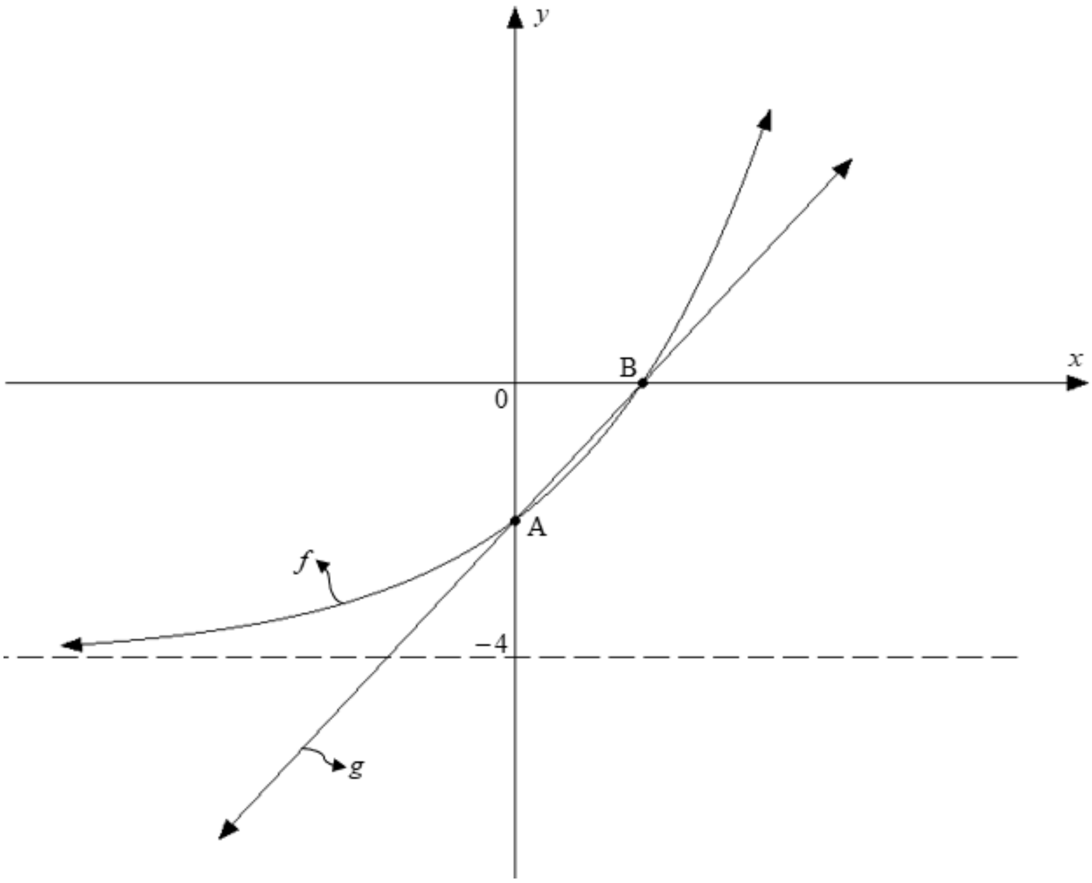
2.4.2 Range of $h$. (2)

2.4.3 Values of $x$ for which $h(x) \geq 0$ (2)

### Question 3

The graphs below represent the functions defined by $f(x) = 2 \cdot a^x + q$ and $g(x) = 2x - 2$

A and B are intercepts with the axis of both $f$ and $g$. Dotted line cuts the $y$-axis at $-4$ .




| | | |
|---|---|---|
| **3.1** | Determine the coordinates of A and B. | (3) |
| **3.2** | Write down the value of $q$. | (1) |
| **3.3** | Hence, determine the equation of $f$. | (3) |
| **3.4** | Write down the value(s) of $x$ for which $f(x) - g(x) = 0$ | (2) |
| **3.5** | Write down the value(s) of $x$ for which $g(x) > f(x)$ | (2) |

### Question 3

Draw sketch graph of the function defined by
$y = f(x) = a \cdot b^{x} + q$, where $b > 0$ and $b \neq 0$ with the following properties:

- $f$ is undefined at the point at the point where $y = -4$
- $f$ is a decreasing function
- Passes through the points $f(0) = -3$ and $(-1;0)$

### Sketching of Semi-circle

Points to remember when sketching Semi-circle

| Function | What to find? | How? |
|---|---|---|
| Semi-circle $y = f(x) = \pm\sqrt{r^2 - x^2}$ | Radius | Use Theorem of Pythagoras |
| | Whether it is above or below the *x*-axis | If $y = f(x) = +\sqrt{r^2 - x^2}$, the graph is above the *x*-axis<br><br>If $y = f(x) = -\sqrt{r^2 - x^2}$, the graph is below the *x*-axis. |

### Example 1

Given the function defined by $h(x) = \sqrt{25 - x^2}$

**1.1** Write down the value of the radius of *h*.

**1.2** Draw the graph of *h*.

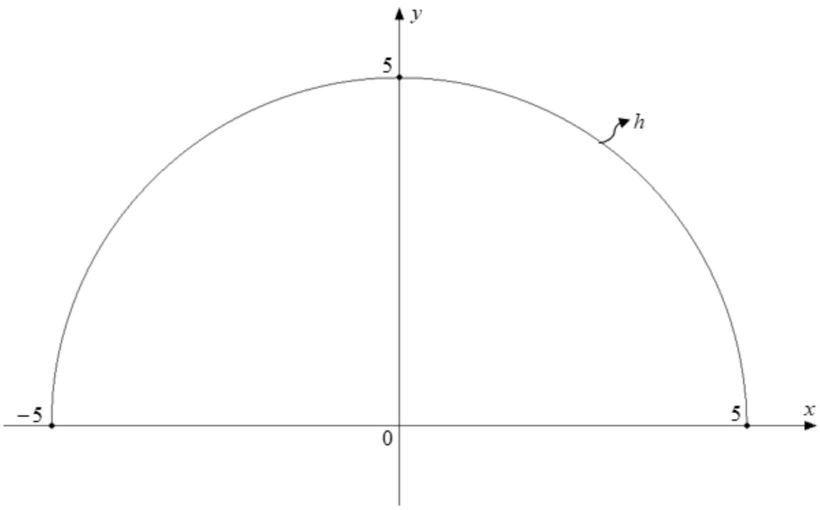
**1.3** Write down the domain of *h*.

## Solution to Example 1

**2.1** $r^2 = 25$

$\therefore r = 5$

**2.2**



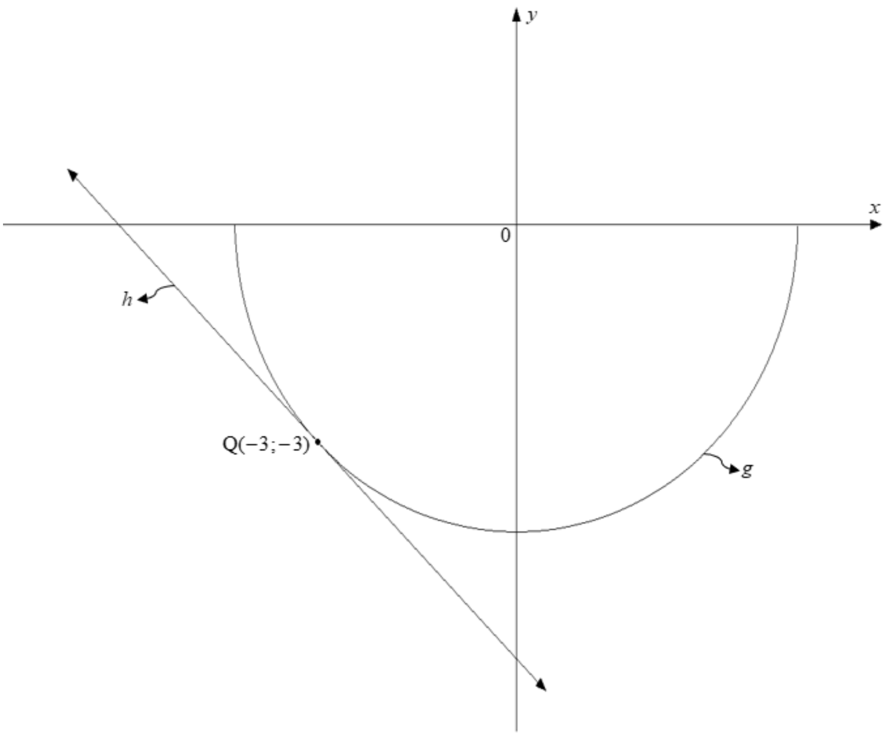

**2.3** $-5 \leq x \leq 5$

**OR**

$x \in [-5;5]$

**Example 2**

In the diagram below, tangent with the function $h(x) = mx + c$ intersect the semi-circle with the function $g(x) = \sqrt{r^2 - x^2}$ at the point $Q(-3;-3)$




Determine

**2.1** Equation of *g*.

**2.2** Equation of *h*.

**2.2** Range of *g*.

## Solution to Example 1

**2.1**

$$x^2 + y^2 = r^2$$
$$(-3)^2 + (-3)^2 = r^2$$
$$r^2 = 18$$
$$\therefore g(x) = -\sqrt{18 - x^2}$$

**2.2**

$$m_r = \frac{-3 - 0}{-3 - 0}$$
$$m_r = 1$$
$$m_r \times m_{\tan} = -1$$
$$\therefore m_{\tan} = -1$$
$$y + 3 = -1(x + 3)$$
$$\therefore h(x) = -x - 6$$

**2.3**

$$-\sqrt{18} \leq y \leq 0$$

**OR**

$$-\sqrt{18} \leq y \leq 0$$

## Practice Questions

### Question 1

Given the function defined by $h(x) = -\sqrt{9 - x^2}$

**1.1** Determine the radius of the circle. (2)

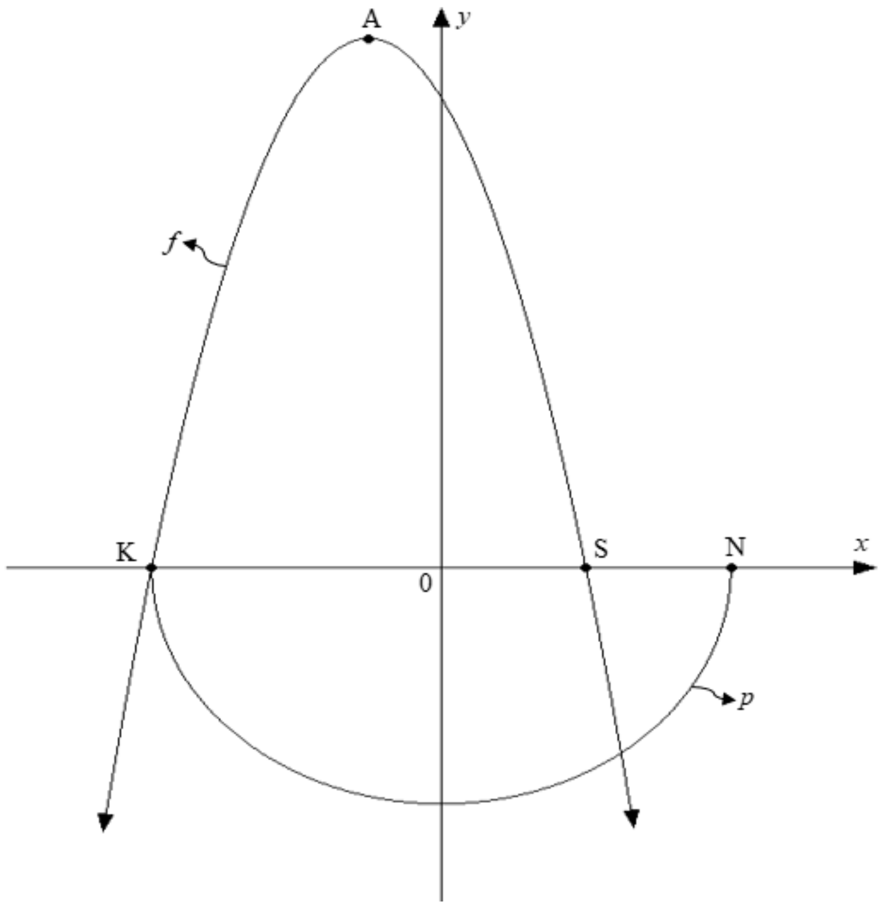
**1.2** Draw the graph of $h$. (2)

**1.3** Write down the range of $h$. (2)

## Question 2

The sketch below represents the function defined by $f(x) = -x^2 - 2x + 8$ and $p(x) = \sqrt{r^2 - x^2}$

S is the $x$-intercept of $f$, while K is the $x$-intercept of both $f$ and $p$.
A and J are the turning point and $y$-intercept of $f$ respectively.
N is the $x$-intercept of $p$ and C is a point on $p$.




**2.1** Determine coordinates of K and S. (5)

**2.2** Determine coordinates of A. (4)

**2.3** Write down the maximum value of $f$. (1)

**2.4** Write down the range of $f$. (2)

**2.5** Determine the equation of $p$. (2)

**2.6** Hence write down the radius of $p$. (1)

## Selected Solutions on Practice Questions

## Question 1 on Parabola and Straight line

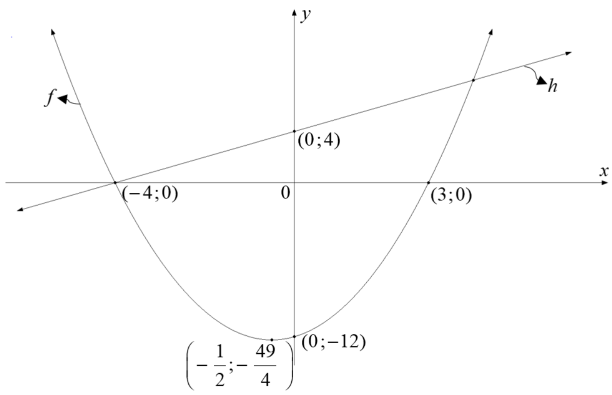
| | | | |
|---|---|---|---|
| **1.1.1** | $y$-intercept ; $x = 0$ | | |
| | $y = f(0) = (0)^2 + 0 - 12 = -12$ | ✓ substituting by 0 | |
| | $y = g(0) = 0 + 4 = 4$ | ✓ *y*-intercept | (2) |
| **1.1.2** | $x$-intercept ; $y = 0$ | | |
| | $x + 4 = 0$ | | |
| | $\therefore x = -4$ | ✓ *x*-intercept | (1) |
| **1.1.3** | $x$-intercepts; $y = 0$ | | |
| | $x^2 + x - 12 = 0$ | | |
| | $(x + 4)(x - 3) = 0$ | ✓ factors | |
| | $\therefore x = -4$ or $x = 3$ | ✓ *x*-intercept | (2) |
| **1.1.4** | $x = -\dfrac{b}{2a}$ | | |
| | $= -\dfrac{1}{2(1)}$ | ✓ substitution | |
| | $x = -\dfrac{1}{2}$ | ✓ value of $x$ | |
| | $f\left(-\dfrac{1}{2}\right) = \left(-\dfrac{1}{2}\right)^2 + \left(-\dfrac{1}{2}\right) - 12$ | ✓ substitution | |
| | $f\left(-\dfrac{1}{2}\right) = -\dfrac{49}{4}$ | | |
| | $\therefore$ turning point is $\left(-\dfrac{1}{2}; -\dfrac{49}{4}\right)$ | ✓ turning point | |
| | **OR** | **OR** | |
| | $\left(-\dfrac{1}{2(1)}; \dfrac{4ac - b^2}{4a}\right)$ | ✓ method/formula | |
| | $\left(-\dfrac{1}{2(1)}; \dfrac{4(1)(-12) - (1)^2}{4(1)}\right)$ | ✓ substitution<br>✓ substitution | |
| | $\therefore$ turning point is $\left(-\dfrac{1}{2}; -\dfrac{49}{4}\right)$ | ✓ turning point | |
| | **OR** | **OR** | |
| | | ✓ $2x + 1 = 0$ | |

$$f^{/}(x) = 2x + 1 = 0$$
$$2x = -1$$
$$x = -\frac{1}{2}$$
$$f(2) = \left(-\frac{1}{2}\right)^2 - 4\left(-\frac{1}{2}\right) + 12$$
$$f(2) = -\frac{1}{2}$$
$$\therefore \text{ turning point is } \left(-\frac{1}{2}; -\frac{49}{4}\right)$$

✓ value of $x$
✓ substitution
✓ turning point

(4)

**1.2**




✓ shape
✓ intercepts
✓ turning point

(3)

**1.3**

$$f(x) - h(x) = 0$$
$$f(x) = h(x)$$
$$x^2 + x - 12 = x + 4$$
$$x^2 - 16 = 0$$
$$(x - 4)(x + 4) = 0$$
$$x = 4 \text{ or } x = -4$$
$$y = 4 + 4 \text{ or } y = -4 + 4$$
$$y = 8 \text{ or } y = 0$$
$\therefore$ points of intersection are
$(4;8)$ and $(-4;0)$

✓ method
✓ equating equation
✓ standard form
✓ factors
✓ values of $x$
✓ points of intersection

(6)

**1.4.1** $y \geq -\frac{49}{4}$

**OR**

✓ critical values
✓ notation

(2)

| | | | |
|---|---|---|---|
| | $\left[-\dfrac{49}{4};\infty\right)$ | | |
| **1.4.2** | $-\dfrac{49}{4}$ | ✓ $-\dfrac{49}{4}$ | |
| | | | (1) |
| **1.4.3** | $x<-4$ or $x>3$<br>**OR**<br>$(-\infty;-4)\cup(3;\infty)$ | ✓ critical values<br>✓ notation | |
| | | | (2) |
| **1.4.4** | $x>3$<br>**OR**<br>$(3;\infty)$ | ✓ critical values<br>✓ notation | |
| | | | (2) |

## Question 4 on Parabola and Straight line

| | | | |
|---|---|---|---|
| **4.1** | $y$-intercept ; $x=0$<br>$5y+2(0)-10=0$<br>$y=2$<br>$\therefore \mathrm{E}(0;2)$ | ✓ coordinates of E | |
| | $x$-intercept ; $y=0$<br>$5(0)+2x-10=0$<br>$\therefore x=5$<br>$\therefore \mathrm{B}(5;0)$ | ✓ coordinates of B | |
| | | | (2) |
| **4.2** | $f(x)=a(x+p)^2+q$<br>$f(x)=a(x-3)^2-4$<br>$0=a(5-3)^2-4$ | ✓ sub. Turning point coordinates<br>✓ sub. $(0;5)$ | |
| | $4a=4$<br>$a=1$<br>$\therefore f(x)=(x-3)^2-4$ | ✓ value of $a$ | |
| | | | (3) |
| **4.3** | $x$-intercepts; $y=0$<br>$(x-3)^2-4=0$<br>$(x-3-2)(x-3+2)=0$ | ✓ equating to 0<br>✓ factors | |
| | $\therefore x=5$ or $x=1$<br>$\mathrm{AB}=5+1=6$ units | ✓ values of $x$<br>✓ length of AB | |
| | | | (4) |

**4.4** $f(x) = (x-3)^2 - 4$

$y$-intercept ; $x = 0$

$y = (0-3)^2 - 4$

$D(0;5)$

$DE = 5 - 2 = 3$ units

✓ coordinates of D
✓ method
✓ length of DE

(3)

**4.5** – 4

✓ – 4

(1)

## Question 1 on Hyperbola

**1.1** $x = 0$ and $y = -1$

✓ $x = 0$
✓ $y = -1$

(2)

**1.2** $x$-intercept ; $y = 0$

$0 = -\frac{6}{x} - 1$

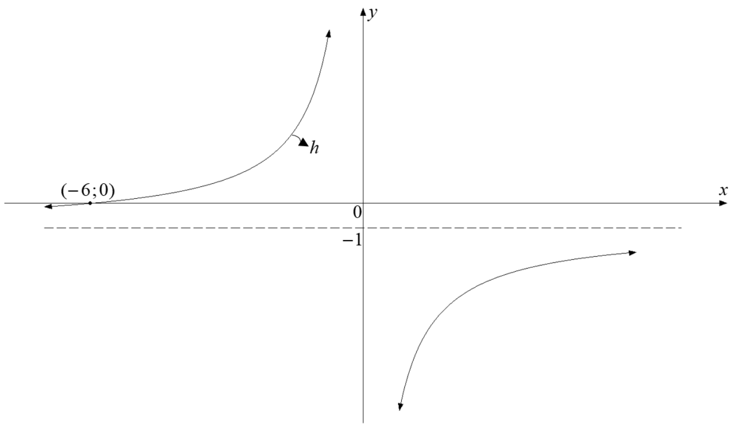
$-1 = \frac{6}{x}$

$\therefore x = -6$

✓ equating to 0
✓ value of $x$

(2)

**1.3**




✓ shape
✓ asymptoes
✓ $x$-intercept

(3)

**1.4** $m < 0$ implies that $m = -1$

$y = -x + c$

Point of intersection of asymptotes is $(0;-1)$

$-1 = 0 + c$

$c = -1$

$\therefore y = -x - 1$

✓ $m = -1$
✓ value of $c$
✓ equation

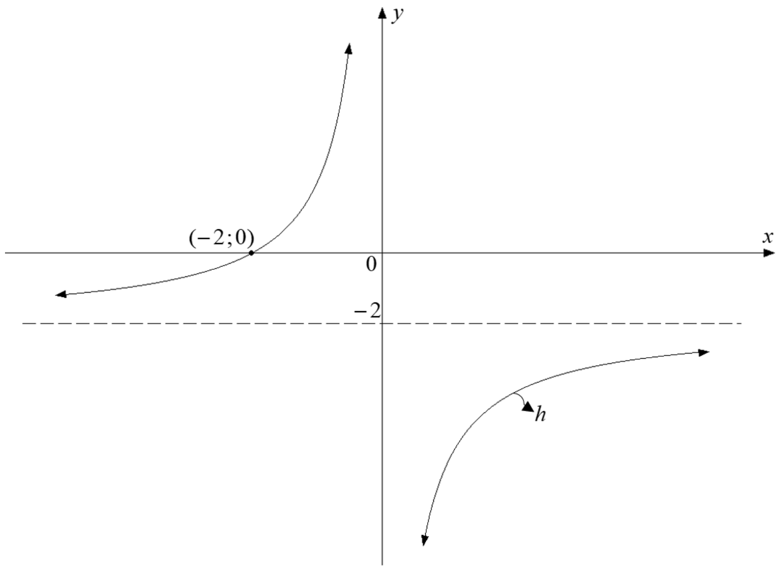
(3)

**1.5** $h(2) = -\dfrac{6}{2} - 1 = -4$ ✓ substitution ✓ $-4$

$\therefore$ the point lies on *h*. (2)

**1.6** $y \in \mathbb{R}\ ,\ y \neq -1$ ✓ range (1)

## Question 3 on Hyperbola




✓ shape
✓ asymptotes
✓ *x*-intercept
(3)

## Question 3 on Exponential function

**1.1** $y = -9$ ✓ $y = -9$ (1)

**1.2**

$$0 = \left(\frac{1}{3}\right)^{x} - 9$$ ✓ $0 = \left(\frac{1}{3}\right)^{x} - 9$

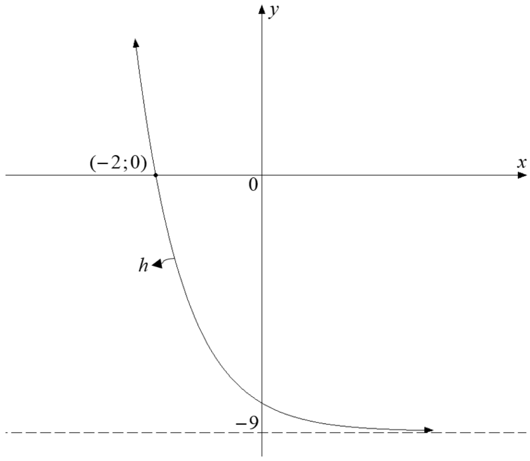
$$3^{-x} = 9$$

$$3^{-x} = 3^{2}$$ ✓ $3^{-x} = 3^{2}$

$$-x = 2$$

$$\therefore x = -2$$ ✓ value of $x$ (3)

**1.3**




✓ shape
✓ asymptotes
✓ *x*-intercept
(3)

**1.4.1** $x \in \mathbb{R}$
**OR**
$x \in (-\infty;\infty)$

✓ domain
(1)

**1.4.2** $y > -9$
**OR**
$y \in (-9;\infty)$

✓ critical values
✓ notation
(2)

**1.4.3** $x \leq -2$
**OR**
$x \in [-2;-\infty)$

✓ critical values
✓ notation
(2)

## 3. EXAMINATION TIPS

- Always have relevant tools (Calculator, Mathematical Set, etc.)
- Read the instructions carefully.
- Thoroughly go through the question paper, check questions that you see you are going to collect a lot of marks, start with those questions in that order because you are allowed to start with any question but finish that question.
- Write neatly and legibly.

## 4. ACKNOWLEDGEMENT

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